A Characterization of Identified Impulse Response Sets in Linear State-Space Models

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Abstract

This paper characterizes, for an arbitrary linear state-space model with autoregressive representation of the observables and for an arbitrary set of identification restrictions, the set of impulse vectors consistent with the proposed identification scheme. For identification procedures that afford point identification this means recovery of the unique identified impulse vector; for set-identifying restrictions, this means recovery of an entire set of identified impulse vectors. Success or failure of a given identification scheme against an underlying data-generating process can then be judged through the overlap (or lack thereof) between the identified set of impulse vectors and the impulse vector of interest. The main application of the methodology is a re-interpretation of the findings in Uhlig (2005). I argue that, against a range of entirely conventional underlying structural models, his identification procedure tends to pick out positive demand and supply shocks jointly masquerading as monetary policy shocks. His failure to reject monetary policy non-neutrality in the data is thus exactly what simple sticky-price models with money non-neutrality would lead us to expect.

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1 Introduction

Structural vector autoregressions (SVARs) are a key part of the applied macroeconomist’s toolkit. Among other things, they have been used for policy analysis and to provide the stylized facts that inform modern dynamic stochastic general equilibrium (DSGE) modeling. As is well known, the key hurdle in the analysis of SVARs is identification. The data allow the researcher to consistently estimate the true reduced-form vector autoregressive system, but each such reduced-form VAR is consistent with a continuum of – in economic terms usually very different – underlying SVARs. Thus, to go from reduced-form VAR to SVAR, the researcher will need to bring in some extra outside information. Such so-called identifying restrictions can take many forms. The most common are direct restrictions on the impulse responses of the estimated VAR. These restrictions can range from exact zero restrictions at short horizons (e.g. Sims (1980)) and zero restrictions at long horizons (e.g. Blanchard and Quah (1993)) to sign restrictions (e.g. Uhlig (2005)), or of course to any combination thereof. Rubio-Ramirez et al. (2010) establish general conditions under which a given combination of identifying restrictions is sufficient to go from estimated reduced-form VAR to some unique underlying SVAR. Early contributions to the SVAR literature by and large restrict attention to identification schemes that afford such exact identification. More recently, however, some researchers have turned away from exact point identification and towards what is known as set identification. In a set-identified SVAR, the researcher uses outside information (usually in the form of sign restrictions) to shrink the set of SVARs consistent with the estimated reduced-form VAR not to a single point, but to some – hopefully small enough – set of admissible SVARs.

Set identification of SVARs has many advantages. First and foremost, the proposed identifying restrictions can be much weaker than those required for point identification. Few researchers actually believe in the stringent zero restrictions required for point identification, in particular if they apply to short-run relations between variables. Sign restrictions, in contrast, are often chosen such that they will enjoy near unanimous support in the profession. Alas, the weakness of the required identifying restrictions is both blessing and curse. Less restrictions mean less power, and so set-identified SVARs may be silent on the question of interest.

Ideally, researchers would select their identification scheme in a way so that, against a broad range of "plausible" underlying economic models, the restrictions are strong enough to get us, with high probability, close to the "true" SVAR, in the sense that the identified structural disturbances are somehow close to the true underlying economic shocks. So far, however, this has only been done in a very crude way. In most applications, researchers are interested in a single structural disturbance and its propagation through the system. The proposed identifying restrictions then usually amount to (sign) restrictions on impulse responses that,
against a range of presumably plausible models, only the shock of interest will satisfy. Clearly this is a minimum requirement for correct structural analysis. Unfortunately, however, it is not sufficient, since combinations of the other disturbances can masquerade as the shock of interest. The problem of masquerading shocks has, in some papers, been mentioned in passing (for example in Uhlig (2005)), but to the best of my knowledge has never been analyzed thoroughly.

To discipline the choice of identifying restrictions, this paper instead proposes the following approach. Start with a range of "plausible" economic models, and consider some set of identifying restrictions. The theoretical contribution of this paper is to exactly characterize, for an arbitrary linear state-space model that admits a (perhaps non-fundamental) VAR representation in observables and for a given identification procedure, the identified set of corresponding SVARs. If the proposed procedure does a good job of recovering the true underlying disturbances of interest across the range of models that the researcher is willing to entertain, then – and I would argue only then – should the proposed set of identifying restrictions be taken to the data. So far researchers have skipped the first step, or at best replaced it with some loose model-based reasoning. This may have, as I will argue later, led to some misleading inference.

There are two different – and conceptually quite distinct – reasons for why the structural shocks in an identified SVAR may bear little relation to the structural shocks in some "true" underlying model. First, the imposed identifying restrictions may simply be wrong (so that some other structural shocks, rather than the shock of interest, are identified) or just too weak (so that an overly large set of shocks is identified). Second, the state-space model may not be invertible, so the reduced-form disturbances in the implied VAR representation do not span the space of structural disturbances. In my general characterization of identified sets I will allow for both threats to identification, and I will show how to interpret a given set of identifying restrictions in both fundamental and non-fundamental VAR representations. This allows us to separately gauge the distorting effects of incorrect identification and non-invertibility.

Even though my analysis is explicitly motivated by the SVAR set identification literature, it of course just as well applies to identification schemes that afford point identification. In such cases, my procedure allows us to see how close the identified set of disturbance vectors is to true model-consistent disturbances, and so to what extent the proposed identifying restrictions distort inference. An analysis of this sort has antecedents in the literature. For example, Carlstrom et al. (2009) show exactly how close, in a simple three-equation New Keynesian model, a simple recursive identification scheme gets us to the true model-implied impulse responses. My analysis is a generalization of theirs, in the sense that I can characterize the identified set for an arbitrary underlying model, as long as it has a VAR representation, either invertible or non-invertible.
The main application of my procedure is to re-interpret the evidence on monetary policy non-neutrality presented in Uhlig (2005). Under his proposed identification procedure, monetary policy neutrality cannot be rejected; in fact, if anything, contractionary monetary policy shocks seem to boost, not lower, output. My motivating question now is as follows: How should someone with a reasonably tight – but certainly non-dogmatic – prior on the non-neutrality of monetary policy update her beliefs after seeing the results of Uhlig (2005)? My procedure provides a formalization of this updating process. I consider the two most conventional DSGE models imaginable – the standard three-equation New Keynesian model, and the Smets-Wouters DSGE – and construct the set of impulse responses consistent with Uhlig’s identification scheme. The identified sets look almost exactly as in the data, with the output response to the identified contractionary monetary policy shock non-significant (but at the median positive) at all horizons. Intuitively, Uhlig’s restrictions allow combinations of positive supply and demand shocks to masquerade as contractionary monetary policy shocks. Prices fall, interest rates rise, and output rises as well. Thus, Uhlig’s results are exactly what we would expect from applying his procedure to data generated from a model in which monetary non-neutrality actually holds. This masquerading occurs in almost the exact same way in the baseline three-equation New Keynesian model and in the Smets-Wouters DSGE, suggesting that the simple demand-supply masquerading intuition is quite robust. Given these insights, the posterior of our hypothetical believer in the non-neutrality of monetary policy should not differ much from her prior, even after seeing Uhlig’s results. Uhlig’s findings are not inconsistent with the neutrality of monetary policy, but they also do not provide (strong) evidence in favor of such neutrality.

I finally pursue three brief extensions of the baseline analysis. First, I consider and re-interpret recently proposed extensions of Uhlig’s baseline identification procedure. My main reference is Arias et al. (2015), who add sign restrictions on the central bank reaction functions to Uhlig’s core restrictions and, with their strictly tighter identification scheme, find the expected negative response of output to contractionary monetary policy disturbances. Interestingly, their proposed extra restrictions are also, in the context of the three-equation New Keynesian model, sufficient to get us much closer to the true model-implied SVAR, in particular as regards the response of output to the identified monetary policy innovation. In the Smets-Wouters model, in contrast, the proposed additional restrictions fail to give us much extra traction; in fact, they are violated in the Smets-Wouters model. The reason for this surprising finding lies, as I will discuss in more detail later, in the tempting, but erroneous confusion of the central bank’s reaction function – one of the equations entering the linear rational expectations model – and the monetary policy equation in the implied VAR(∞). Second, I consider the updating problem of someone with a reasonably right, but again non-dogmatic, prior on the neutrality of monetary policy. I consider a range of money-neutral models and document features of the identified set in these models.
that are somewhat at odds with the identified set in Uhlig (2005). This would suggest that prior
and posterior may, upon observing Uhlig’s results, actually differ more for someone with a belief
in monetary neutrality than non-neutrality. Third, in both the three-equation model and the
Smets-Wouters DSGE, I identify monetary policy shocks using a simple recursive identification
scheme. This exercise illustrates the applicability of my analysis to exact identification schemes
and furthermore, for the application at hand, helps us gauge the limitations of incorrect but (at
least for Smets-Wouters) approximately true identification schemes.

The rest of this paper proceeds as follows. In section 2, I review the identification problem in an
arbitrary linear state-space model with VAR representation. In section 3, I then, for a given set of
identifying restrictions, characterize the identified set of SVARs, first analytically and then, with
analytical tools exhausted, numerically. Section 4 contains the application of my procedure to the
identification of monetary policy disturbances through sign restrictions. Section 5 concludes.

