Asset Pricing under Asymmetric Information
Rational Expectations Equilibrium

Markus K. Brunnermeier

Princeton University

August 17, 2007
A Classification of Market Microstructure Models

• simultaneous submission of demand schedules
  • competitive rational expectation models
  • strategic share auctions

• sequential move models
  • screening models:
    (uninformed) market maker submits a supply schedule first
  • static
    ◦ uniform price setting
    ◦ limit order book analysis
  • dynamic sequential trade models with multiple trading rounds

• signalling models:
  informed traders move first, market maker second
Overview

- Competitive REE (Examples)
  - Preliminaries
    - LRT (HARA) utility functions in general
    - CARA Gaussian Setup
      - Certainty equivalence
      - Recall Projection Theorem/Updating
  - REE (Grossman 1976)
  - Noisy REE (Hellwig 1980)

- Allocative versus Informational Efficiency
- Endogenous Information Acquisition
Utility functions and Risk aversion

- utility functions $U(W)$.
- Risk tolerance, $1/\rho = \text{reciprocal of the Arrow-Pratt measure of absolute risk aversion}$

$$\rho(W) := -\frac{\partial^2 U/\partial W^2}{\partial U/\partial W}.$$ 

- Risk tolerance is linear in $W$ if

$$\frac{1}{\rho} = \alpha + \beta W.$$ 

- also called hyperbolic absolute risk aversion (HARA) utility functions.
### LRT(HARA)-Utility Functions

<table>
<thead>
<tr>
<th>Class</th>
<th>Parameters</th>
<th>$U(W) =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exponential/CARA</td>
<td>$\beta = 0, \alpha = 1/\rho$</td>
<td>$-\exp{-\rho W}$</td>
</tr>
<tr>
<td></td>
<td>$\beta \neq 1$</td>
<td></td>
</tr>
<tr>
<td>generalised power</td>
<td></td>
<td>$rac{1}{\beta-1}(\alpha + \beta W)^{(\beta-1)/\beta}$</td>
</tr>
<tr>
<td>a) quadratic</td>
<td>$\beta = -1, \alpha &gt; W$</td>
<td>$-(\alpha - W)^2$</td>
</tr>
<tr>
<td>b) log</td>
<td>$\beta = +1$</td>
<td>$\ln(\alpha + W)$</td>
</tr>
<tr>
<td>c) power/CRRA</td>
<td>$\alpha = 0, \beta \neq 1, -1$</td>
<td>$rac{1}{\beta-1}(\beta W)^{(\beta-1)/\beta}$</td>
</tr>
</tbody>
</table>
Certainty Equivalent in CARA-Gaussian Setup

\[ U(W) = -\exp(-\rho W), \text{ hence } \rho = -\frac{\partial^2 U(W) / \partial (W)^2}{\partial U(W) / \partial W} \]

\[ E[U(W) | \cdot] = \int_{-\infty}^{+\infty} -\exp(-\rho W)f(W|\cdot)dW \]

where \( f(W|\cdot) = \frac{1}{\sqrt{2\pi\sigma^2_W}} \exp\left[-\frac{(W - \mu_W)^2}{2\sigma^2_W}\right] \)

Substituting it in

\[ E[U(W) | \cdot] = \frac{1}{\sqrt{2\pi\sigma^2_W}} \int_{-\infty}^{+\infty} -\exp\left(-\frac{\rho z}{2\sigma^2_W}\right)dW \]

where \( z = (W - \mu_W)^2 - 2\rho\sigma^2_W W \)
Certainty Equivalent in CARA-Gaussian Setup

Completing squares \( z = (W - \mu_W - \rho \sigma^2_W)^2 - 2\rho(\mu_W - \frac{1}{2}\rho \sigma^2_W)\sigma^2_W \)

Hence, \( E[U(W) \mid \cdot] = - \exp[-\rho(\mu_W - \frac{1}{2}\rho \sigma^2_W)] \times \)

\[
\times \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2_W}} \exp\left(-\frac{(W - \mu_W - \rho \sigma^2_W)^2}{2\sigma^2_W}\right) dW
\]

Trade-off is represented by

\[
V(\mu_W, \sigma^2_W) = \mu - \frac{1}{2}\rho \sigma^2_W
\]
Certainty Equivalent in CARA-Gaussian Setup
More generally, multinomial random variables \( w \sim \mathcal{N}(0, \Sigma) \) with a positive definite (co)variance matrix \( \Sigma \). More specifically,
\[
E[\exp(w^T A w + b^T w + d)] =
\]
\[
= |I - 2\Sigma A|^{-1/2} \exp\left[\frac{1}{2} b^T (I - 2\Sigma A)^{-1} \Sigma b + d\right],
\]
where
- \( A \) is a symmetric \( m \times m \) matrix,
- \( b \) is an \( m \)-vector, and
- \( d \) is a scalar.
Note that the left-hand side is only well-defined if \((I - 2\Sigma A)\) is positive definite.
### Demand for a Risky Asset

