A Characterization of Identified Impulse Response Sets in Invertible Linear State-Space Models

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Abstract

This paper characterizes, for an arbitrary model with invertible vector autoregressive representation and for an arbitrary set of identification restrictions, the set of impulse vectors consistent with the proposed identification scheme. For identification procedures that afford point identification this means recovery of the unique identified impulse vector; for set-identifying restrictions, this means recovery of an entire set of identified impulse vectors. Success or failure of a given identification scheme against an underlying data-generating process can then be judged through the overlap (or lack thereof) between identified set of impulse vectors and the impulse vector of interest. The main application of the methodology is a re-interpretation of the findings in Uhlig (2005). I argue that, against a range of entirely conventional underlying structural models, his identification procedure tends to pick out positive demand and supply shocks jointly masquerading as monetary policy shocks. His failure to reject monetary policy non-neutrality in the data is thus exactly what simple sticky-price models with money non-neutrality would lead us to expect.

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1 Introduction

Structural vector autoregressions (SVARs) are a key part of the applied macroeconomist's toolkit. Among other things, they have been used for policy analysis and to provide the stylized facts that inform modern dynamic stochastic general equilibrium (DSGE) modeling. As is well known, the key hurdle in the analysis of SVARs is identification. The data allow the researcher to consistently estimate the true reduced-from vector autoregressive system, but each such reduced-form VAR is consistent with a continuum of – usually very different – underlying SVARs. Thus, to go from reduced-form VAR to SVAR, the researcher will need to bring in some extra outside information. Such so-called identifying restrictions can take many forms. The most common are direct restrictions on the impulse responses of the estimated VAR. These restrictions can range from exact zero restrictions at short horizons (e.g. Sims (1980)) and zero restrictions at long horizons (e.g. Blanchard and Quah (1993)) to sign restrictions (e.g. Uhlig (2005)), or of course to any combination thereof. Rubio-Ramirez et al. (2010) establish general conditions under which a given combination of identifying restrictions is sufficient to go from estimated reduced-form VAR to some unique underlying SVAR. Early contributions to the SVAR literature by and large restrict attention to identification schemes that afford such exact identification. More recently, however, some researchers have turned away from exact point identification and towards what is known as set identification. In a set-identified SVAR, the researcher uses outside information (usually in the form of sign restrictions) to shrink the set of SVARs consistent with the estimated reduced-form VAR not to a single point, but to some – hopefully small enough – set of admissible SVARs.

Set identification of SVARs has many advantages. First and foremost, the proposed identifying restrictions can be much weaker than those required for point identification. Few researchers actually believe in the stringent zero restrictions required for point identification, in particular if they apply to short-run relations between variables. Sign restrictions, in contrast, are often chosen such that they will enjoy near unanimous support in the profession. Alas, the weakness of the required identifying restrictions is both blessing and curse. Less restrictions mean less power, and so set-identified SVARs may be silent on the question of interest.

Ideally, researchers would select their identification scheme in a way so that, against a broad range of "plausible" underlying economic models, the restrictions are strong enough to get us, with high probability, close to the true SVAR. So far, however, this has only been done in a very crude way. In most applications, researchers are interested in a single structural disturbance and its propagation through the system. The proposed identifying restrictions then usually amount to (sign) restrictions on impulse responses that, against a range of plausible models, only the shock of interest will satisfy. Clearly this is a minimum requirement for correct structural analysis. Unfor-
fortunately, however, it is not sufficient, since combinations of the other disturbances can masquerade as the shock of interest. The problem of masquerading shocks has, in some papers, been mentioned in passing (for example in Uhlig (2005)), but to the best of my knowledge never analyzed thoroughly.

To discipline the choice of identifying restrictions, this paper instead proposes the following approach. Start with a range of "plausible" economic models, and consider some set of identifying restrictions. The theoretical contribution of this paper is to exactly characterize, for models that admit a VAR representation and for a given identification procedure, the identified set of underlying SVARs. If the proposed procedure does a good job of recovering the true SVAR across the range of models that the researcher is willing to entertain, then – and I would argue only then – should the proposed set of identifying restrictions be taken to the data. So far researchers have skipped the first step, or at best replaced it with some loose model-based reasoning. This may have, as I will argue later, led to some misleading inference.

Even though my analysis is explicitly motivated by the set identification literature for SVARs, it of course just as well applies to identification schemes that afford point identification. In such cases, my procedure allows us to see how close the identified SVAR is to true model-implied SVAR, and so to what extent the proposed identifying restrictions distort inference. An analysis of this sort has antecedents in the literature. For example, Carlstrom et al. (2009) show exactly how close, in a simple three-equation New Keynesian model, a simple recursive identification scheme of the sort proposed in Sims (1980) gets us to the true model-implied impulse responses. My analysis is a strict generalization of theirs, in the sense that I can characterize the identified set for an arbitrary underlying model, as long as it has an invertible VAR representation.

The main application of my procedure is to re-interpret the evidence on monetary policy non-neutrality presented in Uhlig (2005). Under his proposed identification procedure, monetary policy neutrality cannot be rejected; in fact, if anything, contractionary monetary policy shocks seem to boost, not lower, output. I interpret his evidence through the lens of the two most conventional DSGE models imaginable: the standard three-equation New Keynesian model, and the Smets-Wouters DSGE. All the intuition will already be in the small-scale model; analysis of the Smets-Wouters then show us that most of the conclusions from the baseline model also survive in a medium-scale DSGE with many bells and whistles. My main finding is that Uhlig’s proposed identification procedure is simply too weak; in other words, it does not pass the baseline check for identification procedures I proposed above. Intuitively, Uhlig’s restrictions allow combinations of positive supply and demand shocks to masquerade as contractionary monetary policy shocks. Prices fall, interest rates rise, and output rises as well. Thus, Uhlig’s results are exactly what we would expect from applying his procedure to data generated from a model in which monetary
non-neutrality actually holds. This masquerading occurs in almost the exact same way in the baseline three-equation New Keynesian model and in the Smets-Wouters DSGE, suggesting that the simple demand-supply masquerading intuition is quite robust. In a very recent contribution, Arias et al. (2015) slightly extend Uhlig’s identification restrictions and, with their tighter set of sign restrictions, find the expected negative response of output to contractionary monetary policy disturbances. Interestingly, their proposed extra restrictions are also, in the context of the three-equation New Keynesian model, sufficient to get us much closer to the true model-implied SVAR, in particular as regards the response of output to the identified monetary policy innovation. In the Smets-Wouters model, in contrast, the proposed additional restrictions fail to give us much extra traction; in fact, as I will show, they are violated in the Smets-Wouters model. The reason for this surprising finding lies, as I will discuss in more detail later, in the tempting, but erroneous confusion of the central bank's reaction function – one of the equations entering the linear rational expectations model – and the monetary policy equation in the implied VAR(∞). In summary, my model-based approach allows us to transparently see why additional restrictions of the sort proposed by Arias et al. (2015) may, but need not, improve identification of monetary policy shocks.