2 Identification in Linear State-Space Models

2.1 From State-Space Model to VAR

Consider the following linear Gaussian state-space system:

\[ s_t = A s_{t-1} + B w_t \]

\[ y_t = C s_{t-1} + D w_t \]  

where \( s_t \) is a \( k \)-dimensional vector of state variables, \( y_t \) is an \( n \times 1 \) vector of observables and
\( w_t \) is an \( m \times 1 \) vector of structural shocks. The disturbances \( w_t \) are Gaussian white noise, with
\( E[w_t] = 0, E[w_t w_t'] = I \) and \( E[w_t w_{t-j}'] = 0 \) for \( j \neq 0 \). We are interested in the propagation of the
structural disturbances through the system.

Under weak conditions, the state-space system (1) - (2) implies a VAR representation for the
observables \( y_t \). The discussion here will be purposefully brief, and the interested reader is referred
to the appendix or to Fernández-Villaverde et al. (2007) for more details. The steady-state Kalman
filter associated with the state-space system is characterized by the pair of Riccati equations

\[ \Sigma_s = (A - KC) \Sigma_s (A - KC)' + BB' + KD'K' - BD'K' - KDB' \]  

\[ K = (A \Sigma_s C' + BD')(C \Sigma_s C' + DD')^{-1} \]

\footnote{The important assumption throughout will be that the structural disturbances come from a distribution uniquely parameterized by its mean and variance. The assumption of an identity covariance matrix is without loss of generality.}
in the Kalman gain $K$ and the forecast error variance $\Sigma_s$. Denote the Kalman-filtered estimate of $s_t$ given information on the observables up to time $t$ by $\hat{s}_t$, and let $u_t \equiv y_t - C\hat{s}_{t-1} = C(s_{t-1} - \hat{s}_{t-1}) + Dw_t$ denote the (Gaussian) forecast error for observables given information up to $t - 1$. Then, assuming that $(A - KC)$ is stable, the system admits the VAR representation in observables

$$y_t = \sum_{j=1}^{\infty} C(A - KC)^{j-1} K\hat{s}_{t-j} + u_t$$

(5)

with disturbance variance $E(u_t u_t') \equiv \Sigma_u = C\Sigma_s C' + DD'$. A state-space system is said to be invertible if knowledge of the observables is sufficient to derive the corresponding values of the hidden states. For the rest of the paper, I strengthen invertibility to also mean that $D$ is invertible. A necessary condition for this is of course $n = m$, i.e. there are as many observables as structural shocks. In that case $\Sigma_s = 0$, $K = BD^{-1}$ and so the VAR representation specializes to

$$y_t = \sum_{j=1}^{\infty} C(A - BD^{-1}C)^{j-1} BD^{-1} y_{t-j} + u_t$$

(6)

where now $u_t = Dw_t$ and so $\Sigma_u = DD'$. Either way, we have arrived at a reduced-form VAR representation

$$y_t = \sum_{j=1}^{\infty} B_j y_{t-j} + u_t$$

(7)

Stacking regressors into the vector $x_t$ and coefficient matrices into the matrix $B_+$, we can compactify notation and write

$$y_t = B_+ x_t + u_t$$

(8)

In the rest of the paper I adopt the notational conventional that, for an arbitrary conformable matrix $M$, $M \cdot B_+$ indicates elementwise multiplication of $M$ and the entries of $B_+$ (the $B_j$).

A structural VAR representation of the same system is

$$A_0 y_t = \sum_{j=1}^{\infty} A_j y_{t-j} + \epsilon_t$$

(9)

where $A_0^{-1} A_0^{-1'} = \Sigma_u$, $\epsilon_t \equiv A_0 u_t$ is Gaussian white noise with $E[\epsilon_t] = 0$, $E[\epsilon_t \epsilon_t'] = I$ and $E[\epsilon_t \epsilon_{t-j}] = 0$ for $j \neq 0$, and $A_j \equiv A_0 B_j$. In more compact notation we have

$$A_0 y_t = A_+ x_t + \epsilon_t$$

(10)
2.2 Identification of Structural VARs

This section provides a brief review of the identification problem for structural VARs. The presentation here draws heavily on Rubio-Ramirez et al. (2010). For the purposes of this section, let the particular structural-form VAR \( (A_0, A_+)^{10} \) be the "SVAR of interest" (or reference SVAR) – i.e., the SVAR we want to identify, for some reason as another. The ideal SVAR has structural disturbances that are equal to the structural disturbances of the underlying state-space model; the "SVAR of interest" for this section is naturally thought of as the SVAR that achieves this goal or, according to some metric, comes closest to it.

The structural form is uniquely characterized by the parameter vector \((A_0, A_+)^{10}\). Following Rothenberg (1971), I say that two structural-form VAR models \((A_0, A_+)^{10}\) and \((\tilde{A}_0, \tilde{A}_+)^{10}\) are observationally equivalent if and only if they imply the same distribution for the vector of observables \(y_t\), for any arbitrary time horizon. In the Gaussian linear environment studied here, this means that two structural VARs are observationally equivalent if and only if they imply the same reduced-form VAR representation \((B_+, \Sigma)^{8}\). It is easy to see that, under this definition of observational equivalence, the reference SVAR \((10)^{10}\) is only identified up to the set \(O(n)\) of orthogonal matrices of size \(n\). For every \(P \in O(n)\), the alternative structural VAR \((\tilde{A}_0, \tilde{A}_+)^{10}\), with \(A_0 = P^{-1}\tilde{A}_0\) and \(A_+ = P^{-1}\tilde{A}_+\) implies the same reduced-form VAR \((8)^{8}\), and so all such \(P\) are equally consistent with the data.\(^2\)

Out-of-sample information can be used to sharpen identification to strict subsets \(O(n)\). Following Rubio-Ramirez et al. (2010), I allow out-of-sample information to take the form of linear restrictions on transformations of the structural parameter space into \(q \times n\) matrices, where \(q > 0\). Denote the transformation by \(f(\cdot)\). Linear restrictions of the transformation can then be represented via \(q \times q\) matrices \(Z_j, S_j\), with \(j = 1, \ldots, n\).

**Definition 1.** Identified Set of SVARs. Consider the reduced-form VAR \((B_+, \Sigma_u)^{8}\). Let \(R\) be an identifying restriction, defined by a transformation \(f(\cdot)\) and a set of restriction matrices \(Z_j, S_j\), \(j = 1, \ldots, n\). Then the identified set of SVARs, \(A_R\), is defined as follows:

\[
A_R \equiv \{\tilde{A}_0 | \tilde{A}_0^{-1}\tilde{A}_0^{-1'} = \Sigma_u, Z_j f(\tilde{A}_0, \tilde{A}_0 \cdot B_+) e_j = 0 \text{ and } S_j f(\tilde{A}_0, \tilde{A}_0 \cdot B_+) e_j \geq 0 \text{ for } 1 \leq j \leq n\}
\]

where \(e_j\) is the \(j\)-th column of the \(n \times n\) identity matrix \(I_n\).

If the SVAR of interest is a particular natural or good reference point then we may be want to define an identified set of rotation matrices with respect to this benchmark.

\(^2\)To get the same reduced-form VAR any invertible matrix \(P\) would do. However, in our definition of the structural VAR, we require the structural disturbances to have unit variance. Thus only orthogonal matrices \(P\) are allowed.
**Definition 2. Identified Set of Rotation Matrices.** Denote the parameters of the reference SVAR by \((A_0, A_+)\). Let \(R\) be an identifying restriction, defined by a transformation \(f(\cdot)\) and a set of restriction matrices \(Z_j, S_j, j = 1, \ldots, n\). Then the identified set of rotation matrices, \(P_R\), is defined as follows:

\[
P_R \equiv \{P \in O(n) \mid Z_j f(PA_0, P \cdot A_+) e_j = 0 \text{ and } S_j f(PA_0, P \cdot A_+) e_j \geq 0 \text{ for } 1 \leq j \leq n\}
\]

where \(e_j\) is the \(j\)-th column of the \(n \times n\) identity matrix \(I_n\).

The \(Z_j\) allow us to impose exact linear restrictions on \(f(\cdot)\), and the \(S_j\) allow us to impose linear sign restrictions.\(^3\) Identifying restrictions may afford point identification, set identification or they may be infeasible.

**Definition 3.** Let \(P_R\) denote the identified set of rotation matrices of a restriction \(R\) with respect to reference SVAR \((A_0, A_+)\). We then say the following:

1. If \(P_R\) is a singleton, then \(R\) affords point identification with respect to \((A_0, A_+)\).
2. If \(P_R\) is empty, then \(R\) is infeasible with respect to \((A_0, A_+)\).
3. If \(P_R\) is neither a singleton nor empty, then \(R\) affords set identification with respect to \((A_0, A_+)\).

Note that set identification is consistent with both \(\mu(P_R) = 0\) and \(\mu(P_R) > 0\), where \(\mu(\cdot)\) denotes the Haar measure on the group of orthogonal rotation matrices. Sign restrictions generically afford set identification with \(\mu(P_R) > 0\), while combinations of sign and zero restrictions generically afford set identification with \(\mu(P_R) = 0\).

An important class of restrictions is what I call **minimally consistent** restrictions.

**Definition 4.** A restriction \(R\) is minimally consistent with respect to a reference set of SVAR parameters by \((A_0, A_+)\) if and only if

\[
I \in P_R
\]

Minimally consistent restrictions are trivially feasible.

### 2.3 Interpreting Identified Structural VARs

In the previous section I reviewed the identification problem under reference to some nebulous "SVAR of interest". To formalize the notion of SVAR disturbances being in some sense "close" to the structural disturbances of the underlying state-space model we need to connect \(\epsilon_t\) and \(w_t\).