- **2 assets**
  - asset
    - bond (numeraire)
    - stock
  - payoff
    - \( v \sim \mathcal{N}(E[v|\cdot], \text{Var}[v|\cdot]) \)
  - endowment
    - \( e_0^i \)
    - \( z^i \)

- **Equation:**
  \[
  P x^i + b^i = P z^i + e_0^i
  \]

- **Final wealth is**
  \[
  W^i = b^i R + x^i v = \left( e_0^i + P(z^i - x^i) \right) R + x^i v
  \]
  - **Mean:** \( (e_0^i + P(z^i - x^i)) R + x E[v|\cdot] \)
  - **Variance:** \( (x^i)^2 \text{Var}[v|\cdot] \)
Demand for a Risky Asset

\[ V(\mu_W, \sigma^2_W) = \mu_W - \frac{1}{2}\rho^i \sigma^2_W \]  
\[ = (e^i_0 + Pz^i)R + x^i(E[v|\cdot] - PR) - \frac{1}{2}\rho^i \text{Var}[v|\cdot](x^i)^2 \]

First order condition: \( E[v|\cdot] - PR - \rho \text{Var}[v|\cdot]x^i = 0 \)

\[ x^i(P) = \frac{E[v|\cdot] - PR}{\rho^i \text{Var}[v|\cdot]} \]

Remarks

- independent of initial endowment (CARA)
- linearly increasing in investor’s expected excess return
- decreasing in investors’ variance of the payoff \( \text{Var}[v|\cdot] \)
- decreasing in investors’ risk aversion \( \rho^i \)
- for \( \rho^i \rightarrow 0 \) investors are risk-neutral and \( x^i \rightarrow +\infty \) or \(-\infty\)
Symmetric Info - Benchmark Model setup:

- \( i \in \{1, \ldots, I\} \) (types of) traders
- CARA utility function with risk aversion coefficient \( \rho^i \)
  (Let \( \eta^i = \frac{1}{\rho^i} \) be trader \( i \)'s risk tolerance.)
- all traders have the same information \( v \sim \mathcal{N}(\mu, \sigma_v^2) \)
- aggregate demand \( \sum_i \frac{E[v] - PR}{\rho^i \text{Var}[v]} = \sum_i \eta^i \tau_v \{ E[v] - PR \} \)
  
  Let \( \eta := \frac{1}{I} \sum_i \eta^i = \frac{1}{I} \sum_i \frac{1}{\rho^i} \) (harmonic mean)
- market clearing \( \eta \tau_v \{ E[v] - PR \} = X^{\text{supply}} \)

\[
P = \frac{1}{R} \left\{ E[v] - \frac{X^{\text{sup}}}{I \eta \tau_v} \right\}
\]

The expected excess payoff \( Q := E[v] - PR = \frac{1}{\eta \tau_v} \frac{X^{\text{sup}}}{I} \)
Symmetric Info - Benchmark

- Trader $i$'s equilibrium demand is

$$x^i(P) = \frac{\eta^i}{\eta} \frac{X^{\text{sup}}}{I}$$

- Remarks:
  - $\frac{\partial P}{\partial E[v]} = \frac{1}{R} > 0$
  - $\frac{\eta^i}{\eta}$ risk sharing of aggregate endowment

$$\frac{x^{i^*}}{x^{i''^*}} = \frac{\eta^i}{\eta^{i''}}$$

- no endowment effects
REE - Grossman (1976) Model setup:

- $i \in \{1, \ldots, I\}$ traders
- CARA utility function with risk aversion coefficient $\rho^i = \rho$
  (Let $\eta^i = \frac{1}{\rho^i}$ be trader $i$’s risk tolerance.)
- Information is dispersed among traders
  trader $i$’s signal is $S^i = \nu + \epsilon^i_S$, where $\epsilon^i_S \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2_\epsilon)$
REE - Grossman (1976)

Step 1: Conjecture price function

\[ P = \alpha_0 + \alpha_S \bar{S}, \text{ where } \bar{S} = \frac{1}{I} \sum_{i} S^i \text{ (sufficient statistics)} \]

Step 2: Derive posterior distribution

\[ E[v|S^i, P] = E[v|\bar{S}] = \lambda E[v] + (1 - \lambda) \bar{S} = \lambda E[v] + (1 - \lambda) \frac{P - \alpha_0}{\alpha_S} \]

\[ \text{Var}[v|S^i, P] = \text{Var}[v|\bar{S}] = \lambda \text{Var}[v] \]

where \( \lambda := \frac{\text{Var}[\epsilon]}{I\text{Var}[v] + \text{Var}[\epsilon]} \)

