Asset Pricing under Asymmetric Information
Strategic Market Order Models

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      ◦ uniform price setting
      ◦ limit order book analysis
  - dynamic sequential trade models with multiple trading rounds
- strategic market order models where the market maker sets prices ex-post
Strategic Market Order Models - Overview

- Kyle (1985) model
  - static version
  - dynamic version (in discrete time)
    - Refresher in Dynamic Programming
  - continuous time version (Back 1992)
- Multi-insider Kyle (1985) version
- Other strategic market order models
Kyle 1985 Model

- Model Setup
  - asset return $v \sim \mathcal{N}(p_0, \Sigma_0)$
  - Agents (risk neutral)
    - Insider who knows $v$ and submit market order of size $x$
    - Noise trader who submit market orders of exogenous aggregate size $u \sim \mathcal{N}(0, \sigma_u^2)$
    - Market maker sets competitive price after observing net order flow $X = x + u$
  - Timing (order of moves)
    - Stage 1: Insider & liquidity traders submit market orders
    - Stage 2: Market Maker sets the execution price
  - Repeated trading in dynamic version
### Kyle 1985 Model — Static Version

<table>
<thead>
<tr>
<th>Single informed trader</th>
<th>(Competitive) Market Maker</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0) Information</strong></td>
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</tr>
<tr>
<td>( v := \text{asset's payoff} )</td>
<td>( X = x + u \text{ batch net order flow} )</td>
</tr>
<tr>
<td><strong>1) Conjecture (price-rule)</strong></td>
<td><strong>1) Conjecture (insider trading rule)</strong></td>
</tr>
<tr>
<td>( p = \mu + \lambda(x + u) )</td>
<td>( x = \alpha + \beta v )</td>
</tr>
<tr>
<td><strong>2) No Updating</strong></td>
<td><strong>2) Updating</strong> ( E[v</td>
</tr>
<tr>
<td><strong>3) Optimal Demand</strong></td>
<td><strong>3) Price Setting Rule</strong></td>
</tr>
<tr>
<td>( \max_x E[(v - p)</td>
<td>v]x )</td>
</tr>
<tr>
<td>( \max_x E[v - \mu - \lambda x</td>
<td>v]x )</td>
</tr>
<tr>
<td>( \text{FOC: } x = -\frac{\mu}{2\lambda} + \frac{1}{2\lambda} v )</td>
<td>( p = p_0 + \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} {x + u - \alpha + \beta E[v]} )</td>
</tr>
<tr>
<td>( \text{SOC: } \lambda &gt; 0 )</td>
<td><strong>4) Correct Beliefs</strong></td>
</tr>
<tr>
<td>( \alpha = -\frac{\mu}{2\lambda}, \beta = \frac{1}{2\lambda} )</td>
<td>( \mu = p_0 \text{ Martingale}, \lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} )</td>
</tr>
</tbody>
</table>
Kyle 1985 Model — Static Version

- solve for unknown coefficients
  - 4 unknown Greeks
  - 4 equations

\[ \lambda = \frac{1}{2} \sqrt{\frac{\Sigma_0}{\sigma_u^2}} \]

- \( \lambda \) (illiquidity) decreases with noise trading, \( \sigma_u^2 \)
- \( \Sigma_0 \) reflect info advantage of insider
Dynamic Programming - A Refresher

- The Problem:
  max for several periods $t = 1, \ldots, T$ (discrete time)

$$\max_{u_t} E_t \left[ \sum_{s=1}^{T} v_s (\tilde{x}_s, u_s) \right] \quad \forall t \Rightarrow \text{(sequential rationality)}$$

under the following law of motion

$$\tilde{x}_{t+1} = f_t (\tilde{x}_t, u_t, \tilde{\varepsilon}_t)$$

$\tilde{x}_t$: vector of state variables (sufficient state space)
$u_t$: vector of control variables
$\varepsilon_t$: vector of random shocks

- Method
  - Backward Induction
  - Dynamic Programming
Dynamic Programming - A Refresher

- Define Value Function

\[ V_t(x_t) := E_t \left[ \sum_{s=t}^{T} v_s(\tilde{x}_s, u_s^*) \right] \]

- \Rightarrow Bellman Equation

\[ \max_{u_t} E_t [v_t(x_t, u_t) + V_{t+1}(x_{t+1})] \]

- Start at final date \( T \)

\[ V_{T+1}(\cdot) := 0 \]

\Rightarrow in \( t = T \)

\[ \max_{u_T} E_T [v_T(x_T, u_T)] \]

\[ \frac{\partial}{\partial u_T} E_T [v_T(x_T, u_T)]=0 \]

\Rightarrow \quad u_T^* = g_T(x_T)