\(^3\) I could in fact represent any given set of exact linear and sign restrictions through only sign restrictions, with weak inequalities in both directions enforcing the exact restriction. However, for the sake of notational clarity, I distinguish between sign and exact restrictions, and so implicitly assume that the sign restriction matrices do not enforce any exact equality.
Proposition 2.1. Consider an arbitrary state-space model of the form (1) - (2) with VAR representation in the observables $y_t$. The identified structural disturbances in the derived SVAR (9) satisfy the following relation to the structural disturbances $w_t$ of the state-space model:

$$
\epsilon_t = A_0 \left\{ D + \begin{pmatrix} C & -C \end{pmatrix} \left( I - \begin{pmatrix} A \\ KC & A - KC \end{pmatrix} L \right)^{-1} \begin{pmatrix} B \\ KD \end{pmatrix} L \right\} w_t
$$

In particular, if the state-space system is invertible, then

$$
\epsilon_t = A_0 D w_t
$$

The proofs of all results are relegated to the appendix. Now define the sequence of matrices $M \equiv \{ M_i \}_{i=0}^{\infty}$, where $M_i$ is the polynomial term on $L^i$ in the matrix polynomial

$$
D + \begin{pmatrix} C & -C \end{pmatrix} \left( I - \begin{pmatrix} A \\ KC & A - KC \end{pmatrix} L \right)^{-1} \begin{pmatrix} B \\ KD \end{pmatrix} L
$$

This sequence of matrices induces the sequence $\rho(A_0) \equiv \{ P_i(A_0) \}_{i=0}^{\infty} \equiv \{ A_0 M_i \}_{i=0}^{\infty}$. It is immediate that the $j,l$th entry of $P_i$ is the loading of the $j$th identified SVAR shock on the $l$th structural disturbance in the state-space model with lag $i$. I formalize this in the following definition.

Definition 5. Identified Set of Structural Disturbance Vectors. Consider the reduced-form VAR $(B, \Sigma_u)$. Let $R$ be an identifying restriction, defined by a transformation $f(\cdot)$ and a set of restriction matrices $Z_j, S_j, j = 1, \ldots, n$. Then the identified set of structural disturbance vectors for shock $j, D_j$, is defined as follows:

$$
D_j \equiv \left\{ \rho_j \equiv \{ p_{ij} \}_{i=0}^{\infty} \mid \exists A_0 \in \mathcal{A}_R \text{ such that } p_{ij} \text{ is the } j \text{th row of } P_i = A_0 M_i \right\}
$$

A $\rho_j \in D_j$ is called admissable.

The interpretation of this identified set is particularly neat if the state-space system is invertible. In that case we have

$$
A_0 \epsilon_t = D w_t
$$

The natural choice for the reference SVAR evidently is $A_0 = D^{-1}$. Then $\epsilon_t = w_t$, so the structural disturbances in the identified SVAR are the structural shocks in the underlying state-space system. It is immediate that, in this case, only the first matrix of the sequence $\rho(A_0)$ is non-zero. So consider now, still for an invertible state-space system, an arbitrary rotation matrix $P \in O(n)$, with

\footnote{In line with this interpretation, the matrix $P^* \equiv \sum_{i=0}^{\infty} P_i(A_0)$ is orthonormal, with $j,l$th entry of $P^*$ representing the total loading of the $j$th identified SVAR on the $l$th structural disturbance, measured over all horizons.}
corresponding $\tilde{A}_0$ defined via $A_0 = P^{-1}\tilde{A}_0$. Evidently the matrix sequence $\rho(\tilde{A}_0)$ also only has the first element non-zero, and furthermore $P = P_0$. Thus the identified set of rotation matrices $P_R$ and the identified set of structural disturbance vectors $D_j$ are intimately connected – the $j$th row of a rotation matrix in $P_R$ is in $D_j$. In what follows, whenever I consider the special case of an invertible state-space system, I use the notational convention of identifying elements of $D_j$ with the first entry in the sequence $\rho_j$.

Most of the subsequent analysis will be conducted with the particular special case of an invertible state-space model in mind, and so with the reference "SVAR of interest" accordingly characterized by $A_0 = D^{-1}$. This makes interpretation easier, but is not essential; all procedures readily generalize to the case of non-fundamental representations and to arbitrary reference matrices $A_0$.

2.4 Identification Through Restrictions on Impulse Responses

Following recent developments in the SVAR literature, I will throughout focus on a certain class of identifying restrictions: restrictions on impulse response functions (IRFs). For a given candidate structural VAR $(\tilde{A}_0, \tilde{A}_+)$, the IRF of $i$-th variable to the $j$-th structural shock at finite horizon $h$ corresponds to the element in row $i$ and column $j$ of the matrix

$$IR_h(\tilde{A}_0, \tilde{A}_+) = (\tilde{A}_0^{-1} J' F_h^j J')'$$

where

$$F_h = \begin{pmatrix} \tilde{A}_1 \tilde{A}_0^{-1} & I_n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}_h \tilde{A}_0^{-1} & 0 & \ldots & I_n \\ \tilde{A}_h \tilde{A}_0^{-1} & 0 & \ldots & 0 \end{pmatrix}$$

and

$$J = \begin{pmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

If the $i$-th variable is in first differences then we can also compute the long-run IRF, given as

$$IR_\infty(\tilde{A}_0, \tilde{A}_+) = (\tilde{A}_0 - \sum_{l=1}^{\infty} \tilde{A}_l')^{-1}$$

I denote specific elements of these matrices by $IR_{t,j,h}$, and entire rows or columns by $IR_{i,-,h}, IR_{-,j,h}$, respectively. Restrictions on IRFs can be expressed through transformation functions $f(\cdot)$ of the

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5In the formulation here the size of the matrix $F_h$ depends on the horizon $h$. This is necessary because I want to allow for VAR($\infty$). If IRFs up to some fixed horizon $H$ are desired, then we could either stick with the varying-size matrices $F_h$ specified above or just take $F_H$ throughout.
following form:

\[
f(\tilde{A}_0, \tilde{A}_+) = \begin{pmatrix}
\cdots \\
IR_h(\tilde{A}_0, \tilde{A}_+) \\
\cdots
\end{pmatrix}
\]

Throughout I will consider restrictions on IRFs up to some fixed horizon \( H \). From here we can now define the **identified set of impulse responses**.

**Definition 6. Identified Set of Impulse Responses.** Consider the reduced-form VAR \((B_+, \Sigma_u)\). Let \( R \) be an identifying restriction, defined by a transformation \( f(\cdot) \) and a set of restriction matrices \( Z_j, S_j, j = 1, \ldots, n \). Then the identified set of impulse responses for variable \( i \) in response to shock \( j \) at horizon \( h \), \( IS_{i,j,h} \) is defined as follows:

\[
IS_{i,j,h} \equiv \{ a \in \mathbb{R} | a = IR_{i,j,h}(\tilde{A}_0, \tilde{A}_0 \cdot B_+), \tilde{A}_0 \in A_R \}
\]

In words, the identified set of impulse responses for variable \( i \) in response to shocks to variable \( j \) at horizon \( h \) consists all horizon-\( h \) impulse responses of variable \( i \) to shock \( j \) that can be generated through a SVAR \((\tilde{A}_0, \tilde{A}_+)\) that lies in the identified set of SVARs \( A_R \). Throughout the rest of the analysis, I will make the following assumption.

**Assumption 2.2.** The identifying restriction \( R \), defined by the transformation \( f(\cdot) \) and the set of restriction matrices \( Z_j, S_j, j = 1, \ldots, n \), is feasible.

Under this assumption the identified set of SVARs and so the identified set of impulse responses are non-empty. Naturally, if \( A_R \) is a singleton, then \( IS_{i,j,h} \) is a singleton for every \( i, j, h \). The goal of the next section is to sharply characterize the identified set of impulse responses whenever the identified set of SVARs \( A_R \) is not a singleton. I will be particularly interested in the case where, for arbitrary reference SVAR \( A_0 \in A_R \), we have that \( \mu(P_R) > 0 \). Clearly, if \( \mu(P_R) > 0 \) for some reference SVAR \( A_0 \in A_R \), then \( \mu(P_R) > 0 \) with any reference SVAR \( \tilde{A}_0 \) consistent with the underlying unique reduced-form VAR (even if \( \tilde{A}_0 \notin A_R \)). I will thus from now on omit explicit mention of the reference SVAR.

### 3 Characterizing the Identified Set

For any given variable \( i \), shock \( j \) and horizon \( h \), the identified set of impulse responses is a subset of \( \mathbb{R} \). The goal of this section is twofold. First, I provide some general analytical statements about the shape of this identified set for a generic identification problem with \( \mu(P_R) > 0 \). Second, I discuss numerical techniques to recover the identified set, applicable to set-identified SVARs with either \( \mu(P_R) > 0 \) or \( \mu(P_R) = 0 \).

\(^6\)As usual, the most natural case to think of is \( A_0 = D^{-1} \) in an invertible state-space system.
3.1 Some Analytical Results

The identified set of impulse responses for variable $i$ in response to shocks to variable $j$ at horizon $h$ is just some subset of $\mathbb{R}$. The goal of this section is to characterize this subset as sharply as possible. Let us start with the upper and lower edges of this set.