The rest of this paper proceeds as follows. In section 2, I review the identification problem in the SVAR generated by an invertible linear state-space model. In section 3, I then, for a given set of identifying restrictions, characterize the identified set of SVARs, first analytically and then, with analytical tools exhausted, numerically. Section 4 contains the application of my procedure to the identification of monetary policy disturbances through sign restrictions. Section 5 concludes.

2 Identification in Linear State-Space Models

2.1 From State-Space Model to VAR

Consider the following linear Gaussian state-space system:

\[ x_t = \Lambda x_{t-1} + \Psi w_t \]  
\[ y_t = \Theta x_{t-1} + \Xi w_t \]  

Here \( w \) collects the structural disturbances, \( x \) collects the state variables of the system and the \( n \)-dimensional vector \( y \) collects the observed variables. Suppose that \( \Xi \) is invertible. Furthermore, I assume that \( w_t | I_{t-1} \sim N(0, \Omega) \), where \( \Omega \) is a diagonal matrix.\(^1\) To proceed further we need to consider the matrix

\[ \Lambda - \Psi \Xi^{-1} \Theta \]

\(^1\)The important assumption throughout will be that the structural disturbances come from a distribution uniquely parameterized by its mean and variance.
If all eigenvalues of this matrix have absolute value less than 1 then the linear state-space system
has a VAR($\infty$) representation with the VAR innovations spanning the structural disturbances.
Plugging (2) into (1) and solving we get:

$$x_t = (I - (\Lambda - \Psi \Xi^{-1} \Theta)L)^{-1} \Psi \Xi^{-1} y_t$$

Plugging this into (2):

$$y_{t+1} = \Theta \sum_{j=1}^{\infty} [\Lambda - \Psi \Xi^{-1} \Theta]^{j-1} \Psi \Xi^{-1} y_{t-j} + \Xi w_t$$

It is time to bring the notation somewhat closer to the standard structural VAR literature. First, in
line with most contributions in this literature, I normalize the variance of the structural disturbances
to 1. We can of course trivially write

$$y_t = \Theta \sum_{j=1}^{\infty} [\Lambda - \Psi \Xi^{-1} \Theta]^{j-1} \Psi \Xi^{-1} y_{t-j} + \Xi \Omega^{1/2} \Omega^{-1/2} w_t$$

Thus $\epsilon_t \equiv \Omega^{-1/2} w_t$ is the desired normalized structural disturbance vector. The VAR($\infty$) repre-
sentation becomes

$$y_t = \Theta \sum_{j=1}^{\infty} [\Lambda - \Psi \Xi^{-1} \Theta]^{j-1} \Psi \Xi^{-1} y_{t-j} + \Xi \Omega^{1/2} \epsilon_t$$

Second, I compactify notation. Define $B_j \equiv [\Lambda - \Psi \Xi^{-1} \Theta]^{j-1} \Psi \Xi^{-1}$ to be the coefficient matrix for
lag $j$, and let $u_t \equiv \Xi \Omega^{1/2} \epsilon_t$. We then have the following reduced-form VAR representation

$$y_t = \sum_{j=1}^{\infty} B_j y_{t-j} + u_t \quad (3)$$

where $u_t|I_t \sim N(0, \Xi \Omega \Xi')$. From now on I will let $\Sigma \equiv \Xi \Omega \Xi'$. Stacking regressors into the vector
$x_t$ and coefficient matrices into the matrix $B_+$, we can compactify even further and write

$$y_t = B_+ x_t + u_t \quad (4)$$

Defining $A_0 = (\Xi \Omega^{1/2})^{-1}$, we get the corresponding structural VAR representation

$$A_0 y_t = \sum_{j=1}^{\infty} A_j y_{t-j} + \epsilon_t \quad (5)$$

where $\epsilon_t|I_t \sim N(0, I)$ and $A_j \equiv (\Xi \Omega^{1/2})^{-1} B_j$. Again stacking regressors and coefficient matrices,
we get the more compact expression

\[ A_0 y_t = A_+ x_t + \epsilon_t \] (6)

2.2 Identification of Structural VARs

This section provides a brief review of the identification problem for structural VARs. The presentation here draws heavily on Rubio-Ramirez et al. (2010). Consider the structural-form VAR (6). The structural form is uniquely characterized by the parameter vector \((A_0, A_+)\).

Following Rothemberg (1971), I say that two structural-form VAR models \((A_0, A_+)\) and \((\tilde{A}_0, \tilde{A}_+)\) are observationally equivalent if and only if they imply the same distribution for the vector of observables \(y_t\), for any arbitrary time horizon. In the Gaussian linear environment studied here, this means that two structural VARs are observationally equivalent if and only if they imply the same reduced-form VAR representation \((B_+, \Sigma)\).

It is easy to see that, under this definition of observational equivalence, the structural-form VAR (6) is only identified up to the set \(O(n)\) of orthogonal matrices of size \(n\). For every \(P \in O(n)\), the alternative structural VAR \((\tilde{A}_0, \tilde{A}_+)\), with \(A_0 = P^{-1} \tilde{A}_0\) and \(A_+ = P^{-1} \tilde{A}_+\) implies the same reduced-form VAR (4), and so all such \(P\) are equally consistent with the data.\(^2\)

Out-of-sample information can be used to sharpen identification to strict subsets \(O(n)\). Following Rubio-Ramirez et al. (2010), I allow out-of-sample information to take the form of linear restrictions on transformations of the structural parameter space into \(k \times n\) matrices, where \(k > 0\). Denote the transformation by \(f(\cdot)\). Linear restrictions of the transformation can then be represented via \(k \times k\) matrices \(Z_j\) and \(S_j\), with \(j = 1, \ldots, n\).

Definition 1. Identified Set of Rotation Matrices. Denote the true set of SVAR parameters by \((A_0, A_+)\). Let \(R\) be an identifying restriction, defined by a transformation \(f(\cdot)\) and a set of restriction matrices \(Z_j, S_j, j = 1, \ldots, n\). Then the identified set of rotation matrices, \(P_R\), is defined as follows:

\[ P_R \equiv \{ P \in O(n) \mid Z_j f(PA_0, PA_+) e_j = 0 \text{ and } S_j f(PA_0, PA_+) e_j \geq 0 \text{ for } 1 \leq j \leq n \} \]

where \(e_j\) is the \(j\)-th column of the \(n \times n\) identity matrix \(I_n\).

The \(Z_j\) allow us to impose exact linear restrictions on \(f(\cdot)\), and the \(S_j\) allow us to impose linear sign restrictions.\(^3\) Identifying restrictions may afford point identification, set identification or they may be infeasible.

\(^2\)To get the same reduced-form VAR any invertible matrix \(P\) would do. However, in our definition of the structural VAR, we require the structural disturbances to have unit variance. Thus only orthogonal matrices \(P\) are allowed.