Step 3: Derive individual demand

\[ x^{i,*}(P) = \frac{E[v|S^i, P] - P(1 + r)}{\rho^i \text{Var}[v|S^i, P]} \]

Step 4: Impose market clearing

\[ \sum_{i} x^{i,*}(P) = X^{\text{supply}} \]
### Informational (Market) Efficiency

- **Empirical Literature**
  - **Form**
  - **Price reflects**
    - *strong* all private and public information
    - *semi strong* all public information
    - *weak* only (past) price information

- **Theoretical Literature**
  - **Form**
  - **Price aggregates/reveals**
    - *fully revealing* all private signals
    - *informational efficient* sufficient statistic of signals
    - *partially revealing* a noisy signal of pooled private info
    - *privately revealing* with one signal reveals suff. stat.
Informational (Market) Efficiency

- $\overline{S}$ sufficient statistic for all individual info sets $\{S^1, ..., S^I\}$.
- Illustration: If one can view price function as
  \[ P(\cdot) : \{S^1, ..., S^I\} \xrightarrow{g(\cdot)} \overline{S} \xrightarrow{f(\cdot)} P \]

- if $f(\overline{S})$ is invertible, then price is informationally efficient
- if $f(\cdot)$ and $g(\cdot)$ are invertible, then price is fully revealing
Remarks & Paradoxa

• Grossman (1976) solved it via “full communication equilibria” (Radner 1979’s terminology)
• ‘unique’ info efficient equilibrium (DeMarzo & Skiadas 1998)
• As $I \to \infty$ (risk-bearing capacity), $P \to \frac{1}{R} E[v]$
• **Grossman Paradox:**
  Individual demand does not depend on individual signal $S^i$'s. How can all information be reflected in the price?
• **Grossman-Stiglitz Paradox:**
  Nobody has an incentive to collect information?
• individual demand is independent of wealth (CARA)
• in equilibrium individual demand is independent of price
• equilibrium is not implementable
Noisy REE - Hellwig 1980

Model setup:

- \( i \in \{1, ..., I\} \) traders
- CARA utility function with risk aversion coefficient \( \rho^i = \rho \)
  (Let \( \eta^i = \frac{1}{\rho^i} \) be trader \( i \)'s risk tolerance.)
- information is dispersed among traders
  trader \( i \)'s signal is \( S^i = \nu + \epsilon^i_S \), where \( \epsilon^i_S \sim^{ind} \mathcal{N}(0, (\sigma^i_\epsilon)^2) \)
- noisy asset supply \( X^{Supply} = u \)
- Let \( \Delta S^i = S^i - E[S^i] \), \( \Delta u = u - E[u] \) etc.
Noisy REE - Hellwig (1980)

**Step 1: Conjecture price function**

\[ P = \alpha_0 + \sum_{i} \alpha_i S^i \Delta S^i + \alpha_u \Delta u \]

**Step 2: Derive posterior distribution** let’s do it only half way through

\[ E[v|S^i, P] = E[v] + \beta_S^i (\alpha) \Delta S^i + \beta_P (\alpha) \Delta P \]

\[ \text{Var}[v|S^i, P] = \frac{1}{\tau^i_{v|S^i, P}} \quad \text{(independent of signal realization)} \]

**Step 3: Derive individual demand**

\[ x^{i,*}(P) = \eta^i \tau^i_{v|S^i, P} \{ E[v|S^i, P] - P(1 + r) \} \]
Noisy REE - Hellwig (1980)

**Step 4: Impose market clearing**

Total demand = total supply (let $r = 0$)

$$
\sum_{i} \eta^{i} \tau^{i}_{\{v|S^{i},P\}}(\alpha) \{ E[v] + \beta^{i}_{S}(\alpha) \Delta S^{i} - \alpha_{0} \beta^{i}_{P}(\alpha) + [\beta^{i}_{P}(\alpha) - 1] P \} = u
$$

... 

$$
P (S^{1}, ..., S^{l}, u) = \frac{\sum_{i} \left( \eta^{i} \tau^{i}_{\{v|S^{i},P\}}(\alpha) \right) \left[ E[v] - \alpha_{0} \beta^{i}_{P}(\alpha) + \beta^{i}_{S}(\alpha) \Delta S^{i} \right] - E[u] - \Delta u}{\sum_{i} \left( 1 - \beta^{i}_{P}(\alpha) \right) \eta^{i} \tau^{i}_{\{v|S^{i},P\}}(\alpha)}
$$
Noisy REE - Hellwig (1980)

