LECTURE 4: RISK PREFERENCES & EXPECTED UTILITY THEORY
specify Preferences & Technology

- evolution of states
- risk preferences
- aggregation

State Prices \( q \)
(or stochastic discount factor/Martingale measure)

derive Asset Prices

Only works as long as market completeness doesn’t change

observe/specify existing Asset Prices

NAC/LOOP

derive Price for (new) asset

relative asset pricing

LOOP

absolute asset pricing

FRM01 Asset Pricing
Lecture 04 Risk Pref & EU (2)
Overview: Risk Preferences

1. **State-by-state dominance**
2. **Stochastic dominance** [DD4]
3. **vNM expected utility theory**
   a) Intuition [L4]
   b) Axiomatic foundations [DD3]
4. **Risk aversion coefficients and portfolio choice** [DD5,L4]
5. **Uncertainty/ambiguity aversion**
6. **Prudence coefficient and precautionary savings** [DD5]
7. **Mean-variance preferences** [L4.6]
State-by-state Dominance

- **State-by-state dominance** is an incomplete ranking

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cost</strong></td>
<td></td>
<td>Payoff $\pi_1 = \pi_2 = 1/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$s = 1$</td>
</tr>
<tr>
<td><strong>Investment 1</strong></td>
<td>-1000</td>
<td>1050</td>
</tr>
<tr>
<td><strong>Investment 2</strong></td>
<td>-1000</td>
<td>500</td>
</tr>
<tr>
<td><strong>Investment 3</strong></td>
<td>-1000</td>
<td>1050</td>
</tr>
</tbody>
</table>

- Investment 3 state-by-state dominates investment 1
- **Recall:** $y, x \in \mathbb{R}^S$
  - $y \geq x \iff y_s \geq x_s$ for each $s = 1, \ldots, S$
  - $y > x \iff y \geq x, y \neq x$
  - $y \gg x \iff y_s > x_s$ for each $s = 1, \ldots, S$
State-by-state Dominance (ctd.)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cost</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Return</strong></td>
<td>$\pi_1 = \pi_2 = 1/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s = 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s = 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Investment 1</strong></td>
<td>-1000</td>
<td>+ 5%</td>
<td>+ 20%</td>
</tr>
<tr>
<td><strong>Investment 2</strong></td>
<td>-1000</td>
<td>-50%</td>
<td>+ 60%</td>
</tr>
<tr>
<td><strong>Investment 3</strong></td>
<td>-1000</td>
<td>+ 5%</td>
<td>+ 60%</td>
</tr>
</tbody>
</table>

- Investment 1 mean-variance dominates 2
- But, investment 3 does *not* mean-variance dominate 1
State-by-state Dominance (ctd.)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$E[\text{Return}]$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Return $\pi_1 = \pi_2 = 1/2$</td>
<td>$s = 1$</td>
<td>$s = 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Investment 4</td>
<td>-1000</td>
<td>+ 3%</td>
<td>+ 5%</td>
<td>4.0%</td>
</tr>
<tr>
<td>Investment 5</td>
<td>-1000</td>
<td>+ 3%</td>
<td>+ 8%</td>
<td>5.5%</td>
</tr>
</tbody>
</table>

- What is the trade-off between risk and expected return?
- Investment 4 has a higher Sharpe ratio $\frac{E[r] - r_f}{\sigma}$ than investment 5 for $r_f = 0$
Overview: Risk Preferences

1. State-by-state dominance
2. **Stochastic dominance**
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   a) Intuition
   b) Axiomatic foundations
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5. Prudence coefficient and precautionary savings
6. Mean-variance preferences
Stochastic Dominance

- **No state-space** – probabilities are not assigned specific states
  - Only applicable for final payoff gamble
    - Not for stocks/lotteries that form a portfolio (whose payoff is final)
  - Random variables before introduction of \((\Omega, \mathcal{F}, P)\)
- **Still incomplete ordering**
  - “More complete” than state-by-state ordering
  - State-by-state dominance \(\Rightarrow\) stochastic dominance
  - Risk preference not needed for ranking!
    - independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. ("risk-preference-free")
- **Next Section:**
  - Complete preference ordering and utility representations
From payoffs per state to probability

<table>
<thead>
<tr>
<th>state</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>$\pi_1$</td>
<td>$\pi_2$</td>
<td>$\pi_3$</td>
<td>$\pi_4$</td>
<td>$\pi_5$</td>
</tr>
<tr>
<td>Payoff x</td>
<td>10</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Payoff y</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

Expressed in “probability lotteries” – only useful for final payoffs

(since some cross correlation information is lost)