\(^3\)I could in fact represent any given set of exact linear and sign restrictions through only sign restrictions, with weak inequalities in both directions enforcing the exact restriction. However, for the sake of notational clarity, I
Definition 2. Let $P_R$ denote the identified set of rotation matrices of a restriction $R$ with respect to a true set of SVAR parameters by $(A_0, A_+).$ We then say the following:

1. If $P_R$ is a singleton, then $R$ affords point identification with respect to $(A_0, A_+).$
2. If $P_R$ is empty, then $R$ is infeasible with respect to $(A_0, A_+).$
3. If $P_R$ is neither a singleton nor empty, then $R$ affords set identification with respect to $(A_0, A_+).$

Note that set identification is consistent with both $\mu(P_R) = 0$ and $\mu(P_R) > 0,$ where $\mu(\cdot)$ denotes the Haar measure on the group of orthogonal rotation matrices. Sign restrictions generically afford set identification with $\mu(P_R) > 0,$ while combinations of sign and zero restrictions generically afford set identification with $\mu(P_R) = 0.$

An important class of restrictions is what I call minimally consistent restrictions.

Definition 3. A restriction $R$ is minimally consistent with respect to a true set of SVAR parameters by $(A_0, A_+)$ if and only if

$$I \in P_R$$

Minimally consistent restrictions are trivially feasible.

2.3 Interpreting $P_R$

For what follows it will be useful to get a better understanding of the identified set of rotation matrices $P_R.$ Recall that the true structural VAR is

$$A_0 y_t = A_+ x_t + \epsilon_t$$

Now consider an arbitrary $P \in P_R.$ This $P$ implies the incorrect SVAR

$$\hat{A}_0 y_t = \hat{A}_+ + P\epsilon_t$$

Thus, the $j$-th structural disturbance according to this incorrect SVAR is actually a linear combination of the $n$ true structural disturbances, with the weights given by the $j$th row of $P$ (or equivalently the $j$th column of $P^{-1}$). In particular, $P = I_n$ allows us to recover the true shocks.

Now suppose we are interested in identification of the $j$-th structural shock. We can define the set of identified structural disturbance vectors as follows.

distinguish between sign and exact restrictions, and so implicitly assume that the sign restriction matrices do not enforce any exact equality.
Definition 4. **Identified Set of Structural Disturbance Vectors.** Denote the true set of SVAR parameters by \((A_0, A_+)|\). Let \(R\) be an identifying restriction, defined by a transformation \(f(\cdot)\) and a set of restriction matrices \(Z_j, S_j, j = 1, \ldots, n\). Then the set of identified structural disturbance vectors for shock \(j\), \(D_j\), is defined as follows:

\[
D_j \equiv \{ p_j \in \mathbb{R}^n | p_j \text{ is the } j\text{th row of } P \text{ and } P \in P_R \}
\]

A \(p_j \in D_j\) is called admissable.

Unfortunately, this identified set of structural disturbance vectors is not always straightforward to interpret, for two reasons. First, different variables may enter the VAR with different scales. Second, some variables may only be minimally related to others, reflected in small values of \(A_+\). Together, these two observations imply that the entries of admissable \(p_j\)'s cannot be directly interpreted as measures of the quantitative significance of the true shock \(k\) for the identified shock \(j\). We thus may in some cases need some kind of normalization rule.

Definition 5. **Normalized Identified Set of Structural Disturbance Vectors.** A normalization rule is a vector \(w \in \mathbb{R}^n_+\) with ||\(w|| = 1\). The normalized identified set of structural disturbance vectors for shock \(j\) with respect to normalization rule \(w\), \(D_j^w\), is defined as follows:

\[
D_j^w \equiv \{ w \cdot p_j | p_j \in D_j \}
\]

2.4 Identification Through Restrictions on Impulse Responses

Following recent developments in the SVAR literature, I will throughout focus on a certain class of identifying restrictions: restrictions on impulse response functions (IRFs). For a given candidate structural VAR \((\tilde{A}_0, \tilde{A}_+)\), the IRF of \(i\)-th variable to the \(j\)-th structural shock at finite horizon \(h\) corresponds to the element in row \(i\) and column \(j\) of the matrix

\[
IR_h(\tilde{A}_0, \tilde{A}_+) = (\tilde{A}_0^{-1} J F^h_h J')
\]

where

\[
F_h = \begin{pmatrix}
\tilde{A}_1 \tilde{A}_0^{-1} & I_n & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{h-1} \tilde{A}_0^{-1} & 0 & \ldots & I_n \\
\tilde{A}_h \tilde{A}_0^{-1} & 0 & \ldots & 0
\end{pmatrix}
\]

and

\[
J = \begin{pmatrix}
I_n \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

4In the formulation here the size of the matrix \(F_h\) depends on the horizon \(h\). This is necessary because I want to allow for VAR(\(\infty\)). If IRFs up to some fixed horizon \(H\) are desired, then we could either stick with the varying-size matrices \(F_h\) specified above or just take \(F_H\) throughout.
If the $i$-th variable is in first differences then we can also compute the long-run IRF, given as

$$IR_\infty(\tilde{A}_0, \tilde{A}_+) = (\tilde{A}_0 - \sum_{i=1}^{\infty} \tilde{A}'_i)^{-1}$$

I denote specific elements of these matrices by $IR_{i,j,h}$, and entire rows or columns by $IR_{i,-}$, $IR_{-,j,h}$, respectively. Restrictions on IRFs can be expressed through transformation functions $f(\cdot)$ of the following form:

$$f(\tilde{A}_0, \tilde{A}_+) = \begin{pmatrix} \ldots \\ IR_h(\tilde{A}_0, \tilde{A}_+) \\ \ldots \end{pmatrix}$$

Throughout I will consider restrictions on IRFs up to some fixed horizon $H$. From here we can now define the identified set of impulse responses.

**Definition 6.** Identified Set of Impulse Responses. Denote the true set of SVAR parameters by $(A_0, A_+)$. Let $R$ be an identifying restriction, defined by a transformation $f(\cdot)$ and a set of restriction matrices $Z_j, S_j, j = 1, \ldots, n$. Then the identified set of impulse responses for variable $i$ in response to shock $j$ at horizon $h$, $IS_{i,j,h}$ is defined as follows:

$$IS_{i,j,h} \equiv \{ a \in \mathbb{R} \mid a = IR_{i,j,h}(PA_0, PA_+), P \in PR \}$$

In words, the identified set of impulse responses for variable $i$ in response to shocks to variable $j$ at horizon $h$ consists all horizon-$h$ impulse responses of variable $i$ to shock $j$ that can be generated through a rotation matrix $P$ that lies in the identified set of rotation matrices. Throughout the rest of the analysis, I will make the following assumption.

**Assumption 2.1.** The identifying restriction $R$, defined by the transformation $f(\cdot)$ and the set of restriction matrices $Z_j, S_j, j = 1, \ldots, n$, is feasible.

Under this assumption the identified set of rotation matrices and so the identified set of impulse responses are non-empty. Naturally, if $PR$ is a singleton, then $IS_{i,j,h}$ is a singleton for every $i, j, h$. The goal of the next section is to sharply characterize the identified set of impulse responses whenever $\mu(PR) > 0$.

