LECTURE 05: ONE PERIOD MODEL
GENERAL EQUILIBRIUM,
EFFICIENCY & AGGREGATION
**FIN501 Asset Pricing**

**Lecture 05 GE, Efficiency, Aggregation (2)**

- **Specify Preferences & Technology**
- **Observe/specify existing Asset Prices**

- **State Prices $$q$$ (or stochastic discount factor/Martingale measure)**

- **Derive Asset Prices**

- **Absolute Asset Pricing**
  - Evolution of states
  - Risk preferences
  - Aggregation

- **Relative Asset Pricing**
  - Only works as long as market completeness doesn’t change

- **Derive Price for (new) asset**

**NAC/LOOP**

**LOOP**
Overview

1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Representative Agent Economy
Representation of Preferences

A (i) complete, (ii) transitive, (iii) continuous [and (iv) relatively stable] preference ordering can be represented by a utility function, i.e.

\[(c_0, c_1, ..., c_S) > (c_0', c_1', ..., c_S') \iff U(c_0, c_1, ..., c_S) > U(c_0', c_1', ..., c_S')\]
Agent’s Optimization

• Consumption vector \((c_0, c_1) \in \mathbb{R}_+ \times \mathbb{R}_+^S\)

• Agent \(i\) has \(U^i: \mathbb{R}_+ \times \mathbb{R}_+^S \rightarrow \mathbb{R}\)
  endowments \((e_0, e_1) \in \mathbb{R}_+ \times \mathbb{R}_+^S\)

• \(U^i\) is quasiconcave \(\{c: U^i(c) \geq v\}\) is convex for each real \(v\)
  \(U^i\) is concave: for each \(0 \leq \alpha \leq 1\),
  \(U^i(\alpha c + (1 - \alpha)c') \geq \alpha U^i(c) + (1 - \alpha)U^i(c')\)

• \(\frac{\partial U^i}{\partial c_0} > 0, \frac{\partial U^i}{\partial c_1} \gg 0\)
Agent’s Optimization

• Portfolio consumption problem

$$\max_{c_0, c_1, h} U^i(c_0, c_1)$$

subject to

(i) $0 \leq c_0 \leq e_0 - p \cdot h$

and

(ii) $0 \leq c_1 \leq e_1 + X'h$

$$\mathcal{L} = U^i(c_0, \tilde{c}_1) - \lambda [c_0 - e_0 + ph] - \tilde{\mu} [\tilde{c}_1 - \tilde{e}_1 - h'X]$$

• FOC

$$\begin{align*}
c_0: & \quad \frac{\partial U^i}{\partial c_0}(c^*) = \lambda, \\
& \quad c_0: \quad \frac{\partial U^i}{\partial c_0}(c^*) = \mu_s \\
h: & \quad \lambda \hat{p} = X \hat{\mu}
\end{align*}$$

$$\Leftrightarrow p^j = \sum_s \frac{\mu_s}{\lambda} x_s^j$$
Agent’s Optimization

\[ p^j = \sum_s \frac{\partial U^i / \partial c_s}{\partial U^i / \partial c_0} x_s^j \]

- For time separable utility function
  \[ U^i(c_0, \tilde{c}_1) = u(c_0) + \delta u(\tilde{c}_1) \]
- And vNM expected utility function
  \[ U^i(c_0, \tilde{c}_1) = u(c_0) + \delta E[u(c)] \]

\[ p^j = \sum_s \pi_s \delta \frac{\partial u^i / \partial c_s}{\partial u^i / \partial c_0} x_s^j \]
Stochastic Discount Factor

\[ p^j = \sum_s \pi_s \delta \frac{\partial u^i / \partial c_s}{\partial u^i / \partial c_0} x_s^j \]

\[ m_s = \frac{q_s}{\pi_s} \]

That is, stochastic discount factor \( m_s = \frac{q_s}{\pi_s} \) for all \( s \)

\[ p^j = \sum_s \pi_s m_s x_s^j = E[mx^j] \]
Agent’s Optimization

To sum up

• Proposition 3: Suppose $c^* \gg 0$ solves problem. Then there exists positive real numbers $\lambda, m_1, \ldots, m_S$, such that

$$\frac{\partial U^i}{\partial c_0} = \lambda$$

$$\frac{\partial U^i}{\partial c_1} = (\mu_1, \ldots, \mu_S)$$

$$\lambda p^j = \sum_s \mu_s x_s^j, \forall j = 1, \ldots, J$$

(The converse is also true.)