<table>
<thead>
<tr>
<th>payoff</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob x</td>
<td>$p_{10} = \pi_1 + \pi_2$</td>
<td>$p_{20} = \pi_3 + \pi_4 + \pi_5$</td>
<td>$p_{30} = 0$</td>
</tr>
<tr>
<td>Prob y</td>
<td>$q_{10} = \pi_1$</td>
<td>$q_{20} = \pi_2 + \pi_3 + \pi_4$</td>
<td>$q_{30} = \pi_5$</td>
</tr>
</tbody>
</table>

Preference $x > y \in \mathbb{R}^S$ expressed in probabilities $p_x > q_y$
### Table 1: Payoff and Probability Distribution

<table>
<thead>
<tr>
<th>Payoffs</th>
<th>Probability 1</th>
<th>Probability 2</th>
<th>Expected Payoff</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>.4</td>
<td>.4</td>
<td>64</td>
<td>44</td>
</tr>
<tr>
<td>100</td>
<td>.6</td>
<td>.4</td>
<td>444</td>
<td>779</td>
</tr>
<tr>
<td>2000</td>
<td>0</td>
<td>.2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Payoff of $10 is equally likely, but can be in different states of the world.
First Order Stochastic Dominance

- **Definition**: Let $F_A(x), F_B(x)$, respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval $[a, b]$. We say that $F_A(x)$ **first order stochastically dominates (FSD)** $F_B(x)$ if and only if for all $x \in [a, b]$
  \[
  F_A(x) \leq F_B(x)
  \]

**Homework**: Provide an example which can be ranked according to FSD, but not according to state dominance.
First Order Stochastic Dominance
<table>
<thead>
<tr>
<th>Payoff</th>
<th>1</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability 3</td>
<td>0</td>
<td>0.25</td>
<td>0.50</td>
<td>0</td>
<td>0</td>
<td>0.25</td>
</tr>
<tr>
<td>Probability 4</td>
<td>0.33</td>
<td>0</td>
<td>0</td>
<td>0.33</td>
<td>0.33</td>
<td>0</td>
</tr>
</tbody>
</table>
Second Order Stochastic Dominance

• **Definition:** Let $F_A(x), F_B(x)$ be two cumulative probability distribution for random payoffs in $[a, b]$. We say that $F_A(x)$ **second order stochastically dominates (SSD)** $F_B(x)$ if and only if for any $x \in [a, b]$

$$\int_{-\infty}^{x} [F_B(t) - F_A(t)]dt \geq 0$$

(with strict inequality for some meaningful interval of values of $t$).
Mean Preserving Spread

$$x_B = x_A + z$$

(where $z$ is independent and has zero mean)

Mean Preserving Spread:
(for normal distributions)

$$\mu = \int x f_A(x) dx = \int x f_B(x) dx$$
Mean Preserving Spread & SSD

- **Theorem:** Let $F_A(x)$ and $F_B(x)$ be two distribution functions defined on the same state space with identical means. Then the following statements are equivalent:
  - $F_A(x)$ SSD $F_B(x)$
  - $F_B(x)$ is a mean-preserving spread of $F_A(x)$
Overview: Risk Preferences

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A Hypothetical Gamble

• Suppose someone offers you this gamble:
  ▪ "I have a fair coin here. I'll flip it, and if it's tails I pay you $1 and the gamble is over. If it's heads, I'll flip again. If it's tails then, I pay you $2, if not I'll flip again. With every round, I double the amount I will pay to you if it turns up tails."

• Sounds like a good deal. After all, you can't lose. So here's the question:
  ▪ How much are you willing to pay to take this gamble?
Proposal 1: Expected Value

- With probability \( \frac{1}{2} \) you get $1, \( \left( \frac{1}{2} \right)^1 \times 2^0 \)
- With probability \( \frac{1}{4} \) you get $2, \( \left( \frac{1}{2} \right)^2 \times 2^1 \)
- With probability \( \frac{1}{8} \) you get $4, \( \left( \frac{1}{2} \right)^3 \times 2^2 \)
- ...

- The expected payoff is given by the sum of all these terms, i.e.

\[
\sum_{t=1}^{\infty} \left( \frac{1}{2} \right)^t \times 2^{t-1} = \sum_{t=1}^{\infty} \frac{1}{2} = \infty
\]
St. Petersburg Paradox

• You should pay everything you own *and more* to purchase the right to take this gamble!

• Yet, in practice, no one is prepared to pay such a high price. Why?