### 3 Characterizing the Identified Set

Suppose that $\mu(PR) > 0$. For any given variable $i$, shock $j$ and horizon $j$, the identified set of impulse responses is a subset of $\mathbb{R}$. The goal of this section is twofold. First, I provide some general analytical statements about the shape of this identified set for a generic identification problem. Second, I discuss numerical techniques to recover the identified set.
3.1 Some Analytical Results

A useful starting point for general statements about the shape of the identified set of impulse responses is a more careful consideration of the identified set of rotation matrices $P_R$. We may write $P_R \equiv O(n) \cap X$, where again $O(n)$ is the set of orthogonal matrices $O(n)$ and $X$ is the set of matrices in $\mathbb{R}^{n \times n}$ such that, for all $X \in X$, $IR_{i,j,h}(XA_0, XA_+)$ satisfies the imposed sign and zero restrictions for all $i, j, h$. For $O(n)$ we have the following two useful results.

**Lemma 3.1.** $O(n)$ is compact.

This is a well-known fact. For the second result recall that a Givens matrix of dimension $n$ with argument $\theta_{i,j}$, written $Q_{i,j}(\theta_{i,j})$, has the following structure:

$$
Q_{i,j}(\theta_{i,j}) = \begin{pmatrix}
\col i & \col j \\
\downarrow & \downarrow \\
1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\row i & \rightarrow & 0 & \ldots & \cos(\theta_{i,j}) & \ldots & -\sin(\theta_{i,j}) & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\row j & \rightarrow & 0 & \ldots & \sin(\theta_{i,j}) & \ldots & \cos(\theta_{i,j}) & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 1 \\
\end{pmatrix}
$$

We then have the following.

**Lemma 3.2.** Let $P$ be an arbitrary $n \times n$ orthogonal matrix. Then there exist $\frac{n(n-1)}{2}$ Givens matrices $Q_{i,j}(\theta_{i,j})$, with $1 \leq i < n$, $i \leq j \leq n$, such that $0 \leq \theta_{i,j} \leq \pi$ for $i < j < n$, $0 \leq \theta_{i,n} \leq 2\pi$ and

$$
P = \prod_{i=1}^{n} \prod_{j=i+1}^{n} Q_{i,j}(\theta_{i,j})
$$

That is, every $P \in O(n)$ can be written as the product of $\frac{n(n-1)}{2}$ Givens matrices.

For future reference I will denote this Givens rotation mapping by $G : \mathbb{R}^{n(n-1)/2} \rightarrow O(n)$.

To characterize $X$, it will be useful to exploit the additional structure afforded by our focus on zero and sign restrictions. For all sign restricted impulse responses, let $l_{i,j,h}$ and $u_{i,j,h}$ denote, respectively, the lower and upper restricted response of variable $i$ to shock $j$ at horizon $h$.\(^5\) Then

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\(^5\)If no constraint is desired, then we can just set $l_{i,j,h} = \min_j IR_{i,j,h}$ and $u_{i,j,h} = \max_j IR_{i,j,h}$.\)
$X$ consists of all matrices $X \in \mathbb{R}^{n \times n}$ such that

\[
IR_{i\cdot,h} \cdot x_j' \geq l_{i,j,h}
\]
\[
IR_{i\cdot,h} \cdot x_j' \leq u_{i,j,h}
\]

for all $i = 1, \ldots, N$, $j = 1, \ldots, H$, $h = 1, \ldots, H$, and where $x_j$ denotes the $j$-th row of $X$. From this characterization the following is immediate.

**Lemma 3.3.** $X$ is compact.

From here we can conclude the following.

**Lemma 3.4.** $P_R$ is compact.

*Proof.* $P_R \equiv O(n) \cap X$, and both $O(n)$ and $X$ are compact, so their intersection is also compact. □

We can use these results to characterize the edges of the identified set of impulse responses.

**Definition 7.** The upper edge of the identified set of impulse responses for variable $i$ in response to shock $j$ at horizon $h$, denoted $IS_{i,j,h}^u$, is defined as follows:

\[
IS_{i,j,h}^u = \sup_{P \in P_R} IR_{i,j,h}(PA_0, PA_+)
\]

Similarly, the lower edge is defined as follows:

\[
IS_{i,j,h}^l = \sup_{P \in P_R} IR_{i,j,h}(PA_0, PA_+)
\]

The first substantive result is that both edges are attained.

**Theorem 3.5.** Let $R$ be a feasible identifying restriction on the impulse responses of the structural VAR $(A_0, A_+)$. Then, for all $i = 1, \ldots, N$, $j = 1, \ldots, H$, and $h = 1, \ldots, H$, there exist matrices $P_{i,j,h}^u$ and $P_{i,j,h}^l$ such that

\[
IS_{i,j,h}^u = IR_{i,j,h}(P_{i,j,h}^u A_0, P_{i,j,h}^u A_+)
\]
\[
IS_{i,j,h}^l = IR_{i,j,h}(P_{i,j,h}^l A_0, P_{i,j,h}^l A_+)
\]

*Proof.* Any matrix $P \in O(n)$ can be written as the product of $\frac{n(n-1)}{2}$ Givens matrices, and so, since matrix multiplication is a continuous operation, $IR_{i,j,h}(P(\theta)A_0, P(\theta)A_+)$ is a continuous function of the $\frac{n(n-1)}{2}$-dimensional vector of Givens angles $\theta$. The objective function thus is continuous. But the constraint set $P_R$ is compact, so suprema and infima are attained. □
In some applications, researchers are mainly interested in the maximal (minimal) admissible response of some variable \(i\) to some structural shock \(j\), at a range of horizons. In other words, they are precisely interested in the edges of the identified set. From the previous result we conclude that we can exactly recover these edges by solving an \(\frac{n(n-1)}{2}\)-dimensional constrained optimization problem. Even for moderately large systems, solutions to problems of this sort can be obtained quite quickly.

More generally, however, researchers will be interested in the entire identified set (or, more specifically, statistics of the identified set beyond minima and maxima). Unfortunately, the upper and lower edges are not sufficient to characterize \(IS_{i,j,h}\), since \(IS_{i,j,h}\) need not be connected.\(^6\) Instead we can only obtain the following results.

**Lemma 3.6.** Let \(R\) be a feasible identifying restriction on the impulse responses of the structural VAR \((A_0, A_+). Then, for all \(i = 1, \ldots, N, j = 1, \ldots, H, and h = 1, \ldots, H, the identified set \(IS_{i,j,h}\) is compact.

**Proof.** \(P_R\) is compact, and \(G\) is a continuous function, so the pre-image of \(P_R\) is a compact subset of \(\mathbb{R}^{n(n-1)/2}\). But \(IR_{i,j,h}(P(\theta)A_0, P(\theta)A_+)\) is a continuous function of \(\theta\), so the identified set \(IS_{i,j,h}\) is also compact.

From there it is easy to arrive at the following conclusion:

**Theorem 3.7.** Let \(R\) be a feasible identifying restriction on the impulse responses of the structural VAR \((A_0, A_+). Then, for all \(i = 1, \ldots, N, j = 1, \ldots, H, and h = 1, \ldots, H, the identified set \(IS_{i,j,h}\) is the union of \(M\) closed intervals \([IS_{i,j,h}^{l,m}, IS_{i,j,h}^{u,m}]\), where \(IS_{i,j,h}^{l,m} \geq IS_{i,j,h}^{l}\) and \(IS_{i,j,h}^{u,m} \leq IS_{i,j,h}^{u}\) for all \(m = 1, \ldots, M\).