**Step 5: Impose rationality**

\[
\begin{align*}
\alpha_0 &= \frac{\sum_i \left( \eta^i \tau^i_{[v|S^i,P]} (\alpha) \right) \left[ E[v] - \alpha_0 \beta^i_p (\alpha) \right] - E[u]}{\sum_i \left( 1 - \beta^i_p (\alpha) \right) \eta^i \tau^i_{[v|S^i,P]} (\alpha)} \\
\alpha^i_s &= \frac{\sum_i \left( \eta^i \tau^i_{[v|S^i,P]} (\alpha) \right)}{\sum_i \left( 1 - \beta^i_p (\alpha) \right) \eta^i \tau^i_{[v|S^i,P]} (\alpha)} \beta^i_s (\alpha) \\
\alpha^i_u &= \frac{-1}{\sum_i \left( 1 - \beta^i_p (\alpha) \right) \eta^i \tau^i_{[v|S^i,P]} (\alpha)} \\
\end{align*}
\]

Solve for root \( \alpha^* \) of the problem \( \alpha = G(\alpha) \).
Noisy REE - Hellwig 1980

Simplify model setup:

- All traders have identical risk aversion coefficient $\rho = 1/\eta$
- Error of all traders’ signals $\epsilon^i_S$ are i.i.d.

**Step 1:** Conjecture price function simplifies to

$$\Delta P = \alpha_S \sum_{i} \frac{1}{l} \Delta S^i + \alpha_u \Delta u$$

**Step 2:** Derive posterior distribution

$$E[v|S^i, P] = E[v] + \beta_S(\alpha) \Delta S^i + \beta_P(\alpha) \Delta P$$

$$\text{Var}[v|S^i, P] = \frac{1}{\tau} \quad \text{(independent of signal realization)}$$

where $\beta$’s are projection coefficients.
Noisy REE - Hellwig (1980)

previous fixed point system simplifies to

\[
\alpha_S = \frac{1}{\sum_i (1 - \beta_P(\alpha))} \beta_S(\alpha)
\]

\[
\alpha_u = \frac{-1}{\eta\tau(\alpha) \sum_i (1 - \beta_P(\alpha))}
\]

To determine \(\beta_S\) and \(\beta_P\), invert Co-variance matrix

\[
\Sigma(S^i, P) = \begin{pmatrix}
\sigma_v^2 + \sigma_\varepsilon^2 & \alpha_S(\sigma_v^2 + \frac{1}{l}\sigma_\varepsilon^2) \\
\alpha_S(\sigma_v^2 + \frac{1}{l}\sigma_\varepsilon^2) & \alpha_S^2(\sigma_v^2 + \frac{1}{l}\sigma_\varepsilon^2) + \alpha_u^2\sigma_u^2
\end{pmatrix}
\]

\[
\Sigma^{-1}(S^i, P) = \frac{1}{D} \begin{pmatrix}
\alpha_S^2(\sigma_v^2 + \frac{1}{l}\sigma_\varepsilon^2) + \alpha_u^2\sigma_u^2 & -\alpha_S(\sigma_v^2 + \frac{1}{l}\sigma_\varepsilon^2) \\
-\alpha_S(\sigma_v^2 + \frac{1}{l}\sigma_\varepsilon^2) & \sigma_v^2 + \sigma_\varepsilon^2
\end{pmatrix}
\]

\[
D = \alpha_S^2 \frac{l-1}{l} (\sigma_v^2 + \frac{1}{l}\sigma_\varepsilon^2) \sigma_\varepsilon^2 + \alpha_u^2\sigma_u^2(\sigma_v^2 + \sigma_\varepsilon^2)
\]
Noisy REE - Hellwig (1980)
Since \( \text{Cov} [v, P] = \alpha_S \sigma_v^2 \) and \( \text{Cov} [v, S^i] = \sigma_v^2 \) leads us to

\[
\beta_P = \frac{1}{D} \alpha_S \frac{I - 1}{I} \sigma_v^2 \sigma_\varepsilon^2
\]

\[
\beta_S = \frac{1}{D} \alpha_u \sigma_u^2 \sigma_v^2
\]

For conditional variance (precision) from projection theorem.

\[
\text{Var} [v | S^i, P] = \frac{1}{D} \left[ D \sigma_v^2 - \left( \alpha_u^2 \sigma_u^2 + \alpha_S^2 \frac{I - 1}{I} \sigma_\varepsilon^2 \right) \sigma_v^4 \right]
\]

\[
= \frac{1}{D} \left[ \alpha_S^2 \frac{I - 1}{I^2} \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \right] \left( \sigma_\varepsilon^2 \right) \sigma_v^2
\]

Hence,
Noisy REE - Hellwig (1980)

\[ \alpha_S = \frac{\alpha_u^2 \sigma_v^2 \sigma_u^2}{(D - \alpha_s \frac{l-1}{l} \sigma_\epsilon^2 \sigma_v^2)l} \]

\[ \alpha_u = -\rho \frac{\left( \alpha_u^2 \sigma_u^2 + \alpha_s^2 \frac{l-1}{l^2} \sigma_\epsilon^2 \right) \sigma_\epsilon^2 \sigma_v^2}{(D - \alpha_s \frac{l-1}{l} \sigma_\epsilon^2 \sigma_v^2)l} \]