• The vector of marginal rate of substitutions $MRS_{s,0}$ is a (positive) state price vector.
Overview

1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Representative Agent Economy
Suppose $c_s'$ is fixed since it can’t be traded

\[ \{ c_0^A, c_1^A: U(c_0^A, c_1^A) = U(e) \} \]

\[ \frac{\partial U(c_0^A, c_1^A)}{\partial c_0^A} d c_0^A + \frac{\partial U(c_0^A, c_1^A)}{\partial c_s^A} d c_s = 0 \]

Slope

\[ -MRS_{0,s}^A = -\frac{\partial U^A}{\partial c_0^A} / \frac{\partial U^A}{\partial c_s^A} \]

A’s indifference curve
\[ c_s^A, c_s^B \]

Diagram showing the relationship between Mr. A and Ms. B with respect to consumption and savings.
\[ q = \frac{1}{MRS_{0,s}} = MRS_{s,0} = -\frac{\partial U^A/\partial c_s^A}{\partial U^A/\partial c_0^A} \]
Set of PO allocations (contract curve)
Pareto Efficiency

• Allocation of resources such that
  – there is no possible redistribution such that
    • at least one person can be made better off
    • without making somebody else worse off

• Note
  – Allocative efficiency ≠ Informational efficiency
  – Allocative efficiency ≠ fairness
Set of PO allocations (contract curve)
\[ \text{MRS}^A = \text{MRS}^B \]
Overview

1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Representative Agent Economy
Welfare Theorems

• *First Welfare Theorem.* If markets are complete, then the equilibrium allocation is Pareto optimal.
  – State price is unique $q$. All $MRS^i(c^*)$ coincide with unique state price $q$.
  – Despite (pecuniary) externalities

• *Second Welfare Theorem.* Any Pareto efficient allocation can be decentralized as a competitive equilibrium.
Knife-edginess of Welfare Theorem

- In multi-period (or multiple good) setting if markets are incomplete, then equilibrium allocation is generically not only Pareto inefficient but also constrained Pareto inefficient.
  - i.e. a social planner can do better even if restricted to the same trading space
  - Pecuniary externalities can lead to wealth shifts
    - With incomplete markets not all MRS are equalized, hence pecuniary externalities generically lead to inefficiencies.
  - ... more when we study multi-period settings
Overview

1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Aggregation/Representative Agent Economy
Representative Agent & Complete Markets

• **Aggregation Theorem 1:** Suppose
  – markets are complete
Then asset prices in economy with *many agents* are identical to an economy with a *single agent/planner* whose utility is

\[ U(c) = \sum_{k} \alpha_k u^k(c) \]

where \( \alpha^k \) is the welfare weight of agent \( k \). and the single agent consumes the aggregate endowment.
Representative Agent & HARA utility world

- **Aggregation Theorem 2:** Suppose
  - riskless annuity and endowments are tradable.
  - agents have common beliefs
  - agents have a common rate of time preference
  - agents have LRT (HARA) preferences with
    \[ R_A(c) = \frac{1}{A_i + Bc} \Rightarrow \text{linear risk sharing rule} \]

Then asset prices in economy with *many agents* are identical to a *single agent* economy with HARA preferences with
\[ R_A(c) = \frac{1}{\sum_i A_i + Bc} \]

- Recall fund separation theorem in Lecture 04.