• Even though the expected payoff is infinite, the distribution of payoffs is not attractive...
  - with 93% probability we get $8 or less;
  - with 99% probability we get $64 or less
Proposal 2

• Bernoulli suggests that large gains should be weighted less. He suggests to use the natural logarithm.
  [Cremer - another great mathematician of the time - suggests the square root.]

\[
\sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \times \ln 2^{t-1} = \ln 2 < \infty
\]

According to this Bernoulli would pay at most \(e^{\ln 2} = 2\) to participate in this gamble.
Representation of Preferences

A preference ordering is (i) complete, (ii) transitive, (iii) continuous and [(iv) relatively stable] can be represented by a utility function, i.e.

$$(c_0, c_1, ..., c_S) > (c'_0, c'_1, ..., c'_S) \iff U(c_0, c_1, ..., c_S) > U(c'_0, c'_1, ..., c'_S)$$

(preference ordering over lotteries – (S + 1)-dimensional space)
Indifference curves in $\mathbb{R}^2$ (for $S = 2$)
Preferences over Prob. Distributions

• Consider $c_0$ fixed, $c_1$ is a random variable
• Preference ordering over probability distributions
• Let
  ▪ $P$ be a set of probability distributions with a finite support over a set $X$,
  ▪ $\succeq$ preference ordering over $P$ (that is, a subset of $P \times P$)
Prob. Distributions

• $S$ states of the world
• Set of all possible lotteries

\[ P = \{ p \in \mathbb{R}^S | p(c) \geq 0, \sum p(c) = 1 \} \]

• Space with $S$ dimensions

• Can we simplify the utility representation of preferences over lotteries?
• Space with $one$ dimension – income
• We need to assume further axioms
Expected Utility Theory

• A binary relation that satisfies the following three axioms if and only if there exists a function $u(\cdot)$ such that

$$p > q \iff \sum p(c)u(c) > \sum q(c)u(c)$$

i.e. preferences correspond to expected utility.
vNM Expected Utility Theory

- **Axiom 1 (Completeness and Transitivity):**
  - Agents have preference relation over $P$ (repeated)

- **Axiom 2 (Substitution/Independence)**
  - For all lotteries $p, q, r \in P$ and $\alpha \in (0,1]$, $p \succeq q \iff \alpha p + (1 - \alpha)r \succeq \alpha q + (1 - \alpha)r$

- **Axiom 3 (Archimedean/Continuity)**
  - For all lotteries $p, q, r \in P$ if $p \succ q \succ r$ then there exists $\alpha, \beta \in (0,1)$ such that,
  $$\alpha p + (1 - \alpha)r > q > \beta p + (1 - \beta)r$$

**Problem:** $p$ you get $100 for sure, $q$ you get $10 for sure, $r$ you are killed
Independence Axiom

- Independence of irrelevant alternatives:

\[ p \succeq q \iff p \succeq r \quad \text{and} \quad q \succeq r \]
Allais Paradox –
Violation of Independence Axiom

\[
\begin{array}{c}
10\% & 10' \\
\downarrow & \downarrow \\
0 & < \\
\uparrow & \uparrow \\
9\% & 15' \\
\end{array}
\]
Allais Paradox – Violation of Independence Axiom

Diagram:
- Left side: 10% probability of 10' and 90% probability of 0.
- Right side: 9% probability of 15' and 90% probability of 0.

Comparison:
- The 10% probability choice is preferred over the 9% probability choice.
- The 90% probability choice is preferred over the 90% probability choice.
Allais Paradox –
Violation of Independence Axiom

10% 10’
100% 10’
10% 0
9% 15’
90% 15’
10% 0
vNM EU Theorem

• A binary relation that satisfies the axioms 1-3 if and only if there exists a function \( u(\cdot) \) such that

\[
p > q \iff \sum p(c)u(c) > \sum q(c)u(c)
\]

i.e. preferences correspond to expected utility.
Risk-Aversion and Concavity

- The shape of the von Neumann Morgenstern (NM) utility function reflects risk preference.
- Consider lottery with final wealth of $c_1$ or $c_2$. 

![Graph showing risk-aversion and concavity]

- $u(c)$
- $u(c_1)$
- $u(c_2)$
- $E[u(c)]$
- $c_1$
- $E[c]$
- $c_2$
Risk-aversion and concavity

- Risk-aversion means that the certainty equivalent is smaller than the expected prize.
  - We conclude that a risk-averse vNM utility function must be concave.

\[
\begin{align*}
\text{Risk-aversion means that the certainty equivalent is smaller than the expected prize.} \\
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\end{align*}
\]

\[
\begin{align*}
\text{Risk-aversion means that the certainty equivalent is smaller than the expected prize.} \\
\text{We conclude that a risk-averse vNM utility function must be concave.}
\end{align*}
\]
Jensen’s Inequality

**Theorem:**

- Let $g(\cdot)$ be a concave function on the interval $[a, b]$, and $x$ be a random variable such that
  \[ P[x \in [a, b]] = 1 \]

- Suppose the expectations $E[x]$ and $E[g(x)]$ exist; then
  \[ E[g(x)] \leq g[E[x]] \]

Furthermore, if $g(\cdot)$ is strictly concavely, then the inequality is strict.
Theorem: Let $F_A(\tilde{x})$, $F_B(\tilde{x})$ be two cumulative probability distribution for random payoffs $\tilde{x} \in [a, b]$. Then $F_A(\tilde{x})$ FSD $F_B(\tilde{x})$ if and only if $E_A[u(\tilde{x})] \geq E_B[u(\tilde{x})]$ for all non decreasing utility functions $U(\cdot)$.

