Supplementary Material for "Retrospective Search: Exploration and Ambition on Uncharted Terrain"

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Abstract

We provide characterization results pertaining to retrospective search with general utility functions. We also illustrate the impacts of exponential discounting costs on optimal retrospective search policies.

1 Beyond Risk Neutrality

In this section, we provide techniques for deriving the optimal policy for retrospective search when agents are risk averse. As a special case, we illustrate the optimal boundary for agents with constant relative risk aversion (CRRA) utilities.

We start by providing the proof of claim 3 in the appendix of the paper. We then deliver an alternative characterization of the stopping boundary.

Recall claim 3 in the appendix states the following:

Claim 3: *Let $w(\cdot)$ be the solution of the following Abel equation of the second kind:*

$$w(M)w'(M) - w(M) = \frac{(\sigma^*)^2}{2c(\sigma^*)} u'(M). \tag{1}$$

The optimal stopping boundary is given by:

$$g(M) = M - H'(M)\frac{(\sigma^*)^2}{2c(\sigma^*)},$$

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where $\frac{4c}{\sigma^2}w(M) = 2\sqrt{\frac{4c}{\sigma^2}(H(M) - u(M))}$.

Proof of Claim 3: We first identify an equivalent ordinary differential equation (ODE) for the stopping boundary. The relationship of this ODE which was originally derived for calculating the value of a stopping problem for a standard Brownian motion with fixed flow costs and the one in Peskir (1998) was noted Obłój (2007). We adapt the ODE to our setting, allowing for search scope and its associated cost.

Lemma 1 *Let* $H(M)$ *be defined as the minimal solution that satisfies* $H(M) \geq u(M)$ *to the differential equation*

$$H(M) - \frac{(\sigma^*)^2}{4c(\sigma^*)}(H'(M))^2 = u(M). \tag{2}$$

Then,

$$g(M) = M - H'(M)\frac{(\sigma^*)^2}{2c(\sigma^*)}.$$

Proof of Lemma 1: In the main text, we showed that the optimal stopping boundary is the maximal solution $g(M) \leq M$ of the following ODE:

$$g'(M) = \frac{u'(M)(\sigma^*)^2}{2c(\sigma^*)(M - g(M))}.$$

We now verify that the specification in the lemma's claim indeed satisfies this ODE.

From the first equality, equation (2), analysis identical to that of Obłój (2007) illustrates that the minimal solution satisfying $H(M) \geq u(M)$ corresponds to $H'(M)$ being chosen as the positive square root as follows:

$$\begin{aligned}
H'(M) &= \sqrt{\frac{4c(\sigma^*)}{(\sigma^*)^2}(H(M) - u(M))} \\
\implies H''(M) &= \frac{\frac{4c(\sigma^*)}{(\sigma^*)^2}(H'(M) - u'(M))}{2\sqrt{\frac{4c(\sigma^*)}{(\sigma^*)^2}(H(M) - u(M))}} \\
\iff H''(M) &= \frac{\frac{2c(\sigma^*)}{(\sigma^*)^2}(H'(M) - u'(M))}{H'(M)} \\
\iff H''(M) &= \frac{2c(\sigma^*)}{(\sigma^*)^2}\left(1 - \frac{u'(M)}{H'(M)}\right).
\end{aligned}$$

Now consider the equation for $g(M)$ in the lemma's claim. It implies that:

$$H'(M) = \frac{(M - g(M))2c(\sigma^*)}{(\sigma^*)^2} \text{ and}$$
$$g'(M) = 1 - \frac{H''(M)(\sigma^*)^2}{2c(\sigma^*)}.$$

Plugging in $H'(M)$ in $H''(M)$ derived above and then plugging $H''(M)$ in the expression for $g'(M)$, we have

$$\begin{aligned} g'(M) &= 1 - \frac{H''(M)(\sigma^*)^2}{2c(\sigma^*)} \\ \iff g'(M) &= \frac{u'(M)}{H'(M)} \\ \iff g'(M) &= \frac{u'(M)(\sigma^*)^2}{2c(\sigma^*)(M - g(M))}. \end{aligned}$$

Our choice of $H(M)$ as the minimal solution further guarantees that $g(M)$ as specified in the Lemma is the maximal solution of this last ODE satisfying $g(M) \leq M$.[1] Thus, we reach our original ODE formulation, which complete's the lemma's proof.