**Definition 7.** The upper edge of the identified set of impulse responses for variable $i$ in response to shock $j$ at horizon $h$, denoted $IS_{i,j,h}^u$, is defined as follows:

$$IS_{i,j,h}^u = \sup_{A_0 \in \mathcal{A}_R} IR_{i,j,h}(A_0, A_0 \cdot B_+)$$

Similarly, the lower edge is defined as follows:

$$IS_{i,j,h}^l = \inf_{A_0 \in \mathcal{A}_R} IR_{i,j,h}(A_0, A_0 \cdot B_+)$$

The first substantive result is that both edges are attained.

**Proposition 3.1.** Let $R$ be a feasible identifying restriction on the reduced-form VAR $(B_+, \Sigma_u)$. Then, for all $i = 1, \ldots, N$, $j = 1, \ldots, N$, and $h = 1, \ldots, H$, there exist matrices $A_{u,i,j,h}^0$ and $A_{l,i,j,h}^0$ such that

$$IS_{i,j,h}^u = IR_{i,j,h}(A_{u,i,j,h}^0, A_{u,i,j,h}^0 \cdot B_+)$$
$$IS_{i,j,h}^l = IR_{i,j,h}(A_{l,i,j,h}^0, A_{l,i,j,h}^0 \cdot B_+)$$

In some applications, researchers are mainly interested in the maximal (minimal) admissible response of some variable $i$ to some structural shock $j$, at a range of horizons. In other words, they are precisely interested in the edges of the identified set. From the previous result we conclude that we can exactly recover these edges by solving an $n(n-1)/2$-dimensional constrained optimization problem. Even for moderately large systems, solutions to problems of this sort can be obtained quite quickly.

More generally, however, researchers will be interested in the entire identified set (or, more specifically, statistics of the identified set beyond minima and maxima). Unfortunately, the upper and lower edges are not sufficient to characterize $IS_{i,j,h}$, since $IS_{i,j,h}^u$ need not be connected.

---

7 Examples of disconnected identified sets of impulse responses can easily be constructed for two-variable VARs. Let the two variables of the VAR be $(x, y)$, and suppose that we are only interested in identifying a single shock. Furthermore suppose that the default SVAR (the starting point for our rotation) implies horizon-0 responses $(1, 0)$ and $(0, 1)$ to the two disturbances. Finally suppose that our sole identifying restriction is that the response of $y$ to the identified shock must lie in $[0.2, 0.5]$. Then the loading on the second shock can lie anywhere in $[0.2, 0.5]$, while the loading on the first shock must lie anywhere in $[-\sqrt{0.96}, -\sqrt{0.75}] \cap [\sqrt{0.75}, \sqrt{0.96}]$, so the identified set for the response of $x$ to the identified shock is disconnected.
**Proposition 3.2.** Let \( R \) be a feasible identifying restriction on the reduced-form VAR \((B_+, \Sigma_u)\). Then, for all \( i = 1, \ldots, N, j = 1, \ldots, N \), and \( h = 1, \ldots, H \), the identified set \( IS_{i,j,h} \) is compact.

This is all we can say at this level of generality. In all my DSGE-based applications the identified set indeed turns out to be equal to \([IS^l_{i,j,h}, IS^u_{i,j,h}]\) for all \( i, j, h \). This result, however, cannot be established ex-ante, so for explicit characterization of the identified set we need rely on numerical techniques.

### 3.2 Some Numerical Procedures

We wish to numerically characterize \( IS_{i,j,h} \) for all \( i, j, h \). Since the mapping from elements of \( D_j \) to elements of \( IS_{i,j,h} \) is linear, we can just as well aim to numerically characterize \( D_j \), and so \( P_R \). The definition of \( P_R \) then suggests two approaches – one deterministic, and one probabilistic.

*Discretization of \( O(n) \).* We can approximate the set of orthogonal rotation matrices \( O(n) \) through some finite subset \( P \). The numerical approximation to \( P_R \) is then \( P \cap P_R \). The potential problem with this procedure is how to construct a "good" discretization. Given the linear mapping from \( D_j \) to \( IS_{i,j,h} \), we would ideally like an algorithm that, for any given \( \epsilon \), creates an \( \epsilon \)-net on \( O(n) \). Another possible approach would be to exploit the well-known fact that Givens rotation matrices form a basis of \( O(n) \). Uniform discretization of the basis intervals \([0, \pi] \), \([0, 2\pi] \) for Givens rotation matrices then translates into a convenient discretization of \( O(n) \).

The efficient construction of \( \epsilon \)-nets appears to be an unsolved problem. Discretization of the Givens basis, in contrast, is conceptually trivial, but quickly becomes computationally infeasible as system size increases. For example, for an \( n \)-dimensional system with \( k \) grid points for every interval \([0, \pi]\) and \( 2k \) points for every interval \([0, 2\pi]\), we would overall need \( k^{n(n-1)/2(n-1)} \) points. Fortunately, in many applications, researchers are only interested in the response to a single structural shock. For an \( n \)-dimensional system, a single structural disturbance vector will be an element of the unit sphere \( S(n) \). The construction of \( \epsilon \)-nets for unit spheres is possible for reasonably small \( n \), but since I am not aware of any algorithms that work for general \( n \) I focus instead on basis discretization.

Any disturbance vector \( p \in S(n) \) can be represented in spherical coordinates through \( n - 1 \) angular coordinates, with the \( n - 1 \)-th coordinate ranging over \([0, 2\pi]\) and all other coordinates ranging over \([0, \pi]\). Thus, for \( k \) gridpoints per \([0, \pi]\) interval, we would overall get \( 2 \cdot k^{n-1} \) vectors in \( S(n) \). An important drawback of this discretization approach is its inability to deal with combinations of zero and sign restrictions. With zero and sign restrictions we generically have \( \mu(P_R) = 0 \), and so with probability 1 all rotation matrices in the discretized basis will lie outside of \( P_R \).
Uniform Draws. We can uniformly draw from \( P_R \). [Rubio-Ramirez et al. (2010)] provide a suitable algorithm for the purely sign-restricted case, and [Arias et al. (2014)] provide an algorithm for the mixture case of sign and zero restrictions. Asymptotically this procedure will of course allow us to perfectly recover the identified set.

**Proposition 3.3.** Let \( P \in P_R \) be generic in the sense that, for all \( i, j, h \), \( IR_{i,j,h}(PA_0, PA_+) \) does not satisfy any sign restrictions with equality. Furthermore let \( n \) be the number of uniform draws with respect to the Haar measure from \( O(n) \) (for the purely sign-restricted case) or the subspace of \( O(n) \) consistent with the imposed zero restrictions (for the mixture case). Then there exists some \( \epsilon > 0 \) such that, for all \( P' \in B_\epsilon(P) \), \( P' \in P_R \), where \( B_\epsilon(P) \) denotes an \( \epsilon \)-ball around \( P \) in the \( L_{2,1} \)-norm.

Thus, asymptotically, uniform draws from \( P_R \) allow us to exactly recover the identified set (as well as any desired statistic thereof). The potential problem with this procedure is how to find bounds on the expected number of draws required to give a "good" characterization of \( D_j \) with sufficiently high probability. It thus remains to provide some finite-sample bounds. For this I will, in line with most of the literature, focus on identification of a single disturbance vector (a single element in \( n \)-dimensional unit sphere \( S(n) \)). Furthermore I now explicitly assume that the state-space system is invertible, so we can interpret rows of rotation matrices as structural disturbance vectors. We can thus, in the definition of the identified of structural disturbance vectors \( D_j \), restrict attention to the first element of the sequence \( \rho_j \), as discussed above. The key result now is the following.

**Proposition 3.4.** Let \( p_j^* \in D_j \) be an admissible structural disturbance vector of interest. Suppose that \( p_j^* \) is generic in the sense that, for \( \epsilon > 0 \) small enough, \( B_\epsilon(p_j^*) \cap S(n) \subset D_j \). Then, for selected small enough \( \epsilon > 0 \) and VAR-dimensionality \( n \), the following table displays the expected number of uniform draws from the unit sphere \( S(n) \) required to generate a draw from \( B_\epsilon(p_j^*) \cap D_j \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \epsilon )</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>20</td>
<td>40</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>54</td>
<td>151</td>
<td>424</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>138</td>
<td>543</td>
<td>2,152</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>348</td>
<td>1,914</td>
<td>10,680</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>864</td>
<td>6,647</td>
<td>52,174</td>
<td></td>
</tr>
</tbody>
</table>

The table presented in the proposition statement above displays the expected number of draws. Another statistic of interest would have been the number of draws required to generate a draw from \( B_\epsilon(p_j^*) \cap D_j \) with probability at least \( 1 - \delta \). This is readily found as \( \ln(\delta)/\ln(1 - q_j^{*\epsilon}) \), where \( q_j^{*\epsilon} \) is the probability of a single draw being in \( B_\epsilon(p_j^*) \cap D_j \) (see appendix). For example, for \( n = 7 \) and \( \epsilon = 0.05 \), we would require 156, 298 draws.
The bounds presented in the statement of the theorem apply directly if we are interested in some special admissible structural shock vector (for example the one corresponding to the maximal response at some horizon) and want to see how long we have to draw to get another, sufficiently close disturbance vector. The probability bounds are, however, also useful for recovery of the entire identified set. Suppose, for example, that we seek to approximate the identified set through a set of \( k \) points on the unit sphere, indexed by \( j \) and denoted \( p_j \). Then we could surround each point \( j \) by a small enough \( \epsilon \)-ball and so partition the unit sphere into \( k + 1 \) regions. We would then be interested in the probability of receiving one draw from each of the \( k \) \( \epsilon \)-balls, where the probability of a draw from each individual ball is the same and equal to

\[
q'_j = \frac{\text{area of } B_\epsilon(p_j) \cap S(n)}{\text{area of } S(n)}
\]

Any statistic of interest (say the number of draws required to get at least one draw from each ball with sufficiently high probability) could then be recovered using standard methods, for example through Stirling numbers of the second kind. Of course, for all these bounds, the required number of draws will seem quite high. Fortunately, however, the computational burden for drawing uniformly from a unit sphere is quite low. For example, 1,000,000 draws in 10-dimensional system only require around 2 seconds on a standard MacBook Pro.