This is all we can say at this level of generality. To then further characterize the identified set for a given system we need to rely on numerical techniques.

### 3.2 Some Numerical Procedures

We wish to numerically characterize \(IS_{i,j,h}\) for all \(i, j, h\). Since the mapping from elements of \(D_j\) to elements of \(IS_{i,j,h}\) is linear, we can just as well aim to numerically characterize \(D_j\), and

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\(^6\)Examples of disconnected identified sets of impulse responses can easily be constructed for two-variable VARs. Let the two variables of the VAR be \((x, y)\), and suppose that we are only interested in identifying a single shock. Furthermore suppose that the default SVAR (the starting point for our rotation) implies horizon-0 responses \((1, 0)\) and \((0, 1)\) to the two disturbances. Finally suppose that our sole identifying restriction is that the response of \(y\) to the identified shock must lie in \([0.2, 0.5]\). Then the loading on the second shock can lie anywhere in \([0.2, 0.5]\), while the loading on the first shock must lie anywhere in \([-\sqrt{0.96}, -\sqrt{0.75}] \cap [\sqrt{0.75}, \sqrt{0.96}]\), so the identified set for the response of \(x\) to the identified shock is disconnected.
so \( P_R \). The definition of \( P_R \) then suggests two approaches – one deterministic, and one probabilistic.

**Discretization of \( O(n) \).** We can approximate the set of orthogonal rotation matrices \( O(n) \) through some finite subset \( P \). The numerical approximation to \( P_R \) is then \( P \cap P_R \). The potential problem with this procedure is how to construct a "good" discretization. Given the linear mapping from \( D_j \) to \( IS_{i,j,h} \), we would ideally like an algorithm that, for any given \( \epsilon \), creates an \( \epsilon \)-net on \( O(n) \). Another possible approach would be to simply discretize the Givens rotation basis for the set of orthogonal matrices, for example through uniform discretization of the basis intervals \([0, \pi], [0, 2\pi]\).

The efficient construction of \( \epsilon \)-nets appears to be an unsolved problem. Discretization of the Givens basis, in contrast, is conceptually trivial, but quickly becomes computationally infeasible as system size increases. For example, for an \( n \)-dimensional system with \( k \) grid points for every interval \([0, \pi]\) and \( 2k \) points for every interval \([0, 2\pi]\), we would overall need \( k^{n(n-1) + (n-1)} \) points. Fortunately, in many applications, researchers are only interested in the response to a single structural shock. For an \( n \)-dimensional system, a single structural disturbance vector will be an element of the unit sphere \( S(n) \). The construction of \( \epsilon \)-nets for unit spheres is possible for reasonably small \( n \), but since I am not aware of any algorithms that work for general \( n \) I focus instead again on basis discretization. Any disturbance vector \( p \in S(n) \) can be represented in spherical coordinates through \( n - 1 \) angular coordinates, with the \( n - 1 \)-th coordinate ranging over \([0, 2\pi]\) and all other coordinates ranging over \([0, \pi]\). Thus, for \( k \) gridpoints per \([0, \pi]\) interval, we would overall get \( 2 \cdot k^{n-1} \) vectors in \( S(n) \). An important drawback of this discretization approach is its inability to deal with combinations of zero and sign restrictions. With zero and sign restrictions we generically have \( \mu(P_R) = 0 \), and so with probability 1 all rotation matrices in the discretized basis will lie outside of \( P_R \).

**Uniform Draws.** We can uniformly draw from \( P_R \). Rubio-Ramirez et al. (2010) provide a suitable algorithm for the purely sign-restricted case, and Arias et al. (2014) provide an algorithm for the mixture case of sign and zero restrictions. The potential problem with this procedure is how to find bounds on the expected number of draws required to give a "good" characterization of \( D_j \) with sufficiently high probability. First, however, I will proceed to prove some asymptotic properties of this approach.

**Lemma 3.8.** Let \( \theta \) be such that \( P(\theta) \in P_R \). Furthermore let \( \theta \) be generic in the sense that, for all \( i,j,h \), \( IR_{i,j,h}(P(\theta)A_0, P(\theta)A_+ + \frac{1}{2}) \) does not satisfy any sign restrictions with equality. Then there exists some \( \epsilon > 0 \) such that, for all \( \theta' \in B_\epsilon(\theta) \), \( P(\theta') \in P_R \), where \( B_\epsilon(\theta) \) denotes an \( \epsilon \)-ball around \( \theta \) in the standard Euclidian metric.

**Proof.** Since \( \theta \) is generic, the result is immediate by the continuity of the constraint set \( X \) together
with the fact that \( P(\theta') \in O(n) \) for any \( \theta' \in B_\epsilon(\theta) \).

**Theorem 3.9.** Let \( \theta \) be such that \( P(\theta) \in P_R \), and let \( n \) be the number of draws of rotation matrices \( P \in P_R \). Then, for \( \epsilon \) small enough, the probability of drawing \( P' \in P_R \cap B_\epsilon(\theta) \) goes to 1 as \( n \to \infty \).

**Proof.** By the previous lemma we conclude that there exists some \( \epsilon' \) such that, for all \( \theta' \in B_{\epsilon'}(\theta) \), \( P(\theta') \in P_R \). Set \( 0 < \epsilon < \epsilon' \). Then \( \mu(B_\epsilon(\theta)) > 0 \), and \( B_\epsilon(\theta) \subset P_R \), so the probability of drawing \( \theta' \in B_\epsilon(\theta) \) is strictly positive. As \( n \to \infty \), the probability of each draw being outside of \( B_\epsilon(\theta) \) converges to 0.

Thus, asymptotically, uniform draws from \( P_R \) allow us to exactly recover the identified set (as well as any desired statistic of the identified set). It now remains to provide some finite-sample bounds. For such finite-sample bounds, it is more instructive to focus on \( \epsilon \)-distances for structural disturbance vectors \( p_j \) (rows of \( P \)) rather than basis vectors \( \theta \). Again I will, in line with most of the literature, focus on identification of a single disturbance vector (a single element in \( n \)-dimensional unit sphere \( S(n) \)). The key result is the following.