Going back to the proof of Claim 3, let $H(M)$ be defined by Lemma 1. As noted by Zaitsev and Polyanin (2002), introducing the transformation $\frac{4c(\sigma^*)}{(\sigma^*)^2}w = 2\sqrt{\frac{4c(\sigma^*)}{(\sigma^*)^2}(H-u)}$, equation (2) transforms into an *Abel equation of the second kind* in w,

$$ww' - w = \frac{(\sigma^*)^2}{2c(\sigma^*)}u'(M).$$

This, together with Lemma 1, completes the proof of Claim 3. ■

We now utilize the formulation offered by Claim 3 to offer methods for solving the optimal stopping boundary for non-linear utilities. As a special case, we apply these

[1]Indeed, notice that our selection of $H'(M)$ implies that

$$H'(M) = \sqrt{\frac{4c(\sigma^*)}{(\sigma^*)^2}\left(H(M) - u(M)\right)}$$

Thus,

$$g(M) = M - \frac{(\sigma^*)}{\sqrt{c(\sigma^*)}}.\sqrt{(H(M) - u(M))},$$

which is decreasing in $H(M) - u(M) \geq 0$.

techniques to identify a closed-form solution for the optimal stopping boundary corresponding to CRRA utilities.

Consider the function $H(M)$ identified in Lemma 1. We can introduce the substitution $y(M) = \frac{1}{w(M)}$ in the formulation (1) of Claim 3, which yields an *Abel equation of the first kind*:

$$y'(M) = -\frac{(\sigma^*)^2}{2c(\sigma^*)}\left(u'(M)\right)M^3 + \left(y(M)\right)^2.$$

Consider now the transformation $y(M) = -\frac{1}{tM'(t)}$ with t as a free variable. This yields an ODE of the *Emden-Fowler type*:

$$M''(t) = -t^{-2}\frac{(\sigma^*)^2}{2c(\sigma^*)}u'(M(t)). \tag{3}$$

This ODE is solved by Panayotounakos and Zarmpoutis (2011) and has the following parametrized solution, with $z = z(t)$ as the free variable. For simplicity, we drop the explicit dependence of $M(t)$ and $z(t)$ on t to get:

$$\frac{\frac{(\sigma^*)^2}{2c(\sigma^*)}u'(M)}{M} = \frac{(3 + C_1 z)z^4}{\left[(2 + C_1 z) \pm \sqrt{(2 + C_1 z)^2 - C_2 z^2}\right]^3},$$

where C_1 and C_2 are constants of integration that parametrize the solution, and z satisfying:

$$t = t(z) = \frac{z}{(2 + C_1 z) \pm \sqrt{(2 + C_1 z)^2 - C_2 z^2}}. \tag{4}$$

From the above two equations we can conclude the following:

$$\frac{\frac{(\sigma^*)^2}{2c(\sigma^*)}u'(M)}{M} = (3z + C_1 z^2)(t(z))^3,$$

with z as a free parameter. Similarly, inverting equation (4), we get:

$$z(t) = \frac{4t}{C_2 t^2 - 2C_1 t + 1}$$

In general, for any utility function, we can attempt getting a parametric solution using equations (4) and (3). However, the term $\frac{\frac{(\sigma^*)^2}{2c(\sigma^*)}u'(M)}{M}$ suggests that some forms are easier to tackle compared to others. In particular, plugging in the CRRA form $u(M) = \frac{M^{1-\rho}}{1-\rho}$, we have

$$(M(t(z)))^{-\rho-1} = \frac{2c(\sigma^*)}{(\sigma^*)^2}(3z + C_1 z^2)(t(z))^3$$

$$\implies M(t(z)) = \left[\frac{2c(\sigma^*)}{(\sigma^*)^2}(3z + C_1 z^2)(t(z))^3\right]^{\frac{1}{-\rho-1}},$$

which, inverting t and z, can be written as:

$$M(t) = \left[\frac{2c(\sigma^*)}{(\sigma^*)^2}(3z(t) + C_1 (z(t))^2) t^3\right]^{\frac{1}{-\rho-1}}.$$

