Asset Pricing under Asymmetric Information Screening Models

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      - uniform price setting
      - limit order book analysis
    - dynamic sequential trade models with multiple trading rounds
- strategic market order models where the market maker sets prices ex-post
Screening Models à la Glosten

1. **Uninformed (risk-neutral) market maker sets whole supply schedule**
   - market making sector is *competitive*
   - oligopolistic market making sector
   - market maker is *monopolist*

2. **Possibly informed trader submits**
   - a single order which is executed at *uniform price*
   - many little orders in order to “walk along the limit order book” (*discriminatory prices*)
Uniform Price Setting - Glosten 1989

- Contrast competitive market maker sector with monopolistic market maker (specialist system NYSE).

- Model setup
  - market maker(s) set price (supply) schedule
  - single trader submits order
    - risk-averse with CARA utility function
    - endowment shock of $u$
    - private signal $S^i = v + \epsilon$
  - two-dimensional screening problem
    Glosten (1989) reduces it to a one-dimensional problem (see later)
Uniform Price Setting - Glosten 1989

- Competitive price schedule: \( P^{CO} = E[v|x] \)
  - Perfect competition
    - \( \Rightarrow \) expected profit for any order size \( x \) is ZERO.
    - Prevents market makers from effectively screening orders
    - \( \Rightarrow \) leads to instability
      - Formally, existence problem for certain parameters
        (Hellwig JET 1994 shows that this is due to unbounded support of type sapce and it existence problem is different to the one in Rothschild & Stiglitz)

- Monopolistic price schedule:
  \[
P^{mo} = \arg \max E[(P^{mo}(x^*(\cdot)) - v)x^*(\cdot)],
\]
  where \( x^*(\cdot) \) is the optimal order size.
  - Principal-agent problem
  - Principal sets menu of contracts \((x, P^{mo}(x))\)
  - Cross-subsidization: large profit from small trades small (-ve) profit from large trades
  - Market with monopolistic setting stays open for larger trade sizes than a market with multiple market markers
Discrim. Pricing (Limit Order Book)
Glosten 1994 - BRM 2000

- "upper tail" conditional expectations for next marginal order $y$
  $$P^{CO}(y) = E[v|x \geq y]$$
- trader who buy only a tiny marginal quantity have to pay a higher (ask) price $\Rightarrow$ small trade spread
- competitive market makers do not know whether trader only buys first marginal unit or continues to buy further units.
- cross-subsidization from small orders to large orders
- limit order book is immune to "cream skimming" of orders by competing exchanges (no advantage of order splitting).
Discrim. Pricing - Biais, Rochet & Martimort

Oligopolistic Market Makers

- oligopolistic screening game (special cases $I = 1, I = \infty$)
- **Stage 1**: risk-neutral market maker(s)
  set supply schedule $p(x)$ (limit order book)
- **Stage 2**: informed trader buy $x = \sum_i x^i$ shares
  - $x^i$ for market maker $i$
  - transfer to mm $i$: $t^i(x^i) = \int_0^{x^i} p(q) dq$, $T(x) = \sum_i t^i(x^i)$
  - trader’s endowment shock $u$
  - trader’s signal $S$, where $v = S + \varepsilon$.
    $\varepsilon \sim \mathcal{N}(0, \sigma^2)$
    $u$ and $S$ have bounded support.
  - trader’s final wealth $W = v(u + x) - \sum_i t^i(x^i)$
BRM: One Dimensional Screening

• **Stage 2:** (ctd.) - “Glosten (1989)-trick”
  - with CARA utility function

\[
E [W | u, S] - \frac{\rho}{2} V [W | u, S]
\]

\[
= (x + u) S - T(x) - \frac{\rho}{2} (x + u)^2 \underbrace{\text{Var} [v | S]}_{= \sigma^2}
\]

\[
= \left( uS - \frac{\rho \sigma^2}{2} u^2 \right) + \left( xS - \rho \sigma^2 xu - \frac{\rho \sigma^2}{2} x^2 - T(x) \right)
\]

\[
= \left( uS - \frac{\rho \sigma^2}{2} u^2 \right) + \left( xS - \rho \sigma^2 xu - \frac{\rho \sigma^2}{2} x^2 - T(x) \right)
\]

independent of \( x \)

\[
\theta x
\]

i.e. \( \theta := S - \rho \sigma^2 u \)

depends on \( x \)

\[ \Rightarrow \text{Info-Rent} \]