**Theorem 3.10.** Let \( p_j^* \in D_j \) be an admissible structural disturbance vector of interest. Suppose that \( p_j^* \) is generic in the sense that, for \( \epsilon > 0 \) small enough, \( B_\epsilon(p_j^*) \cap S(n) \subset D_j \). Then, for selected small enough \( \epsilon > 0 \) and VAR-dimensionality \( n \), the following table displays the expected number of uniform draws from the unit sphere \( S(n) \) required to generate a draw from \( B_\epsilon(p_j^*) \cap D_j \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \epsilon )</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td>20</td>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>54</td>
<td>151</td>
<td>424</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>138</td>
<td>543</td>
<td>2,152</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>348</td>
<td>1,914</td>
<td>10,680</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>864</td>
<td>6,647</td>
<td>52,174</td>
</tr>
</tbody>
</table>

**Proof.** Fix some arbitrary \( p_j^* \). Then, given that we uniformly draw from the unit sphere, the probability of a single draw being in \( B_\epsilon(p_j^*) \cap D_j \), \( q_{j,\epsilon}^* \) is just

\[
q_{j,\epsilon}^* = \frac{\text{area of } B_\epsilon(p_j^*) \cap S(n)}{\text{area of } S(n)}
\]

Both areas can be computed through numerical integration. The expected number of draws then is just \( \frac{1}{q_{j,\epsilon}^*} \). Finally, as long as \( p_j^* \in S(n) \), the numerator is independent of \( p_j^* \), by the properties of the unit sphere. Thus the results in the statement of the theorem are valid for all possible generic \( p_j^* \in D_j \).
The table presented in the theorem statement above displays the expected number of draws. Another statistic of interest would have been the number of draws required to generate a draw from \( B_\epsilon(p_j^*) \cap D_j \) with probability at least \( 1 - \delta \). This is readily found as

\[
\frac{\ln(\delta)}{\ln(1 - q_j^* \epsilon)}
\]

For example, for \( n = 7 \) and \( \epsilon = 0.05 \), we would require 156,298 draws.

The bounds presented in the statement of the theorem apply directly if we are interested in some special admissable structural shock vector (for example the one corresponding to the maximal response at some horizon) and want to see how long we have to draw to get another, sufficiently close disturbance vector. The probability bounds are, however, also useful for recovery of the entire identified set. Suppose, for example, that we seek to approximate the identified set through a set of \( k \) points on the unit sphere, indexed by \( j \) and denoted \( p_j \). Then we could surround each point \( p_j \) by a small enough \( \epsilon \)-ball and so partition the unit sphere into \( k + 1 \) regions. We would then be interested in the probability of receiving one draw from each of the \( k \) \( \epsilon \)-balls, where the probability of a draw from each individual ball is the same and equal to

\[
q_j^* = \frac{\text{area of } B_\epsilon(p_j) \cap S(n)}{\text{area of } S(n)}
\]

Any statistic of interest (say the number of draws required to get at least one draw from each ball with sufficiently high probability) could then be recovered using standard methods, for example through Stirling numbers of the second kind. Of course, for all these bounds, the required number of draws will seem quite high. Fortunately, however, the computational burden for drawing uniformly from a unit sphere is quite low. For example, 1,000,000 draws in 10-dimensional system only require around 2 seconds on a standard MacBook Pro.
4 Application: The Real Effects of Monetary Policy

In this section I apply the methods developed above to the analysis of monetary policy disturbances identified through sign restrictions in the spirit of Uhlig (2005). For clarity of exposition, most of the analysis is conducted with a simple three-equation New Keynesian model. For this model, I present and compare the results of the various numerical procedures discussed section 3. I can then, relying on the results developed in section 2, offer an economic interpretation of the identified set.

4.1 The Baseline Model

The baseline model is a conventional three-equation New Keynesian model:

\[
\begin{align*}
    y_t &= E_t(y_{t+1}) - (r_t - E_t(\pi_{t+1}) - r^*) + \epsilon_t^d \\
    \pi &= \beta E_t(\pi_{t+1}) + \kappa y_t + \epsilon_t^\pi \\
    r_t &= (1 - \phi_r)[r^* + \phi_\pi \pi_{t-1} + \phi_y y_{t-1}] + \phi_r r_t + \epsilon_t^r
\end{align*}
\]

where

\[
\begin{align*}
    \epsilon_t^d &= \rho_d \epsilon_{t-1}^d + \epsilon_t^d \\
    \epsilon_t^\pi &= \rho_\pi \epsilon_{t-1}^\pi + \epsilon_t^\pi \\
    \epsilon_t^r &= \epsilon_t^r
\end{align*}
\]

and \((\epsilon_t^d, \epsilon_t^\pi, \epsilon_t^r)\) \sim \mathcal{N}(0, \Omega), with \(\Omega\) diagonal. \(y_t\) is output, \(r_t\) is the nominal interest rate (the federal funds rate), \(r^*\) is the real natural rate of interest and \(\pi\) is inflation. Steady-state inflation is 0. The model has three disturbances: a demand shock \(\epsilon_t^d\), a cost-push (negative supply) shock \(\epsilon_t^\pi\) and a monetary policy shock \(\epsilon_t^r\). All parameter values are set at conventional levels. Throughout the analysis, the variables will be ordered \(y, \pi, r\) and the shocks are ordered \(\epsilon^d, \epsilon^\pi, \epsilon^r\).

Following the procedure detailed in section 2, we can recover, from the linear state-space representation of this DSGE, the implied reduced-form VAR representation

\[
y_t = \sum_{j=1}^{\infty} B_j y_{t-j} + u_t, \quad u_t|\mathcal{I}_t \sim \mathcal{N}(0, \Sigma)
\]

In the spirit of Uhlig (2005), I define a contractionary monetary impulse vector as follows:

**Definition 8.** The vector \(c \in \mathbb{R}^3\) is called an impulse vector, iff there is some matrix \(C\) so that \(CC^* = \Sigma\) and so that \(c\) is a column of \(C\). A contractionary monetary policy impulse vector is an impulse vector \(c\) so that the model impulse responses to \(c\) satisfy the following restrictions:

1. At horizons \(h = 0, \ldots, H\), the impulse response of prices is non-positive.
2. At horizons \(h = 0, \ldots, H\), the impulse response of the federal funds rate is non-negative.
Uhlig (2005) additionally requires non-borrowed reserves to decline following a contractionary monetary policy. I cannot impose this restriction in my simple three-equation New Keynesian model. However, this extra restriction is at any rate only designed chiefly to disentangle money demand and monetary policy shocks, which by assumption I do not have in my model. Furthermore, Uhlig’s results can be replicated almost perfectly for a three-variable SVAR (output, prices, interest rate) estimated on U.S. data and identified through the restrictions proposed above. In light of this, and to avoid straying too far from accepted baseline models, I have decided to stick with the simple three-equation New Keynesian model.

I set $H = 2$ (two quarters after impact). The results are robust to changing this horizon, for example up to $H = 4$ (one year after impact) or $H = 8$ (two years after impact). To aid interpretation of the subsequent results, it will be useful to briefly consider the true IRFs implied by the model:

Following a contractionary monetary policy shock, the nominal interest rate (federal funds rate) increases, output drops and inflation drops. Since the model does not include any habit formation or other inertia-inducing features, the maximal responses of all variables are on impact.