Let $\frac{2c(\sigma^*)}{(\sigma^*)^2}(3z(t) + C_1(z(t))^2) = P(t)$, so that

$$M(t) = \left[P(t) t^3\right]^{\frac{1}{-\rho-1}}.$$

Recall that

$$\begin{aligned} w(M(t)) = -tM'(t) &= M(t)\frac{1}{1+\rho}\frac{tP'(t) + 3P(t)}{P(t)} \\ &= M(t)\frac{1}{1+\rho}\left(\frac{tP'(t)}{P(t)} + 3\right). \end{aligned}$$

Plugging in the functional form of $z(t)$, we have:

$$P(t) = \frac{2c(\sigma^*)}{(\sigma^*)^2}\left[\frac{16C_1 t^2}{\left(C_2 t^2 - 2C_1 t + 1\right)^2} + \frac{12t}{C_2 t^2 - 2C_1 t + 1}\right],$$

$$P'(t) = \frac{2c(\sigma^*)}{(\sigma^*)^2}\left[-\frac{32C_1 t^2 (2C_2 t - 2C_1)}{\left(C_2 t^2 - 2C_1 t + 1\right)^3} + \frac{32C_1 t}{\left(C_2 t^2 - 2C_1 t + 1\right)^2} - \frac{12t(2C_2 t - 2C_1)}{\left(C_2 t^2 - 2C_1 t + 1\right)^2} + \frac{12}{C_2 t^2 - 2C_1 t + 1}\right].$$

Since $\left[w(M)\right]^2 = \frac{(\sigma^*)^2}{c}(H(M) - u(M))$, we get:

$$H(M(t)) = \frac{M(t)^{1-\rho}}{1-\rho} + \frac{c}{(\sigma^*)^2}\left[M(t)\frac{1}{1+\rho}(\frac{tP'(t)}{P(t)} + 3)\right]^2,$$

$$H(M(t)) = \frac{M(t)^{1-\rho}}{1-\rho} + \frac{c}{(\sigma^*)^2}\left[w(M(t))\right]^2.$$

Substituting $P(t)$ into the expression for $M(t)$ yields:

$$M(t) = \left(\frac{2c(\sigma^*)4t^4 \left(3C_2t^2 - 2C_1t + 3\right)}{(\sigma^*)^2 \left(C_2t^2 - 2C_1t + 1\right)^2} \right)^{-\frac{1}{\rho+1}}.$$

Now, taking the derivative with respect to t generates

$$H'(M)M'(t) = u'(M)M'(t) + \frac{c}{(\sigma^*)^2} \left[2w(M(t))w'(M(t))M'(t)\right].$$

By definition,

$$\frac{d\left[w(M(t))\right]}{dt} = w'(M)M'(t).$$

Recall that $w(M(t)) = -tM'(t)$. Thus,

$$w(M(t))w'(M) = -t\frac{d\left[w(M(t))\right]}{dt}.$$

Therefore, cancelling out $M'(t)$ on both sides, we get

$$H'(M) = u'(M) - \frac{c}{(\sigma^*)^2}\left[2t\frac{d\left[w(M(t))\right]}{dt}\right].$$

Recalling that $-t^2M''(t) = \frac{(\sigma^*)^2}{2c(\sigma^*)}u'(M(t))$ and $w(M(t)) = -tM'(t)$,

$$H'(M) = -\frac{2c(\sigma^*)}{(\sigma^*)^2}w(M).$$

Now, observe that for the stopping boundary to be valid, we need u to be increasing over the domain of the process as otherwise we can potentially have $u(X) > u(M)$. For CRRA, we know u is increasing over $[0, \infty)$ so a natural restriction is to impose that the underlying process never reaches 0. This implies that the problem is only well defined whenever $M_0 = X_0 > \underline{M} = \underline{X} > 0$ such that $g(\underline{M}) = 0$, which we identify below. The restriction that the boundary hits 0 at some $\underline{M}$, namely $g(\underline{M}) = 0$, is what allows us to identify the maximal solution of $g(M) \leq M$ (as noted in Obłój (2007) for diffusions with bounded domain). This defines an additional boundary condition that needs to be satisfied by the ODE. That is,

$$g(\underline{M}) = \underline{M} - H'(\underline{M})\frac{(\sigma*)^2}{2c(\sigma^*)} = 0 \quad \Longrightarrow \quad \underline{M} + w(\underline{M}) = 0.$$

Let $\bar{M}$ be the minimal value of observed maximum such that the agent stops searching whenever $M \geq \bar{M}$. If the agent never stops when reaching the observed maximal value, we let $\bar{M} = \infty$. The relevant domain of parameters t then corresponds to the set T such that for any $M \in \left[\underline{M}, \bar{M}\right]$ there exists $t \in T$ such that $M(t) = M$.