• This reduces it to a one-dimensional screening problem
  - function \( v(\theta) = E[v|\theta] \) of (one-dimensional) type \( \theta \)

\[ 1 \geq v(\theta) \geq 0 \]
BRM: First Best Benchmark

- ex-ante

optimal trading mechanism

\[
\max_{\{\tau(\theta), x(\theta)\}} \int_{\theta}^{\overline{\theta}} \left( \theta x(\theta) - \frac{\rho \sigma^2}{2} x(\theta)^2 - \tau(\theta) \right) f(\theta) d\theta
\]

s.t.

\[
\int_{\theta}^{\overline{\theta}} \left( \tau(\theta) - v(\theta) x(\theta) \right) f(\theta) d\theta = \Pi
\]

- \(\Pi\) determines how surplus is distributed between P and A

\[
\Longrightarrow \max \int_{\theta}^{\overline{\theta}} \left( \theta x(\theta) - \frac{\rho \sigma^2}{2} x(\theta)^2 - v(\theta) x(\theta) - \Pi \right) f(\theta) d\theta
\]

surplus
Asset Pricing under Asym. Information

Screening Models

Classification of Models

Static
- Uniform Price
- Discr. Price (Limit Order Book)
- Contrast

Dynamic
- Sequential Trade
- Herding
- 1987-crash

BRM: First Best Benchmark

for a given $\theta$

$$\theta - \rho \sigma^2 x(\theta) - v(\theta) = 0$$

$$x^*(\theta) = \frac{\theta - v(\theta)}{\rho \sigma^2}$$

$$= E[-u|\theta], \text{ since } u = -\frac{\theta - S}{\rho \sigma^2}$$

- Assume $x^*(\theta) < 0 < x^*(\bar{\theta})$
  $$\implies \exists \theta_0 \text{ s.t. } x^*(\theta_0) = 0$$

- almost all $\theta$-types trade
  (see later that $\forall \theta > \theta_0 \implies \text{buy}$
  $\forall \theta < \theta_0 \implies \text{sell}$)
BRM: Monopolistic Screening

$x^*(\theta)$ and $x_m(\theta)$

Figure: xxx. xx
BRM: Implementable Allocation under Adverse Selection

- social planner must elicit information
- \textbf{Revelation Principle}
  
  Any allocation that can be achieved with non-linear schedules \( T(x) \) can also be achieved with a truthful direct mechanism \( \{\tau(\cdot), x(\cdot)\} \).
- \textbf{Incentive compatibility}

\[
\theta \in \arg \max_{\hat{\theta}} \left( \theta x(\hat{\theta}) - \frac{\rho \sigma^2}{2} x(\hat{\theta})^2 - \tau(\hat{\theta}) \right)
\]

\[
\implies U(\theta) = \max_{\hat{\theta}} \left( \theta x(\hat{\theta}) - \frac{\rho \sigma^2}{2} x(\hat{\theta})^2 - \tau(\hat{\theta}) \right)
\]

\( \{\tau(\cdot), x(\cdot)\} \) transfers and allocation
BRM: Dual (Mirrlees) Approach

\{U(\cdot), x(\cdot)\} informational rent (see Fudenberg & Tirole Ch. 7)

**Lemma 1:**
A pair \{U(\cdot), x(\cdot)\} is implementable iff \(U(\cdot)\) is convex on \([\theta, \bar{\theta}]\), and for a.e. \(\theta\), \(U(\theta) = x(\theta)\),

\[
\frac{dU(\theta, \hat{\theta}(\theta))}{d\theta} = \frac{\partial U}{\partial \theta} = x(\theta).
\]

envelope theorem
BRM: Monopolistic Screening

m.m.(principal) gets \( \int_\theta^{\bar{\theta}} \tau(x(\theta)) - v(\theta)x(\theta) \) replacing \( \tau \) from information rent \( U(\theta) = \theta x(\theta) - \frac{\rho \sigma^2}{2} x^2(\theta) - \tau(x(\theta)) \), the m.m.'s objective becomes

\[
\max \left\{ U(\cdot), x(\cdot) \right\} \int_\theta^{\bar{\theta}} \left\{ [\theta - v(\theta)]x(\theta) - \frac{\rho \sigma^2}{2} [x(\theta)]^2 - U(\theta) \right\} f(\theta) d\theta
\]

subject to

\[
\text{IC} \quad \begin{cases} 
U(\cdot) \text{ is convex on } [\theta, \bar{\theta}] \\
\dot{U}(\theta) = x(\theta) \quad \forall \theta \text{ (almost everywhere)}
\end{cases}
\]

ex-post PC \( U(\theta) \geq 0 \) **ex-post** participation constraints (ex-post: since traders decide after knowing \( \theta \) whether to participate)
BRM: Monopolistic Screening
Dual Approach