\Rightarrow \quad V_T(x_T) = E_T [v_T(x_T, u_T^*)] \]
Dynamic Programming - A Refresher

- at **date** $T - 1$

$$\max_{u_{T-1}, u_T} \mathbb{E}_{T-1} \left[ \sum_{s=T-1}^{T} v_s (\bar{x}_s, u_s) \right]$$

given $V_T (x_T)$

$\Leftrightarrow \max_{u_{T-1}} \mathbb{E}_{T-1} \left[ v_{T-1} (x_{T-1}, u_{T-1}) + V_T (x_T) \right]$ given law of motion

$\Leftrightarrow \max_{u_{T-1}} \mathbb{E}_{T-1} \left[ v_{T-1} (x_{T-1}, u_{T-1}) + V_T (f_{T-1} (x_{T-1}, u_{T-1}, \tilde{\epsilon}_{T-1})) \right]$

$\Rightarrow u^*_{T-1} = g_{T-1} (x_{T-1})$

$\Rightarrow V_{T-1} = \mathbb{E}_{T-1} \left[ v_{T-1} (x_{T-1}, u^*_{T-1}) + V_T (f_{T-1} (x_{T-1}, u^*_{T-1}, \tilde{\epsilon}_{T-1})) \right]$

- and so on for **date** $T - 2$ etc. (and if they didn’t die in the uncertainties they are still solving ...)

- This process is quite time consuming.
Dynamic Programming - A Refresher

Alternative way:

- **Step 1:** “Guess” the general form of the value function

\[ V_{t+1} (x_{t+1}) = H_{t+1} (x_{t+1}) \]

e.g. \( H_{t+1}(x_{t+1}) = \alpha_{t+1} x_{t+1}^2 \)

- **Step 2:** Derive optimal level of current control

\[
\max_{u_t} E_t [ v_t (x_t, u_t) + H_{t+1} (x_{t+1}) ]
\]

\[
\max_{u_t} E_t [ v_t (x_t, u_t) + H_{t+1} (f_t (x_t, u_t, \varepsilon_t)) ]
\]

\[ \Rightarrow u_t^* = \cdots \]

- **Step 3:** Derive value function and check whether it coincides with general value function

\[ V_t (x_t) = E_t [ v_t (x_t, u_t^*) + H_{t+1} (f_t (x_t, u_t^*, \varepsilon_t)) ] \]

\[ \Rightarrow H_t (x_t) = \alpha_t x_t^2 \]
Kyle (1985) — Dynamic Version

Insider

• **Step 1:** Conjectured price setting strategy (pricing rule)

\[
p_n = p_{n-1} + \lambda_n \Delta X_n
\]

\[
= p_{n-1} + \lambda_n (\Delta x_n + \Delta u_n)
\]

(\[\frac{1}{\lambda_t} \approx \text{Liquidity}\])

• **Step 2:** ‘Guess’ Value function for insider’s profit pricing rule is linear \(\rightarrow\) guess quadratic value function)

\[
E[\pi_{n+1} | \tilde{p}_1, \ldots, p_n, v ] = \alpha_n (v - p_n)^2 + \delta_n
\]

Information set up to \(n\)

(expected expected profit from time \(n + 1\) onwards)

\[
\pi_n = E_n [\pi_{n+1} + (v - p_n) \Delta x_n^i]
\]
Kyle (1985) — Dynamic Version

**Insider ctd.**

- **Step 3:** Write Bellman Equation

\[
\max_{\Delta x_n^i} E \left[ (v - p_n) \Delta x_n^i + \alpha_n (v - p_n)^2 + \delta_n \mid p_1, \ldots, p_{n-1}, v \right]_{l_{n-1}}
\]

- **Step 4:** Given insider’s beliefs \( p_n = p_{n-1} + \lambda_n \Delta X_n \)

\[
\max_{\Delta x_n^i} E \left[ (v - p_{n-1} - \lambda_n \Delta x_n^i - \lambda_n \Delta u_n) \Delta x_n^i \right.
\]
\[+ \alpha_n (v - p_{n-1} - \lambda_n \Delta x_n^i - \lambda_n u_n)^2 + \delta_n \mid l_n \]

Take expectations

\[
\max_{\Delta x_n^i} \left[ (v - p_{n-1} - \lambda_n \Delta x_n^i) \Delta x_n^i + \alpha_n (v - p_{n-1} - \lambda_n \Delta x_n^i)^2 \right.
\]
\[+ \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n \]
\[u \Rightarrow p_n \text{ noisy} \]
Kyle (1985) — Dynamic Version