For some $\underline{t} \in T$, the level $\underline{M}$ can be defined parametrically as

$$\underline{M} = M(\underline{t}) = \left(\frac{2c(\sigma^*)4\underline{t}\,(3C_2\underline{t} - 2C_1\underline{t} + 3)}{(\sigma^*)^2\,(C_2\underline{t} - 2C_1\underline{t} + 1)^{\,2}}\right)^{-\frac{1}{\rho+1}}.$$

Plugging this parametric identity into the boundary condition leads to

$$\begin{aligned} &M(\underline{t}) = -w(M(\underline{t})). \\ &\quad \Longrightarrow M(\underline{t}) = -M(\underline{t})\frac{1}{1+\rho}\left(\frac{\underline{t}P'(\underline{t})}{P(\underline{t})} + 3\right). \\ &\quad \Longrightarrow -1 = \frac{1}{1+\rho}\left(\frac{\underline{t}P'(\underline{t})}{P(\underline{t})} + 3\right). \end{aligned}$$

Since $H'(M) = -w(M)$, and $g(M) = M - H'(M)\frac{(\sigma^*)^2}{2c(\sigma^*)}$, for the stopping boundary $g(M)$ to satisfy our requirement that $g(M) \leq M$, we must have that $w(M(t)) \leq 0$ for all $t \in T$ given the choice of C_1 and C_2. This implies that, for all $t \in T$, we must have $-w(M(t)) = tM'(t) \geq 0$, and thus $M'(t)$ has the same sign as t within T. Given our expression for $M(t)$ above, it follows that, for any selection of C_1 and C_2, $0 \notin T$. In fact, our restriction that t and $M'(t)$ coincide in signs implies that there exists $\varepsilon > 0$ small enough such that $(-\varepsilon, \varepsilon) \cap T = \varnothing$. Similarly, for large enough $\tilde{t} > 0$, we have that $(-\infty, -\tilde{t}) \cap T = \varnothing$ and $(\tilde{t}, \infty) \cap T = \varnothing$. From continuity, it follows that $T = [\underline{t}, \bar{t}]$, where $M(\underline{t}) = \underline{M}$ and $M(\bar{t}) = \bar{M}$. From the definition of $\bar{M}$, it follows that $g(\bar{M}) = \bar{M}$ so that $H(M(\bar{t})) = u(M(\bar{t}))$.

Recalling that $H(M) = u(M) + \frac{c(\sigma^*)}{(\sigma^*)^2}w(M)^2$ implies $w(M(\bar{t})) = 0$ as a second boundary condition, and thus

$$w(M(\bar{t})) = 0 = (\frac{\bar{t}P'(\bar{t})}{P(\bar{t})} + 3).$$

Combining the ODE with its boundary conditions, we can pin down a parametric solution of C_1, $\underline{t}$, and $\bar{t}$ as a function of C_2 and therefore an exact solution for the stopping boundary.[2] Qualitatively, the solution implies that $\bar{M}$, the level of the maximal observed value at which search ceases, is decreasing in the degree of risk aversion ρ. Intuitively, as the agent becomes more risk averse, the marginal returns from improving the current maximal value decline. Marginal search costs, however, are unchanged. Those costs then overwhelm the benefit of search at lower values of the maximum value.

2 Introducing Discounting

Suppose that search costs are derived from exponential discounting rather than from flow costs. Formally, consider an agent facing a fixed search scope σ and maximizing an objective of the form $e^{-rt}M_t$, where $r > 0$ is the agent's discount factor. Since $\ln(\cdot)$ is strictly increasing, we can write the agent's optimization problem as:

$$\begin{aligned} &\max_{\tau} \mathbf{E}\left(\ln M_\tau - r\tau\right) \\ dX_t &= \sigma dB_t \\ M_t &= \max_{0\leq s\leq t}\left(X_s \vee M_0\right) \\ X_0 &= M_0 = 0. \end{aligned}$$