(replace $x(\theta)$ with $\dot{U}(\theta)$)

$$\max_{U(\cdot)} B_m \left( U(\cdot), \dot{U}(\cdot) \right)$$

$$:= \int_{\theta}^{\bar{\theta}} \left( [\theta - v(\theta)] \dot{U}(\theta) - \frac{\rho \sigma^2}{2} \dot{U}(\theta)^2 - U(\theta) \right) f(\theta) \, d\theta$$

s.t. $U(\cdot)$ convex
$U(\theta) \geq 0$

Temporarily ignore convexity constraint and check ex-post.
(Sufficient condition: $U(\cdot)$ is convex if

$$\forall \theta > \theta_0 \quad \frac{d}{d\theta} \left( \frac{1-F(\theta)}{f(\theta)} \right) < 0 \quad (18)$$
$$\forall \theta < \theta_0 \quad \frac{d}{d\theta} \left( \frac{F(\theta)}{f(\theta)} \right) > 0 \quad (19)$$
BRM: Monopolistic Screening

\[ \mathcal{L} \left( U, \dot{U} \right) = B_m \left( U, \dot{U} \right) + \int_{\theta}^{\bar{\theta}} U(\theta) \ d\Lambda(\theta) \]

\[ \uparrow \]

\( \infty \) many Lagrange multipliers different from type to type (ex-post constraint)

By complementary slackness condition, support of \( \Lambda \) be constrained in \((U_m)^{-1}(0)\), (\( \theta \)-types which get zero info ret) view \( \Lambda(\theta) \) as c.d.f., i.e., \( \exists \) a measure \( \Lambda \)

\[ \Lambda(\theta) = \int_{\theta}^{\bar{\theta}} \frac{d\Lambda(s)}{\int_{\theta}^{\bar{\theta}} d\Lambda(s)} \] (slight abuse of notation)
BRM: Monopolistic Screening

Aside: Integrating by parts

$$\int_{\theta}^{\overline{\theta}} U(\theta) \, d[\Lambda(\theta) - F(\theta)] = - \int_{\theta}^{\overline{\theta}} \dot{U}(\theta) \, (\Lambda(\theta) - F(\theta)) \, d\theta + U(\overline{\theta})$$

Consequently, \( \max \mathcal{L} \left( U, \dot{U} \right) = \)

$$= \int_{\theta}^{\overline{\theta}} \left( (\theta - v(\theta) + \frac{F(\theta) - \Lambda(\theta)}{f(\theta)}) \dot{U}(\theta) - \frac{\rho \sigma^2}{2} \dot{U}(\theta)^2 \right) f(\theta) \, d\theta$$

$$+ U(\overline{\theta}) (\Lambda(\theta) - 1)$$

max only if \( \Lambda(\theta) = 1 \) (since \( U(\overline{\theta}) \) is arbitrary)

pointwise maximization over \( \dot{U}(\theta) \)

$$\forall \theta \in [\theta, \overline{\theta}] \, , \, x_m(\theta) = \frac{\theta - v(\theta)}{\rho \sigma^2} + \frac{F(\theta) - \Lambda(\theta)}{f(\theta) \rho \sigma^2} \underbrace{x^*(\theta)}_{x^*(\theta)}$$
Complementary slackness condition ($d\Lambda = 0$ for some $\theta$)

\[
\forall \theta \in [\underline{\theta}, \theta^m_b] \quad \Lambda(\theta) = 0
\]
\[
\forall \theta \in [\theta^m_a, \overline{\theta}] \quad \Lambda(\theta) = 1
\]

\[\implies \text{given (18) & (19), } U(\cdot) \text{ is convex and} \]

Proposition 2

\[\exists \theta^m_a > \theta_0 \text{ and } \theta^m_b < \theta \text{ s.t.} \]

(i) for all $\theta \in [\underline{\theta}, \theta^m_b]$, $x_m(\theta) = x^*(\theta) + \frac{F(\theta)}{\rho\sigma^2 f(\theta)}$

(ii) for all $\theta \in [\theta^m_b, \theta^m_a]$, $x_m(\theta) = 0$ (no info rent)