Trick:

Solve for \( h = -\frac{\alpha_u}{\alpha_S} \). (Recall price signal can be rewritten as \( \frac{P - \alpha_0}{\alpha_S} = \sum_i \frac{1}{i} S + \frac{\alpha_u}{\alpha_S} u \).) [noise signal ratio]

\[ h = \frac{\rho \left( h^2 \sigma_u^2 + \frac{l-1}{l^2} \sigma_\epsilon^2 \right) \sigma_\epsilon^2 \sigma_v^2}{h^2 \sigma_v^2 \sigma_u^2} \]

\[ h = \rho \sigma_\epsilon^2 + \frac{1}{h^2} \frac{(l - 1) \sigma_\epsilon^4}{l^2 \sigma_u^2} \]

\( \Rightarrow \) unique \( h > \rho \sigma_\epsilon^2 \)

Increasing in \( h \)

Decreasing in \( h \)
Noisy REE - Hellwig (1980)

Remember that $h$ is increasing in $\rho$.

Back to $\alpha_S$

$$\alpha_S = \frac{\alpha^2_u \sigma^2_v \sigma^2_u}{D - \alpha_s \frac{l-1}{l} \sigma^2_{\varepsilon} \sigma^2_v}$$

multiply by denominator

$$\alpha_S D = \alpha^2_u \sigma^2_v \sigma^2_u + \alpha^2_S \frac{l-1}{l} \sigma^2_{\varepsilon} \sigma^2_v \iff \alpha_S =$$

$$\frac{1}{D} \left[ \alpha^2_u \sigma^2_v \sigma^2_u + \alpha^2_S \frac{l-1}{l} \sigma^2_{\varepsilon} \sigma^2_v \right]$$

Sub in $D = ...$

$$\alpha_S = \frac{\frac{\alpha^2_u}{\alpha^2_S} \sigma^2_v \sigma^2_u + \frac{l-1}{l} \sigma^2_{\varepsilon} \sigma^2_v}{\frac{l-1}{l} \left( \sigma^2_v + \frac{1}{l} \sigma^2_{\varepsilon} \right)\sigma^2_{\varepsilon} + \frac{\alpha^2_u}{\alpha^2_S} \sigma^2_u \left( \sigma^2_v + \sigma^2_{\varepsilon} \right)}$$

$\Rightarrow$ unique $\alpha_S$.

This proves existence and uniqueness of the NREE!
Characterization of NREE

Recall that \( Var \left[ v \mid S^i, P \right] = \frac{1}{D} \left[ \alpha_S^2 \frac{l-1}{l^2} \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_\varepsilon^2 \sigma_v^2 \)

and \( \alpha_S = \frac{1}{D} \left[ \alpha_u^2 \sigma_u^2 + \alpha_S^2 \frac{l-1}{l} \sigma_\varepsilon^2 \right] \sigma_v^2 \)

Hence, \( \alpha_S = Var \left[ v \mid S^i, P \right] \frac{\left[ \alpha_u^2 \sigma_u^2 + \alpha_S^2 \frac{l-1}{l} \sigma_\varepsilon^2 \right]}{\left[ \alpha_S^2 \frac{l-1}{l^2} \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_v^2} \) (notice \( l^2 \) square)

\[ \alpha_S = Var \left[ v \mid S^i, P \right] \frac{1}{\sigma_\varepsilon^2} \left[ \frac{l^2}{l-1} h^2 \sigma_u^2 + l \sigma_\varepsilon^2 \right] \]

\[ Var \left[ v \mid S^i, P \right] \frac{1}{\sigma_\varepsilon^2} \left[ \frac{l^2}{l-1} h^2 \sigma_u^2 + \sigma_\varepsilon^2 + (l-1) \sigma_\varepsilon^2 \right] \]

\[ \alpha_S = Var \left[ v \mid S^i, P \right] \frac{1}{\sigma_\varepsilon^2} \left[ 1 + \frac{(l-1) \sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \frac{l^2}{l-1} h^2 \sigma_u^2} \right] \]

\[ = Var \left[ v \mid S^i, P \right] \tau_\varepsilon \left[ 1 + (l - 1) \frac{\tau_u}{\tau_u + \frac{l^2}{l-1} h^2 \tau_\varepsilon} \right] \]

\[ \underbrace{\tau_u + \frac{l^2}{l-1} h^2 \tau_\varepsilon} \]

\[ := \theta \]

\[ \alpha_S = Var \left[ v \mid S^i, P \right] \tau_\varepsilon \left[ 1 + \theta \right] \theta \text{ is decreasing in } \rho \ (h \text{ is increasing)} \]
Characterization of NREE