Theorem: Let $F_A(\tilde{x})$, $F_B(\tilde{x})$ be two cumulative probability distribution for random payoffs $\tilde{x} \in [a, b]$. Then $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$ if and only if $E_A[u(\tilde{x})] \geq E_B[u(\tilde{x})]$ for all non decreasing concave utility functions $U(\cdot)$.
Certainty Equivalent and Risk Premium

\[ E[u(c + \tilde{Z})] = u \left( c + CE(c, \tilde{Z}) \right) \]

\[ E[u(c + \tilde{Z})] = u \left( c + E[\tilde{Z}] - \Pi(c, \tilde{Z}) \right) \]
Certainty Equivalent and Risk Premium

\[ U(Y) \]

\[ U(Y_0 + Z_1) \]
\[ U(Y_0 + E(\tilde{Z})) \]
\[ EU(Y_0 + \tilde{Z}) \]
\[ U(Y_0 + Z_2) \]

\[ Y_0 \quad Y_0 + Z_1 \quad CE(Y_0 + \tilde{Z}) \quad Y_0 + E(\tilde{Z}) \quad Y_0 + Z_2 \]

Certainty Equivalent and Risk Premium

\[ CE(\tilde{Z}) \quad \Pi \]
Utility Transformations

• General utility function:
  ▪ Suppose $U(c_0, c_1, ..., c_S) > U(c'_0, c'_1, ..., c'_S)$ represents complete, transitive, ... preference ordering,
  ▪ then $V(\cdot) = f(U(\cdot))$, where $f(\cdot)$ is strictly increasing represents the same preference ordering

• vNM utility function
  ▪ Suppose $E[u(c)]$ represents preference ordering satisfying vNM axioms,
  ▪ then $\nu(c) = a + bu(c)$ represents the same. “affine transformation”
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Measuring Risk aversion

A Strictly Concave Utility Function

U(W)

U(Y+h)

U[0.5(Y+h)+0.5(Y-h)]

0.5U(Y+h)+0.5U(Y-h)

U(Y-h)

Y-h

Y

Y+h

w

tangent lines
Arrow-Pratt Measures of Risk aversion

• absolute risk aversion
  \[ R_A(c) = -\frac{u''(c)}{u'(c)} \]

• relative risk aversion
  \[ R_R(c) = -\frac{cu''(c)}{u'(c)} \]

• risk tolerance
  \[ = \frac{1}{R_A} \]
Absolute risk aversion coefficient

\[ R_A = -\frac{u''(c)}{u'(c)} \]

\[
\pi(c, \Delta) = \frac{1}{2} + \frac{1}{4} \Delta R_A(c) + \text{HOT}
\]
Relative risk aversion coefficient

\[ R_R = -\frac{u''(c)}{u'(c)}c \]

\[ \pi(c, \theta) = \frac{1}{2} + \frac{1}{4} \delta R_R(c) + HOT \]

*Homework:* Derive this result.
CARA and CRRA-utility functions

- Constant Absolute RA utility function
  \[ u(c) = -e^{-\rho c} \]

- Constant Relative RA utility function
  \[ u(c) = \begin{cases} 
  \frac{c^{1-\gamma}}{1-\gamma}, & \gamma \neq 1 \\
  \ln[c], & \gamma = 1 
\end{cases} \]
Level of Relative Risk Aversion

\[
\frac{(Y + CE)^{1-\gamma}}{1 - \gamma} = \frac{1}{2} \left( \frac{(Y + 50000)^{1-\gamma}}{1 - \gamma} \right) + \frac{1}{2} \left( \frac{(Y + 100000)^{1-\gamma}}{1 - \gamma} \right)
\]

\[Y = 0\]
- \(\gamma = 0\), CE = 75,000 (risk neutrality)
- \(\gamma = 1\), CE = 70,711
- \(\gamma = 2\), CE = 66,246
- \(\gamma = 5\), CE = 58,566
- \(\gamma = 10\), CE = 53,991
- \(\gamma = 20\), CE = 51,858
- \(\gamma = 30\), CE = 51,209

\[Y = 100000\]
- \(\gamma = 5\), CE = 66,530
Risk aversion and Portfolio Allocation

• No savings decision (consumption occurs only at $t=1$)