(iii) for all $\theta \in (\theta^m_a, \overline{\theta}]$, $x_m(\theta) = x^*(\theta) - \frac{1-F(\theta)}{\rho\sigma^2 f(\theta)}$
BRM: Monopolistic Screening

\( x^*(\theta) \) and \( x_m(\theta) \)

Figure: xxx. xx
BRM: Monopolistic Screening
Price Schedule

for $\theta > \theta^m_a$ we know

\begin{align}
(1) \quad & \theta \geq \theta^m_a \quad U(\theta) = 0 + \int_{\theta^m_a}^{\theta} \dot{U}'(s) \, ds = \int_{\theta^m_a}^{\theta} x(s) \, ds \\
(2) \quad & U(\theta) = \theta x_m(\theta) - \frac{\rho \sigma^2 x_m(\theta)^2}{2} - T(x(\theta)) \\
(1) \Rightarrow (2) \quad & T(x(\theta)) = \theta x_m(\theta) - \frac{\rho \sigma^2 x_m(\theta)^2}{2} - \int_{\theta^m_a}^{\theta} x(s) \, ds
\end{align}

Differentiate w.r.t. $\theta$

\begin{align*}
\frac{\partial T}{\partial x} \frac{\partial x}{\partial \theta} & = x_m(\theta) + \theta \frac{\partial x}{\partial \theta} - \rho \sigma^2 x_m(\theta) \frac{\partial x_m}{\partial \theta} - x_m(\theta) \\
\frac{\partial T}{\partial x} & = \theta - \rho \sigma^2 x_m(\theta)
\end{align*}

We have

\begin{equation}
x_m(\theta) = x^*(\theta) + \frac{F(\theta)}{\rho \sigma^2 f(\theta)} \left( \frac{\theta - \nu(\theta)}{\rho \sigma^2} \right)
\end{equation}
BRM: Monopolistic Screening Price Schedule

\[ \frac{\partial T}{\partial x} = \theta - \theta + v(\theta) - \frac{F(\theta)}{f(\theta)} \]

\[ t_m(x) = \frac{\partial T}{\partial x} = v(\theta) - \frac{F(\theta)}{f(\theta)} \]

for \( \theta < \theta^m_b \) similar steps

\[ \frac{\partial T}{\partial x} = v(\theta) + \frac{1 - F(\theta)}{f(\theta)} \]

Note that

\[ t_m(x = 0^+) = \theta^m_a > \theta^m_b = t_m(x = 0^-) \]

“small trade spread”
BRM: Oligopolistic Screening
Model setup

- order size of trader is *exogenous*
- is double exponentially distributed \( f(x) = \frac{1}{2} ae^{-a|x|} \)
- conditional expectations
  
  - \( E[\cdot | x \geq y] \Rightarrow \) linear schedule in limit order book
  
  - \( E[v|x] = v_0 + \gamma x \text{ assumed} \Rightarrow \) linear uniform price schedule

- \( p^u(x) = v_0 + \frac{l-1}{l-2} \gamma x \) versus \( p^d(x) = v_0 + \frac{I}{I-1} \frac{\gamma}{a} + \gamma x \)
Limit Order Book vs. Uniform Pricing
Röell (1998)

Figure: Limit Order Book.
Limit Order Book vs. Uniform Pricing
Röell (1998)

Figure: Limit Order Book and Uniform Pricing.
A Classification of Market Microstructure Models

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Sequential Trade Models à la Glosten & Milgrom (1985)

- order size is restricted to $x \in \{-1, +1\}$

**Figure:** Bid-Ask Spread.
Sequential Trade Models à la Glosten & Milgrom (1985)

- **Monopolistic Market Maker - Copeland & Galai (1983)**
  - bid-ask spread is partially due to monopoly power
  - partially due to adverse selection
  - difficult to handle in multi-period setting

- **Competitive Market Makers - Glosten & Milgrom (1985)**
  - bid-ask spread is only due to adverse selection
  - multi-period setting
Glosten & Milgrom (1985)

- **Model Setup**
  - value of the stock $v$ and $\tilde{v}$
  - with probability $\alpha$ an informed trader shows up
  - with probability $(1 - \alpha)$ an uninformed trader shows up
  - all traders are chosen from a pool of a continuum of traders, i.e., the probability that they will trade a second time is zero (rule out strategic considerations as in Kyle’s)
  - informed traders know true $\tilde{v}$ → buys if $v > a$ sells if $v < b$.
  - uninformed traders buy with probability $\mu$ and sell with probability $1 - \mu$.
  - Note: Traders can only buy or sell 1 unit (No-Trade is also not allowed!)
Glosten & Milgrom (1985)

