LECTURE 11:
MULTI-PERIOD EQUILIBRIUM MODEL
FIN501 Asset Pricing
Lecture 11 Multi Period Equilibrium Model (2)

specify Preferences & Technology

observe/specify existing Asset Prices

- evolution of states
- risk preferences
- aggregation

NAC/LOOP

State Prices q (or stochastic discount factor/Martingale measure)

NAC/LOOP

Absolute asset pricing

Relative asset pricing

LOOP

derive Asset Prices

derive Price for (new) asset

Only works as long as market completeness doesn’t change
Time-varying $R^*_t$ (SDF)

- If one-period SDF $m_t$ is not time-varying (i.e. distribution of $m_t$ is i.i.d., then
  - Expectations hypothesis holds
  - Investment opportunity set does not vary
  - Corresponding R* of single factor state-price beta model can be easily estimate (because over time one more and more observations about R*)

- If not, then $m_t$ (or corresponding $R^*_t$)
  - depends on state variable
  - multiple factor model
\( R_t^* \) depends on State Variable

- \( R_t^* = R^*(z_t) \), with state variable \( z_t \)
- Example:
  - \( z_t = 1 \) or \( 2 \) with equal probability
  - Idea:
    - Take all periods with \( z_t = 1 \) and figure out \( R^*(1) \)
    - Take all periods with \( z_t = 2 \) and figure out \( R^*(2) \)
  - Can one do that?
    - No – hedge across state variables
- Potential state-variables: predict future return
Dynamic Hedging Demand

• Trade-off
  – Low return realization in next period
    • Good since opportunity going forward is high
      ➢ Invest more
    ➢ Bad since marginal utility is high
      ➢ Consume and invest less
  – High return realization in next period ....

• Utility
  – $\gamma > (\leq) 1$ first (second) effect dominates
  – $\gamma = 1$ (log-utility) both effects offset each other
(Dynamic) Hedging Demand

- Illustration with noise trader risk:
  - Suppose fundamental value is constant $v=1$, but price is noisy (due to noise traders)
  - If the asset is underpriced, e.g. $p=.9$, then it might be even more underpriced in the next period
    - Myopic risk-averse investor: buy some of the asset and push price towards 1, but not fully
    - Forward-looking risk-averse investor: yes, there can be intermediate losses if price declines in next period, but then **investment opportunity set** improves even more i.e. if returns are bad, then I have great opportunity (dynamic hedge)
Static problem = intertemporal problem

• In general ICAPM setting
  – CRRA with $\gamma \neq 1$ and changing investment opportunity sets

• Special cases
  1. CRRA and i.i.d. returns and constant $r^f$
     • SR and LR investors have the same portfolio weights.
     • Fraction of savings that is invested in asset $j$ is time-invariant (Merton 1971)
  2. Log utility and non-i.i.d. returns $\Rightarrow$ same result
Conditional vs. unconditional CAPM

• If $\beta$ of each subperiod CAPM are time-independent, then conditional CAPM = unconditional CAPM

• If $\beta$s are time-varying they may co-vary with $R_m$ and hence CAPM equation does not hold for unconditional expectations.
  - Additional co-variance terms have to be considered!
  - Move from single-factor setting to multi-factor setting
Intertemporal CAPM (ICAPM)

• Merton (1973)
• Bellman equation
  \[ V(W_t, z_t) = \max_c \{ u(c_t) + \delta E_t[V(W_{t+1}, z_{t+1})] \} \]
  – where \( W_{t+1} = R_{t+1}^W (W_t - c_t) \) with \( R_{t+1}^W \) for optimal portfolio
• FOC:
  – \( 0 = u'(c_t) - \delta E_t[V_W(W_{t+1}, z_{t+1})R_{t+1}^W] \)
  – Since \( V_W(W_t, z_t) = \delta E_t[V_W(W_{t+1}, z_{t+1})R_{t+1}^W] \), (envelope theorem)

\[ u'(c_t) = V_W(W_t, z_t) \quad \forall t \]
Deriving ICAPM

• Hence one period pricing equation

\[ E[R_{t+1}^j] - R_{t+1}^f = -\text{Cov}_t\left[ \frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_{t+1}^j \right] \]

• Becomes

\[ E[R_{t+1}^j] - R_{t+1}^f = -\frac{\text{Cov}_t[V_W(W_{t+1}, z_{t+1}), R_{t+1}^j]}{E_t[V_W(W_{t+1}, z_{t+1})]} \]
Deriving ICAPM: First order Approximation

• Around $V_W(W_t, z_t)$
  
  $V_W(W_{t+1}, z_{t+1})$
  $\approx V_W(W_t, z_t) + V_{WW}(W_t, z_t) \Delta W_{t+1} + V_{Wz}(W_t, z_t) \Delta z_t$