4 Application: The Real Effects of Monetary Policy

In this section I apply the methods developed above to the analysis of monetary policy disturbances identified through sign restrictions, as in Uhlig (2005). As reference models I consider a simple three-equation New Keynesian model as well as the canonical Smets-Wouters DSGE. The three-equation New Keynesian model is also used to illustrate the virtues and drawbacks of the various different approaches to numerical characterization of the identified set discussed above.

4.1 A Three-Equation New Keynesian Model

Model Outline. The baseline model is a conventional three-equation New Keynesian model:

\[
y_t = \mathbb{E}_t(y_{t+1}) - \frac{1}{\sigma}(r_t - \mathbb{E}_t(\pi_{t+1}) - r^*) + \epsilon^d_t \tag{IS}
\]

\[
\pi_t = \beta \mathbb{E}_t(\pi_{t+1}) + \kappa y_t + \epsilon^\pi_t \tag{NKPC}
\]

\[
r_t = (1 - \phi_r)[r^* + \phi_x \pi_t + \phi_y y_t] + \phi_r r_{t-1} + \epsilon^r_t \tag{MP}
\]

where

\[
\epsilon^d_t = \rho_d \epsilon^d_{t-1} + \epsilon^d_t, \quad \epsilon^\pi_t = \rho_\pi \epsilon^\pi_{t-1} + \epsilon^\pi_t, \quad \epsilon^r_t = \epsilon^r_t
\]
and \((\epsilon^d_t, \epsilon^\pi_t, \epsilon^r_t)' \sim N(0, \Omega)\), with \(\Omega\) diagonal. \(y\) is real output, \(r\) is the nominal interest rate (the federal funds rate), \(r^*\) is the real natural rate of interest and \(\pi\) is inflation. Steady-state inflation is 0. The model has three disturbances: a demand shock \(\epsilon^d\), a cost-push (negative supply) shock \(\epsilon^\pi\) and a monetary policy shock \(\epsilon^r\). One period in the model corresponds to a quarter. All parameters are set at conventional levels, with a detailed overview of the exact choices of parameter values relegated to the appendix. It suffices to say here that the conclusions of the analysis are materially unaltered for wide sets of possible parameterizations. Throughout the variables will be ordered \(y, \pi, r\) and the shocks are ordered \(\epsilon^d, \epsilon^\pi, \epsilon^r\).

Following the procedure detailed in section 2, we can recover, from the linear state-space representation of this DSGE, the implied reduced-form VAR representation

\[
y_t = \sum_{j=1}^{\infty} B_j y_{t-j} + u_t
\]

In the spirit of Uhlig (2005), I define a contractionary monetary impulse vector as follows:

**Definition 8.** The vector \(\theta \in \mathbb{R}^3\) is called an impulse vector, iff there is some matrix \(\Theta\) so that \(\Theta \Theta' = \Sigma_u\) and so that \(\theta\) is a column of \(\Theta\). A contractionary monetary policy impulse vector is an impulse vector \(\theta\) so that the model impulse responses to \(\theta\) satisfy the following restrictions:

1. At horizons \(h = 0, \ldots, H\), the impulse response of prices is non-positive.
2. At horizons \(h = 0, \ldots, H\), the impulse response of the federal funds rate is non-negative.

Uhlig (2005) additionally requires non-borrowed reserves to decline following a contractionary monetary policy. I cannot impose this restriction in my simple three-equation New Keynesian model. However, this extra restriction is at any rate designed chiefly to disentangle money demand and monetary policy shocks, which by assumption I do not have in my model. Furthermore, Uhlig’s results can be replicated almost perfectly for a three-variable SVAR (output, prices, interest rate) estimated on U.S. data and identified through the restrictions proposed above. In light of this, and to avoid straying too far from accepted baseline models, I have decided to stick with the simple three-equation New Keynesian model.

I set \(H = 4\) (one year after impact). The results are robust to changing this horizon, for example down to \(H = 2\) (two quarters after impact) or up to \(H = 8\) (two years after impact). To aid interpretation of the subsequent results, it will be useful to briefly consider the true IRFs implied by the model:
Following a contractionary monetary policy shock, the nominal interest rate (federal funds rate) increases, output drops and inflation drops. Since the model does not include any habit formation or other inertia-inducing features, the maximal responses of all variables are on impact.

**Recovering the Identified Set.** To recover the identified set of impulse responses I use the various techniques proposed above. The first plot shows the identified set after 100 uniform draws from $P_R$, with the pointwise median response in thick black:

The identified set appears to be connected in that case. However, with 100 draws, we see that there are still quite a lot of visible gaps left; in particular, this suggests that the edges of the identified set of impulse responses are not recovered correctly. Explicit maximization gives the following bounds,
plotted against the results from the random draws for convenience:

The plot suggests that, with as little as 100 random draws, we are already very close to the true boundaries of the identified set. The third and final approach is to discretize the Givens basis for the rotation matrices. I consider 25 gridpoints per \([0, \pi]\) interval, for a total of 2,863 gridpoints. The identified set is plotted below, again with the pointwise median response in thick black:

The edges identified through exact optimization are, at least up to visual inspection, attained perfectly. The identified set seems to be connected. An almost identical picture emerges from 1,000 random draws:
Since random draws are the fastest procedure, I base all my following results on 1,000 uniform draws from $P_R$, and conclude from the evidence above that this offers adequate characterization of the identified set.

**Interpretation.** The plots of the identified set of impulse responses agree remarkably well with those presented in Uhlig (2005) (and replicated in the appendix for convenience), in particular as regards the response of output. In other words, in the textbook model of monetary non-neutrality, Uhlig’s identification procedure gives results analogous to those in real data. So consider an economist with reasonably tight, but non-dogmatic, prior on the short-run non-neutrality of monetary policy. In light of the foregoing results, such an agent should, after having observed Uhlig’s results, not materially revise his beliefs on the non-neutrality of monetary policy.

Why is the identified set of impulse responses displayed above so wide? In particular, why does the identification procedure of Uhlig (2005) provide hardly any shrinkage on the upper end of the initial response of output, even though only monetary policy shocks satisfy the imposed restrictions? To answer this it will be instructive to more carefully analyze what the admissible "monetary policy shocks" actually look like. A (smoothed) plot of the identified shock vectors follows, with the identified shocks ordered according to the initial output response, from biggest to smallest.

---

8Forecast error variance decompositions also look very similar in data and model, and are available upon request.
9Formally, the upper end of the identified set of impulse responses is very close to the maximal possible response, computed by choosing $P \in O(n)$ to maximize the impact response of output to the identified shock.
10For all parameterizations that I considered, demand and supply shocks move prices and interest rates in the same direction. Of course, with a sufficiently large response to output in the Taylor rule, supply shocks will start to move inflation and interest rates in opposite directions. However, the required coefficients are implausibly large.
As we can see, the largest identified output responses correspond to combinations of positive demand and negative cost-push (or positive supply) shocks. That is, positive demand and supply shocks masquerade as contractionary monetary policy shocks, but of course with opposite output implications. Put briefly, positive demand and supply shocks that (1) occur at the same time and (2) are of a certain relative magnitude will, just like a contractionary monetary policy shock, tend to increase interest rates and push down prices. Since demand and supply shocks are much more frequent than monetary policy shocks, conditions (1) and (2) are satisfied often enough relative to the incidence of monetary shocks for masquerading to be a serious threat to identification.\footnote{In light of this it is not surprising that, as we let the volatility of the monetary policy disturbance rise to counterfactually large levels, identification improves substantially.}

As we go to the right, we get closer to the true monetary policy shock; at around the 850th rotation the monetary policy is recovered perfectly. To get a further gauge of how close the identified shocks are to the true monetary policy disturbances I propose to proceed as follows. Since we are chiefly interested in the response of output, and since in the underlying model the largest responses are at the period-1 horizon, I consider the share of the impact output response attributable to the loading of the monetary policy as a quantitative gauge of the quality of the identification. Formally, I consider

\[
\frac{|p_3^{-1}IR_{1,3,1}|}{\sum_{i=1}^3 |p_i^{-1}IR_{1,i,1}|}
\]

where \(p_i^{-1}\) denotes the \(i\)th entry of the identified shock loading and as before \(IR_{i,j,h}\) is the response of variable \(i\) at horizon \(h\) to structural disturbance \(j\). A plot follows.
According to the proposed measure, most identified shocks have very little to do with the object of interest: monetary policy disturbances. As expected from the plot of loadings displayed above, for the largest output responses the monetary policy shock plays hardly any role, while around rotation 850 we get pretty close to recovering the true monetary policy shock.