Figure: Tree.
Glosten & Milgrom (1985)  
Calculating Bid-Ask Spread

- **Buy order**

  \[
  P(\bar{v}) = \theta \\
  P(\text{buy}|\bar{v}) = \alpha + (1 - \alpha) \mu \\
  P(\text{buy}|v) = (1 - \alpha) \mu 
  \]

  **Bayes’ Rule**

  \[
  P(\bar{v}|\text{buy}) = \frac{(\alpha + (1 - \alpha) \mu) \theta}{(\alpha + (1 - \alpha) \mu) \theta + (1 - \alpha) \mu (1 - \theta)} \\
  P(v|\text{buy}) = 1 - P(\bar{v}|\text{buy})
  \]
Glosten & Milgrom (1985) 
Calculating Bid-Ask Spread

- Sell order \( P(\bar{v} | \text{sell}) = \)

\[
= \frac{(1-\alpha)(1-\mu)\theta}{(1-\alpha)(1-\mu)\theta + [\alpha+(1-\alpha)(1-\mu)](1-\theta)}
\]

\[ P(\bar{v} | \text{buy}) > P(\bar{v}) > P(\bar{v} | \text{sell}) \]

\[ P(\bar{v} | \text{buy}) < P(\bar{v}) < P(\bar{v} | \text{sell}) \]

- Market Maker makes zero expected profit
  (potential Bertrand competition)

\[
b = \text{bid} = E[\nu | \text{sell}] = \bar{v}P(\bar{v} | \text{sell}) + \nu P(\nu | \text{sell})
\]

\[
a = \text{ask} = E[\nu | \text{buy}] = \bar{v}P(\bar{v} | \text{buy}) + \nu P(\nu | \text{buy})
\]
Remarks to Glosten & Milgrom (1985)

1. quotes are regret free
2. $\underline{v} < b < a < \overline{v}$
3. $(a - b) \rightarrow$ gain from liquidity traders = loss to insider
4. bid-ask spread $(a - b)$ increases with $\alpha$
5. over time price converge to true value
6. prices follow a martingale $E_t [p_{t+1} | I_t] = p_t$
   (changes in prices are uncorrelated)
7. Simple setting price at $t$ depends only on $\#$ buy orders $-$ $\#$ sell orders
   (sequence of trades does not matter)
8. mid point of bid ask spread $\frac{a + b}{2}$ is not current market maker’s expectation.
Extensions

- Easley and O’Hara (1987)
  - ‘small and large’ order size
    - noise traders submit randomly a small or a large sized order
    - informed traders always prefer large order size (if bid and ask is the same for both order sizes)
      \[ \Rightarrow \text{m.m. will set larger spread for large orders} \]
  - Separating equilibrium
    - Informed traders’ order size is 2
    - Uninformed traders’ order size is 1 and 2 (exogenously given)
      \[ \Rightarrow \text{Spread for small orders} = 0 \]
  - Pooling equilibrium
    - Informed traders’ order size is 1 and 2
    - Uninformed traders’ order size is 1 and 2 (exogenously given)
      \[ \Rightarrow \text{Larger spread for larger orders} \]
Extensions

• “event uncertainty” (also in Easley & O’Hara (1992))
  • with prob \( \gamma \) info is like in Glosten & Milgrom
  • with prob \( (1 - \gamma) \) no news event occurs
    (nobody receives a signal)

• No-Trade \( \rightarrow \) signals that nothing has occurred!
  \( \Rightarrow \) quotes will pull towards \( \frac{1}{2} \)

  updating

  1. whether event has occurred AND
  2. about true value of the stock

• transaction price is still a Martingale
  but no longer Markov!
Herding - Avery & Zemsky (1998)

- Relates Glosten-Milgrom model to herding models (BHW 1992)
- Price adjustment eliminates herding and informational cascades if market maker learns at the same speed as other informed traders.
- Herding can still arise in a more general setting with event uncertainty and a more complicated information structure which guarantees that the market maker learns at a slower speed compared to other traders.
1987-Crash
Jacklin, Kleiden & Pfleiderer (1992)

Figure: Underestimating portfolio insurance traders \( \theta \).