$$\text{Var} \left[ \nu \mid S^i, P \right] = \frac{1}{D} \left[ \alpha_S^2 \frac{l-1}{l^2} \sigma_e^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_e^2 \sigma_v^2 = \frac{\alpha_S^2 \frac{l-1}{l^2} \sigma_e^2 + \alpha_u^2 \sigma_u^2}{\alpha_S^2 \frac{l-1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma_e^2 \right) \sigma_e^2 + \alpha_u^2 \sigma_u^2 \left( \sigma_v^2 + \sigma_e^2 \right)} = \frac{\frac{l-1}{l^2} \sigma_e^2 + h^2 \sigma_u^2}{h^2 \frac{l-1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma_e^2 \right) \sigma_e^2 + h^2 \left( \sigma_v^2 + \sigma_e^2 \right)} = \ldots$$

“price precision”

$$\frac{1}{\text{Var} \left[ \nu \mid S^i, P \right]} = \tau_v + \tau_\epsilon + (l - 1) \theta \tau_\epsilon$$

**Interpretation**

$$\theta = (l - 1) \frac{\tau_u}{\tau_u + \frac{\tau_u^2}{l-1} h^2 \tau_\epsilon} \quad \text{measure of info efficiency}$$

$$\sigma_u^2 \to \infty \quad (\tau_u \to 0): \, \theta \to 0 \text{ price is uninformative (Walrasian equ.)}$$

$$\sigma_u^2 \to 0 \quad (\tau_u \to \infty): \, \theta \to 1 \text{ price is informationally efficient}$$
Remarks to Hellwig (1980)

- Since $\alpha^2_u \neq 0$, $\beta_S \neq 0$, i.e. agents condition on their signal as risk aversion of trader increases the informativeness of price $\theta$ declines.
- Price informativeness increases in precision of signal $\tau_\varepsilon$ and declines in the amount of noise trading $\sigma^2_u$.
- Negative supply shock leads to a larger price increase compared to a Walrasian equilibrium, since traders wrongly partially attribute it to a good realization of $v$.
- Diamond and Verrecchia (1981) is similar except that endowment shocks of traders serve as asymmetric information.
Endogenous Info Acquisition
Model setup:

- $i \in \{1, \ldots, I\}$ traders
- CARA utility function with risk aversion coefficient $\rho$
  (Let $\eta = \frac{1}{\rho}$ be traders’ risk tolerance.)
- no information aggregation - two groups of traders
  - informed traders who have the same signal $S = v + \epsilon_S$
    with $\epsilon_S \sim \mathcal{N}(0, \sigma^2_{\epsilon})$
  - uninformed traders have no signal
- FOCUS on information acquisition
Noisy REE - Grossman-Stiglitz

**Step 1: Conjecture price function**

\[ P = \alpha_0 + \alpha_S \Delta S + \alpha_u \Delta u \]

**Step 2: Derive posterior distribution**

- for informed traders:
  \[ E[v|S, P] = E[v|S] = E[v] + \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \Delta S \]
  \[ \tau[v|S] = \tau_v + \tau_\varepsilon \]

- for uninformed traders:
  \[ E[v|P] = E[v] + \frac{\alpha_S \sigma_v^2}{\alpha_S^2 (\sigma_v^2 + \sigma_\varepsilon^2) + \alpha_u^2 \sigma_\varepsilon^2} \Delta P \]
  \[ \tau[v|P] = \tau_v + \frac{\tau_u}{\tau_u + h^2 \tau_\varepsilon} \tau_\varepsilon \]

where \( h = -\frac{\alpha_u}{\alpha_S} \)

\( := \phi \in [0,1] \)

After some algebra we get

\[ E[v|P] = E[v] + \frac{1}{\alpha_S} \frac{\phi \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon} \Delta P \]
Noisy REE - Grossman-Stiglitz

**Step 3: Derive individual demand**

\[
x^I(P, S) = \eta^I [\tau_v + \tau_\epsilon] \left[ E[v] + \frac{\tau_\epsilon}{\tau_v + \tau_\epsilon} \Delta S - P \right]
\]

\[
x^U(P) = \eta^U [\tau_v + \phi \tau_\epsilon] \left[ E[v] + \frac{1}{\alpha_S} \frac{\phi \tau_\epsilon}{\tau_v + \phi \tau_\epsilon} \Delta P - P \right]
\]