4.2 The Smets-Wouters Model

Model Overview. Much of the intuition from the simple three-equation New Keynesian model carries through to more general underlying model frameworks. In this section I consider the Smets-Wouters model, probably the most well-known example of an empirically successful medium-scale DSGE model. Following Smets and Wouters (2007), and departing from earlier working paper versions of the Smets-Wouters model, I reduce the number of structural shocks to 7, matching the number of observables. This allows me to derive an invertible linear state-space representation and generate a VAR(∞) in output $y$, consumption $c$, the policy rate $r$, investment $i$, employment $l$, inflation $\pi$ and wages $w$. The disturbances are a productivity shock, a financial frictions/net worth shock, a (government) spending shock, an investment shock, a monetary policy shock, a price mark-up shock, and a wage mark-up shock. I solve the model at the mode parameter estimates of Smets and Wouters (2007). For details of the model the interested reader is referred to the original paper Smets and Wouters (2007). All reported results are based on a characterization of the identified set through random drawing of rotation matrices, as in the three-equation model. The identified set under the baseline restrictions of Uhlig (2005) is shown overleaf.
Results and Interpretation. Again we see that the identified set for the output response is very wide, and that the median response is positive. To interpret these findings it is, as above, instructive to consider the corresponding (smoothed) shock loadings.

The results are very similar to those from the baseline model. The largest output responses correspond to positive demand shocks (here positive financial frictions shocks) and positive supply shocks (here negative price mark-up and wage mark-up shocks) masquerading as contractionary monetary policy shocks.\footnote{Forecast error variance decompositions are also quite similar to those reported in Uhlig (2005).}

Note that the above analysis was conducted exclusively in the context of an invertible state-space system derived from the Smets-Wouters model. An interesting alternative benchmark would be to consider the implied non-invertible system in the three standard observables output, inflation...
and the interest rate. Of course in that case we need to consider the entire identified sequence $\rho_j$ rather than just rows of the rotation matrix $P$, substantially complicating visual analysis of the identified set. Fortunately, however, weights decline quite quickly, so it is not overly misleading to only consider entries of $\rho_0$. The first plot shows the identified impulse response sets.

The results look rather similar to those above, with the shapes of the identified sets overall almost identical. The perhaps most obvious difference is the asymmetry in the identified sets, as reflected in the non-central positions of the median responses. A plot of the shock loadings in $\rho_0$ shows that the masquerading story survives as above.

Conclusions. Overall, I draw three conclusions from my application of Uhlig’s identification scheme to the three-equation New Keynesian and Smets-Wouters DSGE models. First, for a non-dogmatic believer in the non-neutrality of monetary policy, Uhlig’s results should hardly affect her beliefs on monetary policy transmission. In both models, and in both invertible and non-invertible representations, identified impulse response sets (and also forecast error variance decompositions) look very similar to the results presented in Uhlig (2005). Second, a natural explanation for the close agreement between model-based results and data-based is the masquerading story. This story is robust across all model frameworks considered. And third, coming back to my earlier comments on the ex-ante merits of a given (sign) identification scheme, the conclusion seems inevitable that the baseline identification scheme in Uhlig (2005) is too weak to allow us to draw sharp conclusions. This suggests that we need to look for stronger – but ideally, in the spirit of Uhlig (2005), still "widely accepted" or "plausible" – identifying restrictions. A recent contribution that goes exactly in that direction is Arias et al. (2015).
4.3 More Restrictions: Arias et al. (2015)

Arias et al. (2015) argue, in agreement with my previous conclusions, that the restrictions of Uhlig (2005) are too weak to adequately identify monetary policy shocks. They thus propose to add to his two identifying restrictions two further restrictions. With these two additional restrictions, the response of output to a contractionary monetary policy shock is, as expected, negative. Of the two new restrictions imposed by the authors, the decisive one turns out to be the following:

3. The contemporaneous reaction of the federal funds rate to output and to the GDP deflator (to prices) is non-negative.

Implementing this proposed additional restriction in the three-equation New Keynesian model or the Smets-Wouters DSGE is straightforward. In either case, the restriction essentially amounts to saying that the entries of the identified candidate \( \tilde{A}_0 \) matrix corresponding to the output and inflation loadings in the monetary policy equation are weakly negative. For example, in the three-equation model, the restriction is just that the entries \((3,1)\) and \((3,2)\) of \( \tilde{A}_0 \) are both weakly negative. To implement this additional restriction I just check whether or not it is satisfied and then discard the proposed draw if the restriction is violated. Before considering the results of this procedure, however, it will prove instructive to briefly review the economic justification of the proposed restriction. It derives from the natural assumption that, in the monetary authority’s reaction function, the loadings on inflation and output should be positive. For concreteness, let us consider the merits and implications of this assumption in the three-equation model. Clearly, with \( \phi_r \in (0, 1) \) and \( \phi_x, \phi_y > 0 \), the assumption is also satisfied in the simple baseline model. However, the restriction of Arias et al. (2015) is on entries of \( A_0 \), the contemporaneous coefficient matrix in the structural VAR(\( \infty \)) implied by the solution of the linear rational expectations system \((IS), (NKPC), (MP)\). Ex-ante, there is no reason that \( \phi_x, \phi_y > 0 \) must necessarily translate into \( A_0(3,1), A_0(3,2) \leq 0 \). In fact, we know that \( A_0 = IR_0(A_0, A_+)^{-1} \). So write

\[
IR_0(A_0, A_+) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}
\]

Recall that the variables are ordered \( y, \pi, r \) and that the shocks are ordered \( \epsilon^d, \epsilon^\pi, \epsilon^r \). We can thus sign the coefficients. \( b, c, f \) are negative, and all other coefficients are positive. \( A_0 \) is then given as

\[
A_0 = \frac{1}{-ceg + bfg + cdh - afh - bdi + aei} \begin{pmatrix} -fh + ei & ch - bi & -ce + bf \\ fg - di & -eg + ai & cd - af \\ -eg + dh & bg - ah & -bd + ae \end{pmatrix}
\]

It is readily seen that the signs of the entries of \( IR_0(A_0, A_+) \) are not sufficient to pin down the signs
of $A_0(3,1), A_0(3,2)$. For all particular parameterizations that I considered, however, we indeed have $A_0(3,1), A_0(3,2) \leq 0$. I will later return to this issue in the context of the Smets-Wouters model. For the moment, it suffices to note that the restrictions of Arias et al. (2015) are consistent with the model and so imposing them may well be reasonable.

Interestingly, in the three-equation model, imposing the additional restrictions of Arias et al. (2015) has the same effects as in their empirical analysis: the identified set of output responses is shrunk dramatically towards negative impact responses. The corresponding plot follows.

To ease comparison with the existing literature, I have also plotted 16% and 84% bounds (rather than only the true edges of the identified set). At this conventional level, the response of output is negative and significant. To understand why the additional restriction works it is instructive to again consider the set of identified disturbance vectors.

---

13 These bounds are just percentiles of the identified set. In the empirical literature, researchers draw many times (say $N$ times) from the reduced-form posterior, search for a rotation that satisfies their imposed sign restrictions, and then compute percentiles from their $N$ impulse responses. This procedure incorporate both sampling and model uncertainty. By construction my approach abstracts from sampling uncertainty, and so I only report bands reflecting the model uncertainty.
As we can see, the loading on the monetary policy shock is high throughout. The loadings on demand and supply shocks, in contrast, are small, and in particular never reach the levels seen under the pure Uhlig restrictions. Intuitively, most of the masquerading combinations of expansionary demand and supply shocks imply counterfactual responses of interest rates to demand and supply shocks. By ruling out these counterfactual responses we get much closer to recovering the true monetary policy disturbance. This becomes even clearer by looking at the share of output directly attributable to the monetary policy shock.

For most rotations the share attributable to the monetary policy shock is reasonably high, but of course not perfect. Imposing the sign restrictions on prices and federal funds for longer horizons improves this somewhat. Still, the median impact response consistently remains somewhat too high, attributable to the combinations of negative demand, supply and monetary policy shocks at the right end of the identified spectrum.

The success the identification scheme of [Arias et al. (2015)] does not extend to the Smets-Wouters model. Interestingly, even though the monetary policy reaction function agrees with the extra restrictions, it turns out that the model-implied $A_0$ does not. The problem of course is that monetary policy reaction function and monetary policy equation in the VAR($\infty$) are not the same thing. In the simple three-equation New Keynesian model the distinction did not matter; in the Smets-Wouters model, it does. As a result, it is not surprising that the restrictions of [Arias et al. (2015)] now do not help; in fact, they make matters worse (as gauged by the proposed "output response share" metric defined earlier) \[14\]

\[14\] Strictly speaking, the extra restriction only makes matters worse in the full invertible version of the Smets-Wouters state-space system. In the three-observables VAR the restrictions help somewhat, but the median response of output is still positive.

26
4.4 "Real" Baseline Models

I would conjecture that many macroeconomists share a reasonably tight prior on the non-neutrality of monetary policy. For such an audience, the foregoing analysis was natural – it tells a coherent story of why we should not be surprised that, under the identification procedure of Uhlig, "neutrality of monetary policy shocks is not inconsistent with the data" (Uhlig (2005)). However, an interesting alternative exercise would be to gauge the performance of Uhlig’s identification procedure in a model with (near) money neutrality. Indeed, if in such models the identified set looks sufficiently different from that identified in [Uhlig (2005)], then we would have prima facie evidence against this alternative set of "real" baseline models.

To this end I consider first the baseline three-equation New Keynesian model with the probability of Calvo price adjustment approaching 1. Under the proposed parameterization, demand shocks and cost-push shocks lead to inflation overshooting at very short horizons. Since I require price and interest rate responses to have the expected sign for up to one year (in his paper Uhlig actually considers horizons up to two years), the identification scheme actually works somewhat better now and identifies monetary policy shocks slightly more reliably. The median impact response is now negative, but identified sets remain quite wide. A plot follows.