• Asset structure
  
  ▪ One risk free bond with net return $r_f$
  
  ▪ One risky asset with random net return $r$

  \[ a = \text{quantity of risky assets} \]

\[
\max_a E \left[ u \left( Y_0 (1 + r_f) + a(r - r_f) \right) \right]
\]

\[
\text{FOC} \Rightarrow E \left[ u' \left( Y_0 (1 + r_f) + a(r - r_f) \right) (r - r_f) \right] = 0
\]
Risk aversion and Portfolio Allocation

• **Theorem 4.1:** Assume $U' > 0$, $U'' < 0$ and let $\hat{a}$ denote the solution to above problem. Then

\[
\hat{a} > 0 \iff E[\tilde{r}] > r_f \\
\hat{a} = 0 \iff E[\tilde{r}] = r_f \\
\hat{a} < 0 \iff E[\tilde{r}] < r_f
\]

• Define $W(a) = E\left[u\left(Y_0(1 + r_f) + a(\tilde{r} - r_f)\right)\right]$. The FOC can then be written $W'(a) = E\left[u'\left(Y_0(1 + r_f) + a(\tilde{r} - r_f)\right)(\tilde{r} - r_f)\right] = 0$.

• By risk aversion $W''(a) = E\left[u''\left(Y_0(1 + r_f) + a(\tilde{r} - r_f)\right)(\tilde{r} - r_f)^2\right] < 0$, that is, $W'(a)$ is everywhere decreasing
  ▪ It follows that $\hat{a}$ will be positive $\iff W'(0) > 0$

• Since $u' > 0$ this implies that $\hat{a} > 0 \iff E[\tilde{r} - r_f] > 0$
  ▪ The other assertion follows similarly
Portfolio as wealth changes

• **Theorem (Arrow, 1971):**
  Let \( \hat{a} = \hat{a}(Y_0) \) be the solution to max-problem above; then:

  i. \( \frac{\partial R_A}{\partial Y} < 0 \) (DARA) \( \Rightarrow \frac{\partial \hat{a}}{\partial Y_0} > 0 \)

  ii. \( \frac{\partial R_A}{\partial Y} = 0 \) (CARA) \( \Rightarrow \frac{\partial \hat{a}}{\partial Y_0} = 0 \)

  iii. \( \frac{\partial R_A}{\partial Y} > 0 \) (IARA) \( \Rightarrow \frac{\partial \hat{a}}{\partial Y_0} < 0 \)
Portfolio as wealth changes

- **Theorem (Arrow 1971):** If, for all wealth levels $Y$,

  i. $\frac{\partial R_R}{\partial Y} = 0$ (CRRA) $\Rightarrow \eta = 1$

  ii. $\frac{\partial R_R}{\partial Y} < 0$ (DRRA) $\Rightarrow \eta > 1$

  iii. $\frac{\partial R_R}{\partial Y} > 0$ (IRRA) $\Rightarrow \eta < 1$

where $\eta = \frac{da/a}{dY/Y}$
Log utility & Portfolio Allocation

\[ u(Y) = \ln Y \]

\[ E \left[ \frac{(\tilde{r} - r_f)}{Y_0(1 + r_f) + a(\tilde{r} - r_f)} \right] = 0 \]

\[ a \frac{Y_0}{Y_0} = \frac{((1 + r_f)[E[\tilde{r}] - r_f])}{-(r_1 - r_f)(r_2 - r_f)} > 0 \]

2 states, where \( r_2 > r_f > r_1 \)

Constant fraction of wealth is invested in risky asset!

*Homework:* show that this result holds for
- any CRRA utility function
- any distribution of \( r \)
Risk aversion and Portfolio Allocation

Theorem (Cass and Stiglitz, 1970): Let the vector \(
\begin{bmatrix}
\hat{a}_1(Y_0) \\
\vdots \\
\hat{a}_J(Y_0)
\end{bmatrix}
\) denote the amount optimally invested in the \(J\) risky assets if the wealth level is \(Y_0\).

Then
\[
\begin{bmatrix}
\hat{a}_1(Y_0) \\
\vdots \\
\hat{a}_J(Y_0)
\end{bmatrix} = \begin{bmatrix} a_1 \\
\vdots \\
a_J \end{bmatrix} f(Y_0)
\]
if and only if either

i. \( u'(Y_0) = (BY_0 + C)^\Delta \) or

ii. \( u'(Y_0) = \xi e^{-\rho Y_0} \)

In words, it is sufficient to offer a mutual fund.
LRT/HARA-utility functions

- Linear Risk Tolerance/hyperbolic absolute risk aversion
  \[ u''(c) \frac{u'(c)}{1} = \frac{1}{A + Bc} \]