**Step 4: Impose market clearing**

Aggregate demand, for a mass of \(\lambda^I\) informed traders and \((1 - \lambda^I)\) uninformed

\[
\begin{align*}
\lambda^I \eta^I [\tau_v + \tau_\epsilon] \left[ E[v] + \frac{\tau_\epsilon}{\tau_v + \tau_\epsilon} \Delta S - P \right] + \\
\left(1 - \lambda^I\right) \eta^U [\tau_v + \phi \tau_\epsilon] \left[ E[v] + \frac{1}{\alpha_S} \frac{\phi \tau_\epsilon}{\tau_v + \phi \tau_\epsilon} \Delta P - P \right] &= u
\end{align*}
\]
Noisy REE - Grossman-Stiglitz

\[ P(S, u) = \frac{(\nu^l + \nu^U) E[v] + \nu^l \frac{\tau_{\varepsilon}}{\tau_v + \tau_{\varepsilon}} \Delta S - \frac{1}{\alpha_S} \frac{\Phi \tau_{\varepsilon}}{\tau_v - \phi \tau_{\varepsilon}} \alpha_0 \nu^U - E[u] - \Delta u}{\nu U \left( 1 - \frac{1}{\alpha_S} \frac{\Phi \tau_{\varepsilon}}{\tau_v - \phi \tau_{\varepsilon}} \right) + \nu^l} \]

Hence, \( h = -\frac{\alpha_u}{\alpha_S} = \left[ \nu^l \frac{\tau_{\varepsilon}}{\tau_v + \tau_{\varepsilon}} \right]^{-1} = \frac{1}{\lambda^l \eta^l \tau_{\varepsilon}}. \)

Hence, \( \phi = \frac{\tau_u \tau_{\varepsilon}}{\tau_v \tau_{\varepsilon} + \frac{1}{(\lambda^l \eta^l)^2}}. \)

Remarks:

- As \( \text{Var}[u] \downarrow 0, \phi \uparrow 1 \)
- If signal is more precise (\( \tau_{\varepsilon} \) is increasing) then \( \phi \) increases (since informed traders are more aggressive)
- Increases in \( \lambda^l \) and \( \eta^l \) also increase \( \phi \)
Noisy REE - Grossman-Stiglitz

**Step 5: Impose rationality**

Solve for coefficients

\[
\alpha_0 = E[v] - \frac{1}{\nu^I + \nu^U} E[u]
\]

\[
\alpha_S = \frac{1}{\nu^U \left(1 - \frac{1}{\alpha_S \frac{\phi \tau_\epsilon}{\tau_V - \phi \tau_\epsilon}}\right)} + \nu^I \frac{\tau_\epsilon}{\tau_V + \tau_\epsilon} \nu^I = \frac{\lambda^I \eta^I + \lambda^U \eta^U \phi}{\nu^I + \nu^U} \tau_\epsilon
\]

\[
\alpha_U = -\frac{1}{\nu^I + \nu^U} \left(1 + \frac{\lambda^U \tau_U}{\lambda^I \tau_I} \phi\right)
\]

Finally let’s calculate

\[
\frac{\tau[v|S]}{\tau[v|P]} = \frac{\tau_V + \tau_\epsilon}{\tau_V + \phi \tau_\epsilon} = 1 + \frac{(1 - \phi) \tau_\epsilon}{\tau_V + \phi \tau_\epsilon}
\]
Information Acquisition Stage - Grossman-Stiglitz (1980)

- Aim: endogenize $\lambda$

- Recall

\[ x^i = \eta^i \tau_{Q|S} E[Q|S], \text{ where } Q = \nu - RP \text{ is excess payoff} \]

- Final wealth is

\[ W^i = \eta^i Q \tau_{Q|S} E[Q|S] + (P u^i + e^i_0) R \]

(CARA $\Rightarrow$ we can ignore second term)

Note $W^i$ is product of two normally distributed variables

Use Formula of Slide 7 or follow following steps:

Conditional on $S$, wealth is normally distributed.

\[
E[W|S] = \eta \tau_{Q|S} E[Q|S]^2
\]

\[
Var[W|S] = \eta^2 \tau_{Q|S} E[Q|S]^2
\]

- the expected utility conditional on $S$

\[
E[U(W)|S] = -\exp\left\{-\frac{1}{\eta} \frac{1}{2} \eta \tau_{Q|S} E[Q|S]^2 - \frac{1}{2} \eta \tau_{Q|S} E[Q|S]^2 \right\}
\]
Information Acquisition Stage - Grossman-Stiglitz (1980)

\[ E[U(W)|S] = -\exp\left\{-\frac{1}{2}\tau_{Q|S}E[Q|S]^2\right\} \]

Integrate over possible \( S \) to get the ex-ante utility.

W.l.o.g. we can assume that \( S = Q + \epsilon \).