The identified sets in the baseline three-equation New Keynesian model and the Smets-Wouters DSGE considered above look somewhat closer to the identified set [Uhlig (2005)], but clearly the evidence is far too weak for us to be drawing conclusions on the relative merits of the different models. Slightly stronger evidence comes from a limiting real version of the Smets-Wouters model. I let Calvo price and wage adjustment probabilities approach 1, and set all other parameters equal to their mode in the estimated sticky-price Smets-Wouters model. Under this parameterization,
a serious problem is that the interest rate response to a monetary policy shock reverts sign very quickly. The response of inflation to most other shocks furthermore again shows an overshooting pattern, and so overall it becomes rather difficult to impose the proposed identification scheme up to reasonable horizons. For example, with the restrictions imposed until the one-year horizon, I need on average around 200 rotation matrices until the restrictions are satisfied, while under the original parameterization the number of required draws rarely exceeded 5. In my own empirical work with sign-identified SVARs I generally find that price and interest rate restrictions, even for reasonably long horizons, are rather easy to satisfy. In light of this, the proposed money-neutral Smets-Wouters model seems misspecified. As a further check I then tried to re-estimate the Smets-Wouters model subject to the constraint that the price adjustment probabilities be close to 1. For this I used the original Smets-Wouters data and imposed their priors on all remaining model parameters. Unfortunately the estimation routine is extremely ill-behaved, suggesting that a money-neutral Smets-Wouters model is not suitable to be taken to the data.

4.5 Recursive Identification Schemes

It is instructive to briefly assess the performance of the conventional recursive identification scheme in the two proposed underlying models. In the three-equation model the derived impulse responses look as follows.

The results are as usual. Output and inflation both show a hump-shaped decline to a contractionary monetary policy shock, with the peak response occurring after a bit less than a year. After around four years, all variables have returned to baseline. Interestingly, the identified shock has loadings $(0.3744, 0.0118, 0.9272)'$. That is, we are actually identifying a linear combination of (mostly) expansionary demand and contractionary monetary policy shocks, with the loading on supply shocks negligible. Even though the sign of the output response is as expected, the value of the
response is severely distorted. Due to the high loading on the demand shock the output response to the identified monetary policy shock is much smaller than in the true model, up to an order of almost 10. Accordingly, it is not surprising that only around 46% of the variation in output is directly attributable to the true monetary policy shock.

In the Smets-Wouters model, in contrast, a Choleski decomposition works remarkably well. I consider the ordering $y, c, i, l, w, \pi, r$. The identified shock vector has a loading of 0.9627 on the monetary policy shock, and the impulse responses look very similar to the true model-implied impulse responses, of course with some discrepancy on impact. The following plot compares the responses of the three key variables.

Given the habit formation implicit in the Smets-Wouters model, it is unsurprising that recursive schemes now work quite well.\footnote{In fact, habit formation is mostly put in to allow the model to match the IRFs derived from SVARs identified through recursive orderings. This does not mechanically mean that a recursive ordering applied to the VAR representation of the Smets-Wouters model has to do a good job in recovering monetary policy shocks, but of course it makes it much more likely than in, say, the baseline three-equation model.}
5 Conclusions

Set identification of SVARs can be both blessing and curse. Researchers want their identifying restrictions to be as weak and universally acceptable as possible, but in doing so they run the risk of actually not being able to say anything. This paper has offered a guideline on how to balance the virtues of agnosticism with the need for sufficiently strong identification. Before researchers decide to take their proposed identification restrictions to the data, they should make sure that, at least within some pre-specified set of plausible underlying structural models, their identification scheme does a reasonable job of recovering the truth. Whether or not this is so can be assessed through the general methods developed in this paper. More broadly, the methods developed here can also help gauge the extent to which various exact (i.e, point-identifying) identification schemes can allow successful recovery of the structural disturbance of interest.

The particular application that I consider is the debate surrounding the real effects of monetary policy interventions, in particular in light of the influential contribution of [Uhliz (2005)]. Other interesting applications may lie in the econometric analysis of oil price shocks or technology shocks, both routinely identified through sign (or sign and zero) restrictions.
References


A Appendix

A.1 Derivation of VAR Representation

The discussion in this section heavily draws on Fernández-Villaverde et al. (2007). Consider the following linear Gaussian state-space system:

\[ s_t = As_{t-1} + Bw_t \]  \hspace{1cm} (A.1)
\[ y_t = Cs_{t-1} + Dw_t \]  \hspace{1cm} (A.2)

The time-invariant innovations representation of this state-space system

\[ \hat{s}_t = A\hat{s}_{t-1} + Ku_t \]  \hspace{1cm} (A.3)
\[ y_t = C\hat{s}_{t-1} + u_t \]  \hspace{1cm} (A.4)

where \( u_t \) is a Gaussian forecast error with \( \mathbb{E}[u_t] = 0 \), \( \Sigma_u \equiv \mathbb{E}[u_t u_t'] = C\Sigma_s C' + DD' \) and \( \mathbb{E}(u_t u_{t-j}) = 0 \) for \( j \neq 0 \), and where \( K, \Sigma_s \) satisfy the Riccati equations

\[ \Sigma_s = (A - KC)\Sigma_s (A - KC)' + BB' + KDD'K' - BD'K' - KDB' \]  \hspace{1cm} (A.5)
\[ K = (A\Sigma_s C' + BD')(C\Sigma_s C' + DD')^{-1} \]  \hspace{1cm} (A.6)

Sufficient conditions for the existence of this time-invariant innovations representation are given in Fernández-Villaverde et al. (2007). From the innovations representation we get an MA(\( \infty \)) representation for \( y_t \):

\[ y_t = I + C(I - AL)^{-1}KL \]  \hspace{1cm} u_t

Matrix inversion gives

\[ [I - C(I - (A - KC)L)^{-1}KL]y_t = u_t \]

or, assuming that \( A - KC \) is stable, the desired VAR representation

\[ y_t = \sum_{j=1}^{\infty} C(A - KC)^{j-1}Ky_{t-j} + u_t \]  \hspace{1cm} (A.7)

If the state-space system is invertible then \( \Sigma_s = 0 \), so \( K = BD^{-1} \), and straightforward manipulations give

\[ y_t = \sum_{j=1}^{\infty} C(A - BD^{-1}C)^{j-1}BD^{-1}y_{t-j} + u_t \]  \hspace{1cm} (A.8)

It now remains to prove Proposition 2.1.
Proof of Proposition 2.1. Recall that \( u_t = C(s_{t-1} - \hat{s}_{t-1}) + Dw_t \). Using this, we can combine the state-space representation (A.1) - (A.2) and the innovations representation (A.3) - (A.4) to get

\[
\begin{pmatrix}
    s_t \\
    \hat{s}_t
\end{pmatrix} =
\begin{pmatrix}
    A & 0 \\
    KC & A - KC
\end{pmatrix}
\begin{pmatrix}
    s_{t-1} \\
    \hat{s}_{t-1}
\end{pmatrix} +
\begin{pmatrix}
    B \\
    KD
\end{pmatrix} w_t \tag{A.9}
\]

\[
    u_t =
    \begin{pmatrix}
        C \\
        -C
    \end{pmatrix}
    \begin{pmatrix}
        s_{t-1} \\
        \hat{s}_{t-1}
    \end{pmatrix} + Dw_t \tag{A.10}
\]

Solving the (A.9) for \((s_t, \hat{s}_t)\)' and plugging into (A.10) we get

\[
    u_t = \left\{ D + \begin{pmatrix}
        C \\
        -C
    \end{pmatrix} \left( I - \begin{pmatrix}
        A & 0 \\
        KC & A - KC
    \end{pmatrix} L \right)^{-1} \begin{pmatrix}
        B \\
        KD
    \end{pmatrix} L \right\} w_t \tag{A.11}
\]

Recalling that \( \epsilon_t \equiv A_0 u_t \) we get the first part of the proposition.

If instead the state-space system is invertible then from the construction of the VAR representation we have \( u_t = Dw_t \) and so the second part of the proposition follows. \( \blacksquare \)
A.2 Analytical and Numerical Characterization of the Identified Set

This section of the appendix contains the main arguments for the propositions in sections 3.1 (on the analytical characterization of identified sets) and 3.2 (on the numerical characterization of identified sets).

Analytical Results. A useful starting point for general statements about the shape of the identified set of impulse responses is a more careful consideration of the identified set of rotation matrices $P_R$. We may write $P_R \equiv O(n) \cap X$, where again $O(n)$ is the set of orthogonal matrices $O(n)$ and $X$ is the set of matrices in $\mathbb{R}^{n \times n}$ such that, for all $X \in X$, $IR_{i,j,h}(XA_0, XA_+)$ satisfies the imposed sign and zero restrictions for all $i, j, h$. For $O(n)$ we have the following two useful results.

**Lemma A.1.** $O(n)$ is compact.

This is a well-known fact in group theory. For the second result recall that a Givens matrix of dimension $n$ with argument $\theta_{i,j}$, written $Q_{i,j}(\theta_{i,j})$, has the following structure:

$$Q_{i,j}(\theta_{i,j}) = \begin{pmatrix}
& & \text{col } i & \text{col } j & \\
\downarrow & \downarrow & \\
1 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\text{row } i \rightarrow & 0 & \ldots & \cos(\theta_{i,j}) & \ldots & -\sin(\theta_{i,j}) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\text{row } j \rightarrow & 0 & \ldots & \sin(\theta_{i,j}) & \ldots & \cos(\theta_{i,j}) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 1
\end{pmatrix}$$

We then have the following.