**Insider ctd.**

- **Step 5:** maximize

  \[ \text{FOC: } (v - p_{n-1}) - 2\lambda_n \Delta x^i_n - 2\alpha_n \lambda_n (v - p_{n-1}) + 2\alpha_n \lambda_n^2 \Delta x^i_n = 0 \]

  \[ \Delta x^i_n = \frac{1 - 2\alpha_n \lambda_n}{2\lambda_n (1 - \alpha_n \lambda_n)} (v - p_{n-1}) \]

  \[ := \beta_n \Delta t_n \]

  \[ \text{SOC: } \lambda_n (1 - \alpha_n \lambda_n) > 0 \]

- **Step 6:** Check whether ‘guessed’ value fcn is correct

  \[ E [\pi | I_{n-1}] = \max_{\Delta x^i_n} E \left[ (v - p_n) \Delta x^i_n + \alpha_n (\tilde{v} - \tilde{p}_n)^2 + \delta_n | I_{n-1} \right] \]

  \[ = \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1}, \text{ where} \]

  \[ \alpha_{n-1} = \frac{1}{4\lambda_n (1 - \alpha_n \lambda_n)}, \delta_{n-1} = \delta_n + \alpha_n (\lambda_n)^2 \sigma_u^2 \Delta t_n \]


Kyle (1985) — Dynamic Version

**Market Maker** (Filtering Problem)

- **Step 1:** MM’s belief about insider’s strategy
  \[
  \Delta x^i_n = \beta_n \Delta t_n (v - p_{n-1})
  \]
  \[
  \Delta X_n = \beta_n \Delta t_n (v - p_{n-1}) + \Delta u_n
  \]
  \[
  Var[\Delta u_n] = \sigma_u^2 \Delta t_n
  \]

- **Step 2:** MM’s filtering problem
  By definition:
  \[
  p_{n-1} : = E [v | \Delta X_1, \ldots, \Delta X_{n-1}]
  \]
  \[
  \Sigma_{n-1} : = Var [v | \Delta X_1, \ldots, \Delta X_{n-1}]
  \]
  \[
  E [\Delta X_n | \Delta X_1, \ldots, \Delta X_{n-1}] = \beta_n \Delta t_n E [(v - p_{n-1}) + \Delta u_n | \cdot \cdot \cdot]
  \]
  \[
  Var [\Delta X_n | \cdot \cdot \cdot] = (\beta_n \Delta t_n)^2 \Sigma_{n-1} + \sigma_u^2 \Delta t_n
  \]
  \[
  Cov [v, \Delta X_n | \cdot \cdot \cdot] = E [v (\beta_n \Delta t_n (v - p_{n-1}) + \Delta u_n | \cdot \cdot \cdot] = \beta_n \Delta t_n \Sigma_{n-1}
  \]
Kyle (1985) — Dynamic Version

Now we have all ingredients for the Projection Theorem

\[ p_n = p_{n-1} + \frac{\beta_n \Delta t_n \Sigma_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma^2_u} \Delta X_n \]

\[ : = \lambda_n \]

\[ \sum_n = V [v| \cdots \Delta X_n] = \Sigma_{n-1} - \frac{(\beta_n \Delta t_n)^2 \Sigma^2_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma^2_u} \]

\[ = \frac{\sigma^2_u \Sigma_{n-1}}{(\beta_n)^2 \Delta t_n \Sigma_{n-1} + \sigma^2_u} \]

\[ \Rightarrow \]

\[ \lambda_n = \frac{\beta_n \Delta t_n \Sigma_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma^2_u} \]

\[ \sum_n = (1 - \lambda_n \beta_n \Delta t_n) \Sigma_{n-1} = \frac{\sigma^2_u \lambda_n}{\beta_n} \]

\[ \Rightarrow \lambda_n = \frac{\beta_n \Sigma_n}{\sigma^2_u} \]
Step 3: Equate coefficients $\alpha_n, \beta_n, \delta_n, \sum_n$

\[
\begin{align*}
\beta_n \Delta t_n &= \frac{1 - 2\alpha_n \lambda_n}{2\lambda_n(1 - \alpha_n \lambda_n)} \\
\alpha_{n-1} &= \frac{1}{4\lambda_n(1 - \alpha_n \lambda_n)} \\
\delta_{n-1} &= \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n \\
\sum_n &= \sigma_u^2 \sum_{n-1} \\
\lambda_n &= \ldots
\end{align*}
\]

Solve recursive system of equations.

Interpretation of Equilibrium

- restrain from aggressive trading
  - price impact in current trading round
  - price impact in all future trading rounds
- ...

Generalizations of Kyle (1985)

- Multiple Insiders
  - all have same information
  - all hold different information
  - information is correlated
    ⇒ see Foster & Viswanathan, JF 51, 1437-1478

- Risk averse insiders
  - CARA utility

- etc. etc.