4.2 Recovering the Identified Set

To recover the identified set of impulse responses I use the various techniques proposed above. The first plot (overleaf) shows the identified after 100 uniform draws from $P_R$, with the pointwise median response in thick black:
The identified set appears to be connected in that case. However, with 100 draws, we see that there are still quite a lot of visible gaps left; in particular, this suggests that the edges of the identified set of impulse responses are not recovered correctly. Explicit maximization gives the following bounds, plotted against the results from the random draws for convenience:

The plot suggests that, with as little as 100 random draws, we are already very close to the true boundaries of the identified set. The third and final approach is to discretize the Givens basis for
the rotation matrices. I consider 25 gridpoints per $[0, \pi]$ interval, for a total of 2,863 gridpoints. The identified set is plotted below, again with the pointwise median response in thick black:

The edges identified through exact optimization are, at least up to visual inspection, attained perfectly. The identified set seems to be connected. An almost identical picture emerges from 1,000 random draws:

Since random draws are the fastest procedure, I base all my following results on 1,000 uniform draws from $P_R$, and conclude from the evidence above that this offers adequate characterization of the identified set.
4.3 Interpretation

The graphs above show clearly that, in the simple three-equation DSGE model, the proposed variation on Uhlig’s baseline identification procedure fails to identify the true underlying monetary policy disturbance. There is huge uncertainty surrounding the initial response of output, and the median response is, as in the original analysis of Uhlig, positive. In fact, compared with a completely unrestricted reduced-form VAR, Uhlig’s restrictions do not provide much shrinkage on the upper end of the initial response of output – that is, the upper bound is close to the upper bound attained through maximization over all of $O(n)$.

It will be instructive to more carefully analyze what the admissible "monetary policy shocks" actually look like. Fortunately, since the variables and impulse responses have similar scales, we can meaningfully interpret direct loadings of shocks (i.e. the columns of $P^{-1}$), without the need for any normalization rule. A (smoothed) plot of the resulting loadings follows overleaf, with the identified shocks according to the initial output response, from biggest to smallest.

![Graph showing identified shocks]

As we can see, the largest identified output responses correspond to combinations of positive demand and negative cost-push (or positive supply) shocks. That is, as conjectured initially, positive demand and supply shocks masquerade as contractionary monetary policy shocks, but of course with opposite output implications. As we go to the right, we get closer to the true monetary policy shock; at around the 700th rotation the monetary policy is recovered perfectly. To get a further gauge of how close the identified shocks are to the true monetary policy disturbances I propose to proceed as follows. Since we are chiefly interested in the response of output, and since in the underlying model the largest responses are at the period-1 horizon, I consider the share of the impact output response attributable to the loading of the monetary policy as a quantitative gauge of the quality of the identification. Formally, I consider

\[
\frac{|p_{3}^{-1}IR_{1,3,1}|}{\sum_{i=1}^{3}|p_{i}^{-1}IR_{1,i,1}|}
\]
where \( p_i^{-1} \) denotes the \( i \)th entry of the identified shock loading and as before \( IR_{i,j,h} \) is the response of variable \( i \) at horizon \( h \) to structural disturbance \( j \). A plot follows.

According to the proposed measure, most identified shocks have very little to do with the object of interest: monetary policy disturbances. As expected from the plot of loadings displayed above, for the largest output responses the monetary policy shock plays hardly any role, while around rotation 700 we get pretty close to recovering the true monetary policy shock.

4.4 More Restrictions: Arias et al. (2015)

Arias et al. (2015) argue, very much in agreement with my claims here, that the restrictions of Uhlig (2005) are too weak to adequately identify monetary policy shocks. They thus propose to add to his two identifying restrictions two further restrictions. With these two additional restrictions, the response of output to a contractionary monetary policy shock is, as expected, negative. Of the two new restrictions imposed by the authors, the decisive one turns out to be the following:

3. The contemporaneous reaction of the federal funds rate to output and to the GDP deflator (to prices) is non-negative.

Implementing this proposed additional restriction in my model is straightforward. Essentially it amounts to saying that the entries (3, 1) and (3, 2) of the identified candidate \( \tilde{A}_0 \) matrix are both weakly negative. If this restriction is violated then I just discard the proposed draw. Before considering the results of this procedure, however, it will prove instructive to briefly review the economic justification of the proposed restriction. It derives from the natural assumption that, in the monetary authority’s reaction function, the loadings on inflation and output should be positive. Clearly, with \( \phi_r \in (0, 1) \) and \( \phi_\pi, \phi_y > 0 \), this assumption is also satisfied in my simple baseline model.

However, the restriction of Arias et al. (2015) is on entries of \( A_0 \), the contemporaneous coefficient matrix in the structural VAR(\( \infty \)) implied by the solution of the linear rational expectations system \((IS), (NKPC), (MP)\). Ex-ante, there is no reason that \( \phi_\pi, \phi_y > 0 \) must necessarily translate into
$A_0(3,1), A_0(3,2) \leq 0$. In fact, we know that $A_0 = IR_0(A_0, A_+)^{-1}$. So write

$$IR_0(A_0, A_+) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Recall that the variables are ordered $y, \pi, r$ and that the shocks are ordered $\epsilon^d, \epsilon^\pi, \epsilon^r$. We can thus sign the coefficients. $b, c, f$ are negative, and all other coefficients are positive. $A_0$ is then given as

$$A_0 = 1 - \frac{1}{ceg + bfg + cdh - afh - bdi + aei} \begin{pmatrix} -fh + ei & ch - bi & -ce + bf \\ fg - di & -cg + ai & cd - af \\ -eg + dh & bg - ah & -bd + ae \end{pmatrix}$$

It is readily seen that the signs of the entries of $IR_0(A_0, A_+)$ are not sufficient to pin down the signs of $A_0(3,1), A_0(3,2)$. For all particular parameterizations that I considered, however, we indeed have $A_0(3,1), A_0(3,2) \leq 0$. I will later return to this issue in the context of the Smets-Wouters model. For the moment, it suffices to note that the restrictions of Arias et al. (2015) are consistent with the model and so imposing them may well be reasonable.

Interestingly, in the three-equation model, imposing the additional restrictions of Arias et al. (2015) has the same effects as in their empirical analysis: the identified set of output responses is shrunk dramatically towards negative impact responses. The corresponding plot follows overleaf.

To ease comparison with the existing literature, I have also plotted 16% and 84% bounds (rather
than only the true edges of the identified set). At this conventional level, the response of output is negative and significant. To understand why the additional restriction works it is instructive to again consider the set of identified disturbance vectors.

As we can see, the loading on the monetary policy shock is high throughout. The loadings on demand and supply shocks, in contrast, are small, and in particular never reach the levels seen under the pure Uhlig restrictions. Intuitively, most of the masquerading combinations of expansionary demand and supply shocks reply counterfactual responses of interest rates to demand and supply shocks. By ruling out these counterfactual responses we get much closer to recovering the true monetary policy disturbance. This becomes even clearer by looking at the share of output directly attributable to the monetary policy shock.

For most rotations the share attributable to the monetary policy shock is reasonably high, but of course not perfect. Imposing the sign restrictions on prices and federal funds for longer horizons

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7These bounds are just percentiles of the identified set. In the empirical literature, researchers draw many times (say \( N \) times) from the reduced-form posterior, search for a rotation that satisfies their imposed sign restrictions, and then compute percentiles from their \( N \) impulse responses. This procedure incorporates both sampling and model uncertainty. By construction my approach abstracts from sampling uncertainty, and so I only report bands reflecting the model uncertainty.
improves this somewhat. However, the median impact response consistently remains somewhat too high, attributable to the combinations of negative demand, supply and monetary policy shocks at the right end of the identified spectrum.