- Special Cases
  - \( B = 0, A > 0 \) CARA
    \[ u(c) = \frac{1}{B-1} (A + Bc)^{\frac{B-1}{B}} \]
  - \( B \neq 0, \neq 1 \) Generalized Power
    - \( B = 1 \) Log utility
      \[ u(c) = \ln[A + Bc] \]
    - \( B = -1 \) Quadratic Utility
      \[ u(c) = -(A - c)^2 \]
    - \( B \neq 1, A = 0 \) CRRA Utility function
      \[ u(c) = \frac{1}{B-1} (Bc)^{\frac{B-1}{B}} \]
Overview: Risk Preferences

1. State-by-state dominance
2. Stochastic dominance
3. vNM expected utility theory
   a) Intuition
   b) Axiomatic foundations
4. Risk aversion coefficients and portfolio choice
5. Uncertainty/ambiguity aversion
6. Prudence coefficient and precautionary savings
7. Mean-variance preferences
Digression: Subjective EU Theory

• Derive perceived probability from preferences!
  ▪ Set $S$ of prizes/consequences
  ▪ Set $Z$ of states
  ▪ Set of functions $f(s) \in Z$, called acts (consumption plans)

• Seven SAVAGE Axioms
  ▪ Goes beyond scope of this course.
Digression: Ellsberg Paradox

- 10 balls in an urn
  - Lottery 1: win $100 if you draw a red ball
  - Lottery 2: win $100 if you draw a blue ball
- Uncertainty: Probability distribution is not known
- Risk: Probability distribution is known (5 balls are red, 5 balls are blue)

- Individuals are “uncertainty/ambiguity averse” (non-additive probability approach)
Digression: Prospect Theory

• Value function (over gains and losses)

• Overweight low probability events

• Experimental evidence
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Introducing Savings

- Introduce savings decision: Consumption at $t = 0$ and $t = 1$
- **Asset structure 1:**
  - risk free bond $R^f$
  - NO risky asset with random return
- Increase $R^f$:
  - **Substitution effect:** shift consumption from $t = 0$ to $t = 1$
    $\Rightarrow$ save more
  - **Income effect:** agent is “effectively richer” and wants to consume some of the additional richness at $t = 0$
    $\Rightarrow$ save less
  - For log-utility ($\gamma = 1$) both effects cancel each other
Savings: Euler Equation

for CRRA: \( u(c) = \frac{c^{1-\gamma}}{1-\gamma} \)

- \( \max_{c_0, c_1} u(c_0) + \delta u(c_1) \)
  - s.t. \( c_1 = R^f (e_0 - c_0) + e_1 \)
- \( \max_{c_0} u(c_0) + \delta u(R^f e_0 + e_1) \)
- FOC: \( 1 = \delta \left( \frac{c_1}{c_0} \right)^{-\gamma} R^f \)

- \( r^f \approx \ln R^f = - \ln \left( \frac{u'(c_1)}{u'(c_0)} \right) - \ln \delta \)

for log: \( u(c) = \ln c \) & \( e_1 = 0 \)

\[
\begin{align*}
c_0 &= \frac{1}{\delta(\delta+1)} [e_0 + \frac{1}{R} e_1] \\
c_1 &= \left(1 - \frac{1}{\delta(\delta+1)}\right) [Re_0 + e_{-1}] \\
\end{align*}
\]

for \( e_1 = 0 \) saving does not depend on (risk of) \( R^f: \quad 1 = \delta \left( \frac{c_0}{R^f (e_0 - c_0)} \right) R^f \)
Intertemporal Elasticity of Substitution

\[ IES := \frac{\partial \ln \left( \frac{c_1}{c_0} \right)}{\partial r} = - \frac{\partial \ln \left( \frac{c_1}{c_0} \right)}{\partial \ln \left( \frac{u'(c_1)}{u'(c_0)} \right)} \]

• For CRRA \( u(c) = \frac{c^{1-\gamma}}{1-\gamma} \)

\[ IES = \frac{1}{\gamma} \]
Investment Risk

- Savings decision: Consumption at $t = 0$ and $t = 1$
- No endowment risk at $t = 1$
- Asset structure 2: (no portfolio choice yet)
  - Single risky asset only
  - No risk-free asset

- Theorem (Rothschild and Stiglitz, 1971):
  For $R^B = R^A + \varepsilon$, where $E[\varepsilon] = 0$ and $\varepsilon \perp R^A$, then respective savings $s^A, s^B$ out of initial wealth level $W_0$ are
  - If $\frac{\partial R_R}{\partial W_0} \leq 0$ and $R_R > 1$, then $s^A < s^B$.
  - If $\frac{\partial R_R}{\partial W_0} \geq 0$ and $R_R < 1$, then $s^A > s^B$. 
Investment Risk with Portfolio and Savings Decision