• One obtains

$$E[R_{t+1}^j] - R_{t+1}^f$$

$$= -\gamma \text{Cov}_t[\Delta W_{t+1}, R_{t+1}^j] + \frac{V_{Wz}}{E_t[V_W]} \text{Cov}_t[\Delta z_{t+1}, R_{t+1}^j]$$

– $\gamma$ is relative risk aversion coefficient of $V$
– Second term is additional “risk factor”
Approximate ICAPM Campbell

- CRRA Agent

\[ 1 = E_t \left[ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{j,t+1} \right] \]

- 2nd order Taylor Approximation, where

\[ V_{cc} \equiv \text{var}_t [\Delta c_{t+1}] = \text{var}[\Delta c_{t+1} - E_t \Delta c_{t+1}] \text{ etc.} \]

\[ 0 = \log \delta - \gamma E_t \Delta c_{t+1} + E_t r_{j,t+1} + \frac{1}{2} \left[ \gamma^2 V_{cc} + V_{jj} - 2\gamma V_{cj} \right] \]

- Note that this implies \( E_t [\Delta c_{t+1}] = \mu_m + \frac{1}{\gamma} E_t [r_{m,t+1}] \)

- With \( \mu_m = \frac{1}{\gamma} \log[\delta] + \frac{1}{2} \left[ \gamma V_{cc} + \frac{1}{\gamma V_{mm}} - 2V_{cm} \right] \)

- \[ = \frac{1}{\gamma} \log[\delta] + \frac{1}{2} \gamma \text{var}_t [\Delta c_{t+1} - \sigma r_{m,t+1}] \]
Consumption

- Budget constraint
  \[ W_{t+1} = R_{m,t+1}(W_t - C_t) \Rightarrow \frac{W_{t+1}}{W_t} = R_{m-t+1} \left( 1 - \frac{C_t}{W_t} \right) \]

- In logs
  \[ \Delta w_{t+1} = r_{m,t+1} + \log[1 - \exp(c_t - w_t)] \]

- 1st Order Taylor Approximation (1)
  \[ \log[1 - \exp[x_t]] \approx \log[1 - \exp[\bar{x}]] - \frac{\exp[\bar{x}]}{1 - \exp[\bar{x}]} (x_t - \bar{x}) \]
  \[ \Delta w_{t+1} \approx r_{m,t+1} + k + (1 - \rho^{-1})(c_t - w_t) \]

Where \( \rho \equiv 1 - \exp(\bar{c} - \bar{w}) \)
Consumption Innovations

• Since $\Delta w_{t+1} = \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1})$ (Simply an identity)

$$\Rightarrow c_t - w_t = \sum_{\tau=1}^{\infty} \rho^\tau (r_{m,\tau+j} - \Delta c_{t+\tau}) + \frac{\rho k}{1 - \rho}$$

• Taking expectations on both sides yields (2)

$$c_t - w_t = E_t \left[ \sum_{\tau=1}^{\infty} \rho^\tau (r_{m,\tau+j} - \Delta c_{t+\tau}) + \frac{\rho k}{1 - \rho} \right]$$

• Combining (1) and (2)

$$c_{t+1} - E_t c_{t+1}$$

$$= (E_{t+1} - E_t) \sum_{\tau=0}^{\infty} \rho^\tau r_{m,t+1+\tau} - (E_{t+1} - E_t) \sum_{\tau=1}^{\infty} \rho^\tau \Delta c_{t+1+\tau}$$
Consumption Innovations

• Combining previous result with \( E_t[\Delta c_{t+1}] = \mu_m + \frac{1}{\gamma} E_t[r_{m,t+1}] \)

\[
\Rightarrow c_{t+1} - E_t c_{t+1} = r_{m,t+1} - E_t r_{m,t+1} + \left(1 - \frac{1}{\gamma}\right) (E_{t+1} - E_t) \sum_{\tau=0}^{\infty} \rho^\tau r_{m,t+1+\tau}
\]

• Finally, this implies that \( V_{jc} = V_{jm} + \left(1 - \frac{1}{\gamma}\right) V_{jh} \)

– Here, we define \( V_{ih} = \text{cov}_t[r_{j,t+1}, (E_{t+1} - E_t) \sum_{\tau=0}^{\infty} \rho^\tau r_{m,t+1+\tau}] \)

– This is the covariance of the asset with a “hedge” portfolio
ICAPM

• For a risk-free asset the log-Euler equation simplifies to

\[ 0 = \log \delta - \gamma E_t \Delta c_{t+1} + r_{f,t+1} + \frac{1}{2} \gamma^2 V_{cc} \]

• Then we can write the Consumption CAPM

\[ E_t r_{j,t+1} - r_{f,t+1} = - \frac{V_{jj}}{2} + \gamma V_{jc} \]

• And finally the Intertemporal CAPM

\[ E_t r_{j,t+1} - r_{f,t+1} = - \frac{V_{jj}}{2} + \gamma V_{jm} + (\gamma - 1)V_{jh} \]