LECTURE 06: SHARPE RATIO, BONDS, & THE EQUITY PREMIUM PUZZLE
Money, Bonds vs. Stocks
Sharpe Ratios and Bounds

• Consider a one period security available at date $t$ with payoff $x_{t+1}$. We have

$$p_t = E_t[m_{t+1}x_{t+1}]$$

or

$$p_t = E_t[m_{t+1}]E_t[x_{t+1}] + \text{cov}[m_{t+1}, x_{t+1}]$$

• For a given $m_{t+1}$ we let $R_{t+1}^f = \frac{1}{E_t[m_{t+1}]}$
  
  – Note that $R_{t+1}^f$ will depend on the choice of $m_{t+1}$ unless there exists a riskless portfolio
  
  – $R_{t+1}$ is the return from $t$ to $t + 1$, typically measurable w.r.t. $\mathcal{F}_{t+1}$. (An exception is $R_f$, which is measurable w.r.t. $\mathcal{F}_t$, but we stick with subscript $t + 1$.)
Sharpe Ratios and Bounds (ctd.)

• Hence

\[ p_t = \frac{1}{R_{t+1}} E_t [x_{t+1}] + \frac{\text{cov}[m_{t+1}, x_{t+1}]}{R_{t+1}} \]  

– Positive correlation with the discount factor adds value, i.e. decreases required return
in Returns

\[ E_t [m_{t+1} x_{t+1}] = p_t \]

– Divide both sides by \( p_t \) and note that \( \frac{x_{t+1}}{p_t} = R_{t+1} \)

\[ E_t [m_{t+1} R_{t+1}] = 1 \]

– Using \( R_{t+1}^f = 1/E_t [m_{t+1}] \), we obtain

\[ E_t [m_{t+1} (R_{t+1} - R_{t+1}^f)] = 0 \]

– \( m \)-discounted expected excess return for all assets is zero.
in Returns

- Since $E_t[m_{t+1}(R_{t+1} - R_{t+1}^f)] = 0$
  $\text{cov}_t[m_{t+1}, R_{t+1} - R_{t+1}^f]$
  $$= -E_t[m_{t+1}]E_t[R_{t+1} - R_{t+1}^f]$$

- That is, risk premium or expected excess return
  $$E_t[R_{t+1} - R_{t+1}^f] = -\frac{\text{cov}_t[m_{t+1}, R_{t+1}]}{E_t[m_{t+1}]}$$

is determined by its covariance with the stochastic discount factor.
Sharpe Ratio

- Multiply both sides with portfolio $h$

$$E_t[(R_{t+1} - R_{t+1}^f)h] = -\frac{\text{cov}_t[m_{t+1}, R_{t+1}h]}{E_t[m_{t+1}]}$$

$$E_t[(R_{t+1} - R_{t+1}^f)h] = -\frac{\rho(m_{t+1}, R_{t+1}h)\sigma(R_{t+1}h)\sigma(m_{t+1})}{E_t[m_{t+1}]}$$

- NB: All results also hold for unconditional expectations $E[\cdot]$ 

- Rewritten in terms of Sharpe Ratio $= ...$

$$-\frac{\sigma(m_{t+1})}{E[m_{t+1}]}\rho(m_{t+1}, R_{t+1}h) = \frac{E[(R_{t+1} - R_{t+1}^f)h]}{\sigma(R_{t+1}h)}$$
Hansen-Jagannathan Bound

– Since $\rho \in [-1,1]$ we have

$$\frac{\sigma(m_{t+1})}{E[m_{t+1}]} \geq \sup_h \left[ \frac{E[(R_{t+1} - R^f_{t+1})h]}{\sigma(R_{t+1}h)} \right]$$

• **Theorem (Hansen-Jagannathan Bound):**
  The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
Hansen-Jagannathan Bound

- Theorem (Hansen-Jagannathan Bound): The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
  - Can be used to easily check the “viability” of a proposed discount factor
  - Given a discount factor, this inequality bounds the available risk-return possibilities
  - The result also holds conditional on date $t$ info
Hansen-Jagannathan Bound

\[
\begin{align*}
\text{expected return} & \uparrow \\
R_f & \quad \text{available portfolios} \\
\sigma & \downarrow \\
\end{align*}
\]

slope $\sigma (m) / E[m]$
Assuming Expected Utility

- \( c_0 \in \mathbb{R}, c_1 \in \mathbb{R}^S \)
- \( U(c_0, c_1) = \sum_s \pi_s u(c_0, c_{1,s})U(c_0, c_1) \)
  \[
  \partial_0 u = \left( \frac{\partial u(c_0^*, c_{1,1}^*)}{\partial c_0}, \ldots, \frac{\partial u(c_0^*, c_{1,S}^*)}{\partial c_0} \right)
  \]
  \[
  \partial_1 u = \left( \frac{\partial u(c_0^*, c_{1,1}^*)}{\partial c_{1,1}}, \ldots, \frac{\partial u(c_0^*, c_{1,S}^*)}{\partial c_{1,S}} \right)
  \]