• Savings decision: Consumption at $t = 0$ and $t = 1$
• No endowment risk at $t = 1$, $e_1 = 0$
• Asset structure 3: portfolio shares $\alpha^j$

\[
\max_{c_0, c_1, \alpha_0} u(c_0) + \delta E_0 [u(c_1)]
\]

s.t. $W_1 = \sum \alpha_0^j R_1^j (W_0 - c_0)$

\[
u'(c_0) = E_0 [\delta u'(c_1) R^j] \forall j
\]
Investment Risk: Excess Return

\[ 1 = E_0 \left[ \delta \frac{u(c_1)}{u(c_0)} R^j \right] \forall j \]

- For CRRA
  \[ 1 = \delta E_0 \left[ \left( \frac{c_1}{c_0} \right)^{-\gamma} R^j \right] \]

- In “log-notation”: \( c_t \equiv \log c_t, r^j_t \equiv \log R^j_t \)
  \[ 1 = \delta E_0 \left[ e^{-\gamma(c_1-c_0)+r^j} \right] \]

- Assume \( c_t, r^j_t \sim \mathcal{N} \)
  \[ 1 = \delta [e^{-\gamma E_0[\Delta c_1]+E_0[r^j]+\frac{1}{2}Var_0[-\gamma \Delta c_1+r^j]}] \]

  \[ 0 = \ln \delta - \gamma E_0[\Delta c_1] + E_0[r^j] + \frac{\gamma^2}{2} Var_0[\Delta c_1] + \frac{1}{2} Var_0[r^j] - \gamma Cov_0[\Delta c_1, r^j] \]

- For risk free asset:
  \[ r^f = -\ln \delta + \gamma E_0[\Delta c_1] - \frac{\gamma^2}{2} Var_0[\Delta c_1] \]

- Excess return of any asset:
  \[ E_0[r^j] + \frac{1}{2} Var_0[r^j] - r^f = \gamma Cov_0[\Delta c_1, r^j] \]
Investment Risk: Portfolio Shares

- Excess return
  \[ E_0[r^j] + \frac{1}{2} Var_0[r^j] - r^f = \gamma Cov_0[\Delta c_1, r^j] \]

- If consumption growth \( \Delta c_1 = \Delta w_1 \) wealth growth

- \( Cov_0[\Delta w_1, r^j] = Cov_0[\alpha^j r, r^j] = \alpha^j Var_0[r^j] \)

- Hence, optimal portfolio share
  \[ \alpha_0^j = \frac{E_0[r^j] + \frac{1}{2} Var_0[r^j] - r^f}{\gamma Var_0[r^j]} \]
Making $\Delta w_1$ Linear in $c_0 - w_0$

- $W_1 = \sum \alpha_0^j R_1^j (W_0 - c_0)$  
  recall $e_1 = 0$
- $\frac{w_1}{w_0} = \sum \alpha_0^j R_1^j (1 - \frac{c_0}{w_0})$  
  let $R_1^p = \sum \alpha_0^j R_1^j$
- In “log-notation”: $\Delta w_1 = r_1^j + \log(1 - e^{c_0 - w_0})$

- Linearize using Taylor expansion around $c - w$

- $\Delta w_1 = r_1^j + k + \left(1 - \frac{1}{\rho}\right) (c_0 - w_0)$
  - Where $k \equiv \log \rho + (1 - \rho) \log \frac{1-\rho}{\rho}$, $\rho = 1 - e^{c-w}$

Hint: in continuous time this approximation is precise
Endowment Risk: Prudence and Pre-cautionary Savings

• Savings decision
  Consumption at $t = 0$ and $t = 1$

• Asset structure 2:
  ▪ No investment risk: riskfree bond
  ▪ Endowment at $t = 1$ is random (background risk)

• 2 effects: Tomorrow consumption is more volatile
  ▪ consume more today, since it’s not risky
  ▪ save more for precautionary reasons
Prudence and Pre-cautionary Savings

• Risk aversion is about the willingness to insure ...
• ... but not about its comparative statics.
• How does the behavior of an agent change when we marginally increase his exposure to risk?
• An old hypothesis (J.M. Keynes) is that
  ▪ people save more when they face greater uncertainty
  ▪ precautionary saving
• Two forms:
  ▪ Shape of utility function \( u''' \)
  ▪ Borrowing constraint \( a_t \geq -b \)
Precautionary Savings 1: Prudence

• Utility maximization $u(c_0) + \delta u E_0[u(c_1)]$
  - Budget constraint: $c_1 = e_1 + (1 + r)(e_0 - c_0)$
  - Standard Euler equation: $u'(c_t) = \delta(1 + r) E_t[u'(c_{t+1})]$

• If $u''' > 0$, then Jensen’s inequality implies:
  - $\frac{1}{\delta(1+r)} = \frac{E_t[u'(c_{t+1})]}{u'(c_t)} > \frac{u'(E_t[c_{t+1}])}{u'(c_t)}$
  - Increase variance of $e_1$ (mean preserving spread)
  - Numerator $E_t[u'(c_{t+1})]$ increases with variance of $c_{t+1}$
  - For equality to hold, denominator has to increase $c_t$ has to decrease, i.e. savings has to increase precautionary savings