This is equivalent to the optimization problem analyzed in the previous section taking the utility to be $u(M) = \ln(M)$ (with constant search costs of r). As it turns out, there is a readily available parametrized solution to the ODE specified in equation (2) in Section 1.6.3.13 of Zaitsev and Polyanin (2002). Let

$$F = \left[\int e^{\pm z^2} dz + C\right]^{-1}.$$

Then the solution in parametric form is given by

$$\begin{aligned} M(z) &= \frac{\sigma}{\sqrt{c}} F e^{\pm z^2} \\ H(z) &= \left[(2z \pm Fe^{\pm z^2})^2 \pm 4log(\frac{\sigma}{\sqrt{r}}F) - 4z^2\right], \end{aligned}$$

[2]Relevant *Mathematica* code is available from the authors upon request.

with τ as a parameter and C as a constant of integration to be chosen. From Lemma 1, one needs to find the minimal $H(M) \geq u(M)$ that satisfies these equalities in order to obtain the closed-form solution for the optimal stopping boundary $g(M)$.

One could also extend our analysis to a setting in which the agent experiences discounting of rewards, on top of the flow costs due to experimentation. Discounting effectively translates into a termination rate of the process X, often referred to as the *killing rate*. Introducing discounting then follows very closely the analysis of Peskir (1998) and Pedersen (2000).

Consider the discounted search problem with discount rate $r > 0$. That is, consider an agent facing the following optimization problem:

$$\max_{\tau, \{\sigma_t\}_{t=0}^{\tau}} \mathbf{E}\left(e^{-r\tau} M_\tau - \int_0^\tau [e^{-rt} c(\sigma_t)] dt\right)$$

$$\begin{aligned} dX_t &= \sigma_t dB_t \\ M_t &= \max_{0 \leq s \leq t} (X_s \vee M_0) \\ X_0 &= M_0 = 0. \end{aligned}$$

In order to introduce "killing" of the original diffusion X_t, we add a new state Δ to the original state space $\mathbb{R}$ and define the killed process $\hat{X}$. Once the process is "killed," it takes value Δ and stays at Δ. To make the distinction between Δ and possible final rewards clear, we assume that $\Delta < M_0$. The probability that the process $\hat{X}$ gets killed at time t is given by $1 - e^{-rt}$. Similarly, define $\hat{M}_t = \max_{0 \leq s \leq t}(\hat{X}_s \vee M_0)$. We can then write the agent's problem in a more familiar form, with a slightly different diffusion.

$$\max_{\tau, \{\sigma_t\}_{t=0}^{\tau}} \mathbf{E}\left(\hat{M}_\tau - \int_0^\tau [c(\sigma_t)] dt\right).$$

By definition of killing, the infinitesimal generator of the process $\hat{Z} = (\hat{M}, \hat{X})$, denoted by $\mathcal{A}_{\hat{Z}}$, is equal to

$$\mathcal{A}_{\hat{Z}} = \mathcal{A}_Z - r.$$

From Claim 1 in the Appendix of the main text, we already know that $\mathcal{A}_Z = \mathcal{A}_X$. We can therefore write the HJB as follows:

$$0 = \sup_{\sigma_t} \left\{ \hat{M}_t - V(\hat{Z}_t), \mathcal{A}_x^{\sigma_t} V(\hat{Z}) - c(\sigma_t) - rV(\hat{Z}) \right\}.$$

The process $\hat{Z}$ behaves identically to the process Z, except when it is killed, which happens with an exponential rate r. If the process has been killed at time t, we know that $V(\Delta, M_t) = M_t$. If the process is not killed, again we must have a stopping rule of the form $\tau^* = \inf\{t \geq 0 : X_t \leq \hat{g}(M_t)\}$.

An analytical characterization of $\hat{g}(M)$ is challenging. Unlike in our original setting, the stopping boundary has to account for random and exogenous stops corresponding to the hazard rate induced by discounting. As noted in Peskir (1998), this stopping boundary still satisfies

$$\begin{aligned}
&\frac{(\sigma_t)^2}{2}\frac{\partial^2 V}{\partial X^2} = c(\sigma_t) \text{ for } \hat{g}(M) < x < M && \text{(Continuation Region)}\\
&V(M,X)|_{x=\hat{g}(M)} = M && \text{(Value Matching)}\\
&\frac{\partial V(M,\hat{g}(M))}{\partial X} = 0 && \text{(Smooth Pasting).}
\end{aligned}$$

However, the maximality principle may no longer be valid, making it difficult to identify the right solution to this ODE.

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