4.5 Recursive Identification

As discussed above, my procedure can also be used to interpret identification schemes offering exact (point) identification. The following plot shows the derived impulse responses.

The results are as usual. Output and inflation both show a hump-shaped decline to a contractionary monetary policy shock, with the peak response occurring after a bit less than a year. After around four years, all variables have returned to baseline. Interestingly, the identified shock has loadings $(0.3744, 0.0118, 0.9272)'$. That is, we are actually identifying a linear combination of (mostly) expansionary demand and contractionary monetary policy shocks, with the loading on supply shocks negligible. Even though the sign of the output response is as expected, the value of the response is severely distorted. Due to the high loading on the demand shock the output response to the identified monetary policy shock is much smaller than in the true model, up to an order of almost 10. Accordingly, it is not surprising that only around 46% of the variation in output is directly attributable to the true monetary policy shock.

4.6 Extension: Smets-Wouters Model

Much of the intuition from the simple three-equation New Keynesian model considered above carries through to more general underlying model frameworks. In this section I consider the Smets-Wouters model, probably the most well-known example of an empirically successful medium-scale DSGE model. Following Smets and Wouters (2007), and departing from earlier working paper versions of
the Smets-Wouters model, I reduce the number of structural shocks to 7, matching the number of observables. This allows me to derive an invertible linear state-space representation and generate a VAR(∞) in output y, consumption c, the policy rate r, investment i, employment l, inflation \( \pi \) and wages w. The disturbances are a productivity shock, a financial frictions/net worth shock, a (government) spending shock, an investment shock, a monetary policy shock, a price mark-up shock, and a wage mark-up shock. I solve the model at the mode parameter estimates of Smets and Wouters (2007). The identified set under the baseline restrictions of Uhlig (2005) looks as follows:

Again we see that the identified set for the output response is very wide, and that the median response is positive. To interpret these findings it is again instructive to consider the corresponding (smoothed) shock loadings.
The results are very similar to those from the baseline model. The largest output responses correspond to positive demand shocks (here positive financial frictions shocks) and positive supply shocks (here negative price mark-up and wage mark-up shocks) masquerading as contractionary monetary policy shocks. The masquerading interpretation is thus robust across a range of conventional models.

Finally it is instructive to briefly consider alternative identification schemes, in particular the extra restrictions of Arias et al. (2015) and simple recursive orderings. Interestingly, even though the monetary policy reaction function agrees with the restrictions of Arias et al. (2015), it turns out that the model-implied \( A_0 \) does not. The problem of course is that monetary policy reaction function and monetary policy equation in the VAR(\( \infty \)) are not the same thing. In the simple three-equation New Keynesian model the distinction did not matter; in the Smets-Wouters model, it does. As a result, it is not surprising that the restrictions of Arias et al. (2015) now do not help; in fact, they make matters worse (as gauged by the proposed "output response share" metric defined earlier).

A Choleski decomposition, in contrast, works remarkably well. I consider the ordering \( y, c, i, l, w, \pi, r \). The identified shock vector has a loading of 0.9627 on the monetary policy shock, and the impulse responses look very similar to the true model-implied impulse responses, of course with some discrepancy on impact. Given the habit formation implicit in the Smets-Wouters model, it is unsurprising that recursive schemes now work quite well.\(^8\)

### 4.7 "Real" Baseline Models

The preceding sections essentially amount to the following conclusion: Against a range of plausible sticky-price models, the identification restrictions proposed by Uhlig (2005) are too weak to correctly identify monetary policy shocks. For an economist with a reasonably tight prior on the non-neutrality of monetary policy, these results will be comforting – they tell a coherent story of why we should not be surprised that, under the identification procedure of Uhlig, "neutrality of monetary policy shocks is not inconsistent with the data" (Uhlig (2005)). Since many macroeconomists (I would conjecture) share such a prior, the earlier focus on conventional sticky-price models was natural for a broad audience. However, an interesting alternative exercise would be to gauge the performance of Uhlig’s identification procedure in a model with (near) money neutrality.

---

\(^8\)In fact, habit formation is mostly put in to allow the model to match the IRFs derived from SVARs identified through recursive orderings. This does not mechanically mean that a recursive ordering applied to the VAR representation of the Smets-Wouters model has to do a good job in recovering monetary policy shocks, but of course it makes it much more likely than in, say, the baseline three-equation model.
To this end I consider first the baseline three-equation New Keynesian model with the probability of Calvo price adjustment approaching 1. Under the proposed parameterization, demand shocks and cost-push shocks lead to inflation overshooting at very short horizons. Since I require price and interest rate responses to have the expected sign for up to one year (in his paper Uhlig actually considers horizons up to two years), the identification scheme actually works very well in recovering monetary policy shocks and so sharply identifies the extremely weak but negative response of output to contractionary monetary policy shocks. The output response is negative for all admissible rotations, and the fraction of the output response attributable to the monetary policy shock is above 0.9 for most of the identified shock vectors.

A limiting real version of the Smets-Wouters model is somewhat harder to interpret. I let Calvo price and wage adjustment parameters approach 1, and set all other parameters equal to their mode in the estimated sticky-price Smets-Wouters model. Under this parameterization, a serious problem is that the interest rate response to a monetary policy shock reverts sign very quickly. The response of inflation to most other shocks furthermore again shows an overshooting pattern, and so overall it becomes rather difficult to impose the proposed identification scheme up to reasonable horizons. For example, with the restrictions imposed until the one-year horizon, I need on average around 200 rotation matrices until the restrictions are satisfied, while under the original parameterization the number of required draws rarely exceeded 5. In my own empirical work with sign-identified SVARs I generally find that price and interest rate restrictions, even for reasonably long horizons, are rather easy to satisfy. In light of this, the proposed money-neutral Smets-Wouters model seems misspecified. As a further check I then tried to re-estimate the Smets-Wouters model subject to the constraint that the price adjustment probabilities be close to 1. For this I used the original Smets-Wouters data and imposed their priors on all remaining model parameters. Unfortunately the estimation routine is extremely ill-behaved, suggesting that a money-neutral Smets-Wouters model is not suitable to be taken to the data.

5 Conclusions

Set identification of SVARs can be both blessing and curse. Researchers want their identifying restrictions to be as weak and universally acceptable as possible, but in doing so they run the risk of actually not being able to say anything. This paper has offered a guideline on how to balance the virtues of agnosticism with the need for sufficiently strong identification. Before researchers decide to take their proposed identification restrictions to the data, they should make sure that, at least within some pre-specified set of plausible underlying structural models, their identification scheme does a reasonable job at recovering the truth. Whether or not this is so can be assessed
through the general methods developed in this paper. More broadly, the methods developed here can also help gauge the extent to which various exact (i.e., point-identifying) identification schemes can allow successful recovery of the structural disturbance of interest.

The particular application that I consider is the debate surrounding the real effects of monetary policy interventions, in particular in light of the influential contribution of Uhlig (2005). Other interesting applications may lie in the econometric analysis of oil price shocks or technology shocks, both routinely identified through sign (or sign and zero) restrictions.
References