- Stochastic discount factor
  \[
  m = \frac{\text{MRS}}{\pi} = \frac{\partial_1 u}{E[\partial_0 u]} \in \mathbb{R}^S
  \]
Time-Separable

- **Digression:** if utility is in addition time-separable
  \[ u(c_0, c_1) = v(c_0) + v(c_1) \]
- Then
  \[
  \partial_0 u = \left( \frac{\partial v(c_0^*)}{\partial c_0}, \ldots, \frac{\partial v(c_0^*)}{\partial c_0} \right),
  \partial_1 u = \left( \frac{\partial v(c_{1,1}^*)}{\partial c_{1,1}}, \ldots, \frac{\partial v(c_{1,S}^*)}{\partial c_{1,S}} \right)
  \]
- And
  \[
  m_s = \frac{1}{\pi_s} \frac{\pi_s v'(c_{1,s})}{v'(c_0)} = \frac{v'(c_{1,s})}{v'(c_0)}
  \]
A simple example

- $S = 2, \pi_1 = \frac{1}{2}$
- 3 securities with $x^1 = (1,0), x^2 = (1,0), x^3 = (1,1)$
- Let $m = \left(\frac{1}{2}, 1\right), \sigma = \frac{1}{4} = \sqrt{\frac{1}{2} \left(\frac{1}{2} - \frac{3}{4}\right)^2 + \frac{1}{2} \left(1 - \frac{3}{4}\right)^2}$
- Hence, $p^1 = \frac{1}{4}, p^2 = \frac{1}{2} = p^3 = \frac{3}{4}$ and
- $R^1 = (4,0), R^2 = (0,2), R^3 = \left(\frac{4}{3}, \frac{4}{3}\right)$
- $E[R^1] = 2, E[R^2] = 1, E[R^3] = \frac{4}{3}$
Example: Where does SDF come from?

• “Representative agent” with
  – Endowment: 1 in date 0, (2,1) in date 1
  – Utility $EU(c_0, c_1, c_2) = \sum_s \pi_s (\ln c_0 + \ln c_{1,s})$
  – i.e. $u(c_0, c_{1,s}) = \ln c_0 + \ln c_{1,s}$ (additive) time separable $u$-function

• $m = \frac{\partial_1 u(1,2,1)}{E[\partial_0 u(1,2,1)]} = \left( \frac{c_0}{c_{1,1}}, \frac{c_0}{c_{1,2}} \right) = \left( \frac{1}{2}, 1 \right) = \left( \frac{1}{2}, 1 \right)$
  since endowment=consumption
  – Low consumption states are high “m-states”
  – Risk-neutral probabilities combine true probabilities and marginal utilities.
Equity Premium Puzzle

- Recall $E[R^i] - R^f = -R^f \text{cov}[m, R^i]$
- Now: $E[R^j] - R^f = -\frac{R^f \text{COV}[\partial_1 u, R^j]}{E[\partial_0 u]}$
- Recall Hansen-Jaganathan bound

$$\frac{\sigma(m)}{E[m]} \geq \left| \frac{E[R - R^f]}{\sigma(R)} \right|; E[m] = \frac{1}{R^f}$$

$$\sigma(m) \geq \frac{1}{R^f} \left| \frac{E[R - R^f]}{\sigma(R)} \right|$$
Equity Premium Puzzle (ctd.)

\[
\sigma \left( \frac{\partial_1 u}{E[\partial_0 u]} \right) \geq \frac{1}{R^f} \left| \frac{E[R - R^f]}{\sigma(R)} \right|
\]

Equity Premium Puzzle
- high observed Sharpe ratio of stock market indices
- low volatility of consumption
- \( \Rightarrow \) (unrealistically) high level of risk aversion
Equity Premium or Low Risk-free Rate Puzzle?

• Suppose we allow for sufficiently high risk aversion s.t.
  \[ \sigma \left( \frac{\partial_1 u}{E[\partial_0 u]} \right) \geq \frac{1}{R^f} \left| \frac{E[R - R^f]}{\sigma(R)} \right| \]

• New problem emerges:
  – Strong force to consumption smooth over time (low intertemporal elasticity of consumption (IES)) due to concavity of utility function
  – vNM utility function
    • smoothing over states = smoothing over time
    • CRRA gamma = 1/ IES
  – Model predicts much higher risk-free rate
Equity Premium or Low Risk-free Rate Puzzle?

• Solution:
  – depart from vNM utility preference representation
  – Found preference representation that allows split
    • Risk-aversion
    • Intertemporal elasticity of substitution

• Kreps-Porteus (special case: vNM)
• Epstein-Zin (special case: CRRA vNM)
Digression: Preference for the timing of uncertainty resolution

Kreps-Porteus

\[ U_0(x_1, x_2(s)) = W(x_1, E[U_1(x_1, x_2(s))]) \]

Early (late) resolution if \( W(P_1, \ldots) \) is convex (concave)

Do you want to know whether you will get cancer at the age of 55 now?