• Prudence refers to curvature of $u'$, i.e. $P = -\frac{u'''}{u''}$
Precautionary Savings 1: Prudence

• Does not directly follow from risk aversion, involves $u'''$
  ▪ Leland (1968)
• Kimball (1990) defines **absolute prudence** as

$$P(c) := -\frac{u''''(c)}{u''(c)}$$

• Precautionary saving if any only if prudent.
  ▪ important for comparative statics of interest rates.
• DARA ⇒ Prudence

$$\frac{\partial \left( -\frac{u'''}{u'} \right)}{\partial c} < 0, \quad -\frac{u'''}{u''} > -\frac{u''}{u'}$$
Precautionary Savings 2: Future Borrowing Constraint

- Agent might be concerned that he faces *borrowing constraints* in some state in the future
- Agents engage in precautionary savings (self-insurance)
- In Bewley (1977) idiosyncratic income shocks, mean asset holdings $\text{mean}[a]$ (across individuals) result from individual optimization

![Graph](image-url)
Precautionary Savings 1: Prudence

• **Asset structure 3:**
  – No risk free bond
  – One risky asset with random gross return $R$

• **Theorem (Rothschild and Stiglitz, 1971):** Let $\tilde{R}_A, \tilde{R}_B$ be two return distributions with identical means such that $\tilde{R}_B = \tilde{R}_A + e$, where $e$ is white noise, and let $s_A, s_B$ be the savings out of $Y_0$ corresponding to the return distributions $\tilde{R}_A, \tilde{R}_B$ respectively.

  - If $R'_R(Y) \leq 0$ and $R_R(Y) > 1$, then $s_A < s_B$
  - If $R'_R(Y) \geq 0$ and $R_R(Y) < 1$, then $s_A > s_B$
Precautionary Savings 1: Prudence

\[
P(c) = -\frac{u'''(c)}{u''(c)}
\]

\[
P(c)c = -\frac{cu''''(c)}{u''(c)}
\]

• **Theorem:** Let \( \tilde{R}_A, \tilde{R}_B \) be two return distributions such that \( \tilde{R}_A \) SSD \( \tilde{R}_B \), let \( s_A \) and \( s_B \) be, respectively, the savings out of \( Y_0 \). Then,

- \( s_A \geq s_B \iff cP(c) \leq 2 \) and conversely,
- \( s_A < s_B \iff cP(c) > 2 \)
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Mean-variance Preferences

• Early research (e.g. Markowitz and Sharpe) simply used mean and variance of return
• Mean-variance utility often easier than vNM utility function
• ... but is it compatible with vNM theory?
• The answer is yes ... approximately ... under some conditions.
Mean-Variance: quadratic utility

Suppose utility is quadratic, $u(c) = ac - bc^2$

Expected utility is then

$$E[u(c)] = aE[c] - bE[c^2]$$

Thus, expected utility is a function of the mean $E[c]$ and the variance $\text{var}[c]$. Only.
Mean-Variance: joint normals

• Suppose all lotteries in the domain have normally distributed prized. (independence is not needed).
  ▪ This requires an infinite state space.

• Any linear combination of jointly normals is also normal.

• The normal distribution is completely described by its first two moments.

• Hence, expected utility can be expressed as a function of just these two numbers as well.
Mean-Variances: small risks

- Let $f: \mathbb{R} \to \mathbb{R}$ be a smooth function. The Taylor approximation is
  
  $$f(x) \approx f(x_0) + f'(x_0) \frac{(x - x_0)^1}{1!} + f''(x_0) \frac{(x - x_0)^2}{2!} + \cdots$$

- Use the Taylor approximation for $E[u(x)]$
Mean-Variance: small risks

- Since $E[u(w + x)] = u(c^{CE})$, this simplifies to $w - c_{CE} \approx R_A(w) \frac{\text{var}(x)}{2}$
  - $w - c_{CE}$ is the risk premium
  - We see here that the risk premium is approximately a linear function of the variance of the additive risk, with the slope of the effect equal to half the coefficient of absolute risk.
Mean-Variance: small risks

- Same exercise can be done with a multiplicative risk.
- Let $y = gw$, where $g$ is a positive random variable with unit mean.
- Doing the same steps as before leads to
  
  $$1 - \kappa \approx R_R(w) \frac{\text{var}[g]}{2}$$

- where $\kappa$ is the certainty equivalent growth rate, $u(\kappa w) = E[u(gw)]$.
- The coefficient of relative risk aversion is relevant for multiplicative risk, absolute risk aversion for additive risk.