**Lemma A.2.** Let $P$ be an arbitrary $n \times n$ orthogonal matrix. Then there exist $\frac{n(n-1)}{2}$ Givens matrices $Q_{i,j}(\theta_{i,j})$, with $1 \leq i < n$, $i \leq j \leq n$, such that $0 \leq \theta_{i,j} \leq \pi$ for $i < j < n$, $0 \leq \theta_{i,n} \leq 2\pi$ and

$$P = \prod_{i=1}^{n} \Pi_{j=i+1}^{n} Q_{i,j}(\theta_{i,j})$$

That is, every $P \in O(n)$ can be written as the product of $\frac{n(n-1)}{2}$ Givens matrices.

For future reference I will denote this Givens rotation mapping by $G : \mathbb{R}^{n(n-1)/2} \rightarrow O(n)$.

To characterize $X$, it will be useful to exploit the additional structure afforded by our focus on zero and sign restrictions. For all sign restricted impulse responses, let $l_{i,j,h}$ and $u_{i,j,h}$ denote,
respectively, the lower and upper restricted response of variable $i$ to shock $j$ at horizon $h$. Then $X$ consists of all matrices $X \in \mathbb{R}^{n \times n}$ such that

\[
IR_{i,j,h} \cdot x_j' \geq l_{i,j,h} \\
IR_{i,j,h} \cdot x_j' \leq u_{i,j,h}
\]

for all $i = 1, \ldots, N$, $j = 1, \ldots, H$, $h = 1, \ldots, H$, and where $x_j$ denotes the $j$-th row of $X$. From this characterization the following is immediate.

**Lemma A.3.** $X$ is compact.

From here we can conclude the following.

**Lemma A.4.** $P_R$ is compact.

**Proof.** $P_R \equiv O(n) \cap X$, and both $O(n)$ and $X$ are compact, so their intersection is also compact.

We can use these prove that the edges of the identified set of impulse responses are attained.

**Proof of Proposition 3.1.** Any matrix $P \in O(n)$ can be written as the product of $\frac{n(n-1)}{2}$ Givens matrices, and so, since matrix multiplication is a continuous operation, $IR_{i,j,h}(P(\theta)A_0, P(\theta)A_+)$ is a continuous function of the $\frac{n(n-1)}{2}$-dimensional vector of Givens angles $\theta$. The objective function thus is continuous. But the constraint set $P_R$ is compact, so suprema and infima are attained.

The second important result is the compactness of the identified set.

**Proof of Proposition 3.2.** $P_R$ is compact, and $G$ is a continuous function, so the pre-image of $P_R$ is a compact subset of $\mathbb{R}^{n(n-1)/2}$. But $IR_{i,j,h}(P(\theta)A_0, P(\theta)A_+)$ is a continuous function of $\theta$, so the identified set $IS_{i,j,h}$ is also compact.

**Numerical Results.** Consider an arbitrary $P \in O(n)$. For the purposes of the arguments in this section it will prove convenient to consider $\theta$ such that $P = G(\theta)$, where $G$ is the Givens rotation mapping $G : \mathbb{R}^{n(n-1)/2} \rightarrow O(n)$. To prove Proposition 3.3 which is stated for a generic $P \in P_R$, we will instead consider the corresponding $G(\theta)$.

**Lemma A.5.** Let $\theta$ be such that $G(\theta) \in P_R$. Furthermore let $\theta$ be generic in the sense that, for all $i, j, h$, $IR_{i,j,h}(G(\theta)A_0, G(\theta)A_+)$ does not satisfy any sign restrictions with equality. Then there exists some $\epsilon > 0$ such that, for all $\theta' \in B_\epsilon(\theta)$, $G(\theta') \in P_R$, where $B_\epsilon(\theta)$ denotes an $\epsilon$-ball around $\theta$ in the standard Euclidian metric.

**Proof.** Since $\theta$ is generic, the result is immediate by the continuity of the constraint set $X$ together with the fact that $G(\theta') \in O(n)$ for any $\theta' \in B_\epsilon(\theta)$.

\footnote{If no constraint is desired, then we can just set $l_{i,j,h} = \min_j IR_{i,j,h}$ and $u_{i,j,h} = \max_j IR_{i,j,h}$.}
From here we obtain the desired limit result.

Lemma A.6. Let $\theta$ be such that $G(\theta) \in P_R$, and let $n$ be the number of uniform draws of rotation matrices $G \in P_R$ with respect to the Haar measure from $O(n)$ (for the purely sign-restricted case) or the subspace of $O(n)$ consistent with the imposed zero restrictions (for the mixture case). Then, for $\epsilon$ small enough, the probability of drawing $P' \in P_R \cap B_\epsilon(\theta)$ goes to 1 as $n \to \infty$.

Proof. By the previous lemma we conclude that there exists some $\epsilon'$ such that, for all $\theta' \in B_{\epsilon'}(\theta)$, $G(\theta') \in P_R$. Set $0 < \epsilon < \epsilon'$. Then $\mu(B_{\epsilon}(\theta)) > 0$, and $B_{\epsilon}(\theta) \subset P_R$, so, since draws are uniform from $O(n)$ (or the relevant subspace), the probability of drawing $\theta' \in B_{\epsilon}(\theta)$ is strictly positive. As $n \to \infty$, the probability of each draw being outside of $B_\epsilon(\theta)$ converges to 0.

The previous lemma almost equivalent to Proposition 3.3, with the sole difference that Proposition 3.3 is couched in terms of orthogonal rotation matrices and so the $L_{2,1}$-norm, rather than basis vectors $\theta$ and so the standard Euclidian norm. This final connection is however readily established.

Proof of Proposition 3.3. $P = G(\theta)$ is a continuous function of the rotation angle $\theta$. The desired result is thus immediate from the previous lemma.

Finally it remains to prove Proposition 3.4.

Proof of Proposition 3.4. Fix some arbitrary $p_j^*$. Then, given that we uniformly draw from the unit sphere, the probability of a single draw being in $B_\epsilon(p_j^*) \cap D_j$, denoted $q_j^{*,\epsilon}$, is just

$$q_j^{*,\epsilon} = \frac{\text{area of } B_\epsilon(p_j^*) \cap S(n)}{\text{area of } S(n)}$$

Both areas can be computed through numerical integration. The expected number of draws then is just $\frac{1}{q_j^{*,\epsilon}}$. Finally, as long as $p_j^* \in S(n)$, the numerator is independent of $p_j^*$, by the properties of the unit sphere. Thus the results in the statement of the theorem are valid for all possible generic $p_j^* \in D_j$.\[\Box\]
A.3 Model Details

Three-Equation New Keynesian Model. The textbook derivation of the three-equation New Keynesian model is presented in Galí (2008). I specialize his environment to assume a constant natural rate of output as well as constant returns to scale. Compared to the baseline framework I add demand and cost-push shocks and consider a slightly more general monetary policy rule. The parameterization of the model is as follows:

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>INTERPRETATION</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>discount factor</td>
<td>0.995</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>inverse of IES</td>
<td>1</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>inverse Frisch elasticity</td>
<td>1</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>demand elasticity</td>
<td>5</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Calvo parameter</td>
<td>0.75</td>
</tr>
<tr>
<td>$\phi_\pi$</td>
<td>inflation coefficient (T. rule)</td>
<td>1.5</td>
</tr>
<tr>
<td>$\phi_y$</td>
<td>output coefficient (T. rule)</td>
<td>0.1</td>
</tr>
<tr>
<td>$\phi_r$</td>
<td>rate persistence (T. rule)</td>
<td>0.8</td>
</tr>
<tr>
<td>$\rho_\pi$</td>
<td>persistence cost-push shock</td>
<td>0.6</td>
</tr>
<tr>
<td>$\rho_d$</td>
<td>persistence demand shock</td>
<td>0.7</td>
</tr>
<tr>
<td>$\sigma_\pi$</td>
<td>std monetary policy shock</td>
<td>0.3</td>
</tr>
<tr>
<td>$\sigma_d$</td>
<td>std demand shock</td>
<td>1</td>
</tr>
</tbody>
</table>

The slope of the Phillips curve, $\kappa$, satisfies $\kappa = \frac{(1-\theta)(1-\theta \beta)}{\theta}(\sigma + \varphi)$. The natural rate of interest rate $r^*$ by the time invariance of the natural rate of output satisfies $r^* = -\log(\beta)$.

Results are fully robust to substantial perturbations of most parameter values. The most important parameters for the results reported in the text are the relative standard deviations of the disturbances as well as the coefficients on the monetary policy rule.

Smets-Wouters Model. My implementation of the Smets-Wouters is based on Dynare replication code kindly provided by Johannes Pfeifer.

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17 The absolute values of course do not matter since the model is linear.
18 The code is available at [https://sites.google.com/site/pfeiferecon/dynare](https://sites.google.com/site/pfeiferecon/dynare)
A.4 The Identified Set in Uhlig (2005)

The identified sets for the output response in the various estimated VARs in Uhlig (2005) look as follows.

All impulse responses are computed at the OLS estimates of the VAR. The plots then display a range of impulse responses consistent with the imposed sign restrictions. These are in place for one quarter in the top left panel, two quarters in the bottom left panel, one year in the top right panel, and two years in the bottom right panel. As we can see the median response of output is positive, but very imprecisely estimated, as in the three-equation New Keynesian and Smets-Wouters models considered in the text.