Normal density \( \phi(S) = \sqrt{\frac{\tau}{2\pi}} \exp\left\{-\frac{1}{2}\tau S[\Delta S]^2\right\} \)

\[
E[U(W)] = -\int_S \sqrt{\frac{\tau[S]}{2\pi}} \exp\left\{-\frac{1}{2}\left[\tau_{Q|S}E[Q|S]^2 + \tau S (\Delta S)^2\right]\right\} \text{d}S
\]

Term in square bracket is

\[
\left(\tau_Q + \tau_\epsilon\right) \left(E[Q] + \frac{\tau_\epsilon}{\tau_Q + \tau_\epsilon} \Delta S\right)^2 + \frac{\tau_Q\tau_\epsilon}{\tau_Q + \tau_\epsilon} (\Delta S)^2
\]

simplifies to

\[
\tau_Q E[Q]^2 + \tau_\epsilon (\Delta S + E[Q])^2
\]
Information Acquisition Stage - Grossman-Stiglitz (1980)

Hence, \( E [U(W)] = 
- \exp \left\{ - \frac{\tau Q E[Q]^2}{2} \right\} \int_S \sqrt{\frac{\tau S}{2\pi}} \exp\left\{ - \frac{1}{2} \left[ \tau \epsilon \left( \Delta S + E[Q]\right)^2 \right] \right\} ds 
\]

Define \( y := \sqrt{\tau \epsilon} (\Delta S + E[Q]) \)

\[
E [U(W)] = - \exp \left\{ - \frac{\tau Q E[Q]^2}{2} \right\} \sqrt{\frac{\tau S}{\tau \epsilon}} \int_S \sqrt{\frac{\tau \epsilon}{2\pi}} \exp\left\{ - \frac{1}{2} y^2 \right\} ds 
\]

Letting \( k = - \exp \left\{ - \frac{\tau Q E[Q]^2}{2} \right\} \sqrt{\tau Q} \) and noting that \( \tau S = \frac{\tau Q \tau \epsilon}{\tau Q + \tau \epsilon} \), we have

\[
E [U(W)] = \frac{k}{\sqrt{\tau [Q|S]}} = \frac{k}{\sqrt{\tau Q + \tau \epsilon}}
\]
Willingness to Pay for Signal

General Problem (No Price Signal)

- Without price signal and signal $S$, agent's expected utility

$$E[U(W)] = \frac{k}{\sqrt{\tau Q}}$$

- If the agent buys a signal at a price of $m_S$ his expected utility is

$$E[U(W - m_S)] = E[- \exp(-\rho(W - m_S))] = E[- \exp(-\rho(W)) \exp(\rho m_S)] = \frac{k}{\sqrt{\tau[Q|S]}} \exp(\rho m_S)$$

- Agent is indifferent when

$$\frac{k}{\sqrt{\tau Q}} = \frac{k}{\sqrt{\tau[Q|S]}} \exp(\rho m_S)$$

$$\Rightarrow$$ willingness to pay

$$m_S = \eta \ln \left( \sqrt{\frac{\tau[Q|S]}{\tau Q}} \right)$$

- Willingness to pay depends on the improvement in precision.
Information Acquisition Stage - Grossman-Stiglitz (1980)

- Every agent has to be indifferent between being informed or not.

\[
\text{cost of signal } c = \eta \ln \left( \sqrt{\frac{\tau[v|S]}{\tau[v|P]}} \right) = \eta \ln \left( \sqrt{\frac{\tau v + \tau \varepsilon}{\tau v + \phi \tau \varepsilon}} \right)
\]

(previous slide)

This determines \( \phi = \frac{\tau_u \tau \varepsilon}{\tau_u \tau \varepsilon + \left( \frac{1}{\lambda I \eta} \right)} \), which in turn pins down \( \lambda^I \).

- Comparative Statics (using IFT)
  - \( c \uparrow \Rightarrow \phi \downarrow \)
  - \( \eta \uparrow \Rightarrow \phi \uparrow \) (extreme case: risk-neutrality)
  - \( \tau \varepsilon \uparrow \Rightarrow \phi \uparrow \)
  - \( \sigma_u^2 \uparrow \Rightarrow \phi \uparrow \) (number of informed traders \( \uparrow \))
  - \( \sigma_u^2 \downarrow 0 \Rightarrow \) no investor purchases a signal
Information Acquisition Stage

- Further extensions:
  - purchase signals with different precisions (Verrecchia 1982)
  - Optimal sale of information
    - photocopied (newsletter) or individualistic signal (Admati & Pfleiderer)
    - indirect versus direct (Admati & Pfleiderer)
Endogenizing Noise Trader Demand

- endowment shocks or outside opportunity shocks that are correlated with asset
- welfare analysis
  - more private information $\rightarrow$ adverse selection
  - more public information $\rightarrow$ Hirshleifer effect (e.g. genetic testing)
- see papers by Spiegel, Bhattacharya & Rohit, and Vives (2006)
Tips & Tricks

- risk-neutral competitive fringe observing limit order book $L$
  $p = E[v | L(\cdot)]$
  - separates risk-sharing from informational aspects