Some Technical Results

We first present some technical results that will be useful in establishing the properties of the functional $U\left(\left\{ C_{T}, L_{T}\right\} _{T=0}^{\infty}\right)$.

**Definition B1** Let $X$ and $Z$ be Banach spaces and $G : X \rightarrow Z$ be a vector-valued mapping. Suppose that $G$ is continuously (Fréchet) differentiable in the neighborhood of $x_{0}$ with the derivative denoted by $G'(x_{0})$. Then $x_{0}$ is said to be a **regular point** of $G$ if $G'(x_{0})$ maps $X$ onto $Z$.

**Lemma B1** Let $X$ and $Z$ be Banach spaces. Consider the maximization problem of

$$P(u) = \max_{x \in X} f(x)$$  \hspace{1cm} (B1)

subject to

$$g_{0}(x) \leq u$$  \hspace{1cm} (B2)

and

$$G(x) \leq 0$$  \hspace{1cm} (B3)

where $f : X \rightarrow R$ and $g_{0} : X \rightarrow R$ are real-valued functions and $G : X \rightarrow Z$ is a vector-valued mapping and $0$ is the zero of the Banach space $Z$. Suppose that $f$ is concave and $g_{0}$ is convex, and moreover that the solution at $u = 0$, $x_{0}$, is a regular point. Let $\mu$ be any multiplier of (B2). Then $\mu$ is a subgradient of $P(0)$.

**Proof.** This lemma is a direct generalization of Proposition 6.5.8 of Bertsekas, Nedic and Ozdaglar (2003, p. 382) to an infinite dimensional maximization problem.

**Theorem B1** Let $X$ and $Z$ be Banach spaces. Consider the maximization problem of

$$P(u) = \max_{x \in X} f(x)$$

subject to

$$G(x) \leq 0 + u$$

where $f : X \rightarrow R$ is a real-valued concave function and and $G : X \rightarrow Z$ is a convex vector-valued mapping and $0$ is the zero of the vector space $Z$ and $u$ is a perturbation. Suppose that $x_{0}$ is a solution to this program. Suppose also that $x_{0}$ is a regular point of $G$ and that $f$ and $G$ are continuously (Fréchet) differentiable in the neighborhood of $x_{0}$. Then $P(0)$ is differentiable.
Proof. From Lemma B1, it follows that if there is a unique multiplier, $P$ has a unique subgradient and is thus differentiable. Proposition 4.47 in Bonnans and Shapiro (2000) establishes that under a weaker constraint qualification condition than regularity, this problem has a unique multiplier. 

**Theorem B2** Let $X$ be a compact metric space, then the space of probability measures defined on $X$ is a compact metric space with the weak topology.

**Proof.** See Parthasarathy (1967, p. 45). 

**Randomizations**

We next introduce randomizations to show concavity and differentiability of $U(\{C_t, L_t\}_{t=0}^{\infty})$. To simplify notation, in this appendix, we suppress dependence on public histories $h_t$. The original maximization problem without randomization is to maximize (3.10) subject to (3.11), (3.12), and (3.14) as stated in Proposition 2. Recall also that $\theta_t \in \Theta$, where $\Theta$ is at finite set (with $N + 1$ elements). Therefore $\Theta^t$ for any $t < \infty$ is also a finite set. Consider next the functions $c_t : \Theta^t \to \mathbb{R}^+$ and $l_t : \Theta^t \to [0, l]$. By definition, these functions assign values to a finite number of points in the set $\Theta^t$ for any $t < \infty$, thus can simply be thought of as vectors of $(N(N + 1))^t$ dimension. Moreover

$$\int c_t(\theta^t) dG(\theta^t) \leq \bar{Y}, K_{t+1} \leq \bar{Y} \text{ and } x_t \leq \bar{Y},$$

(B4)

where $\bar{Y} = F(\bar{Y}, l) < \infty$. Therefore, $X_t = \{c_t(\theta^t), l(\theta^t), K_{t+1}, x_t\}$ is a vector (of dimension $(N(N + 1))^{2t} + 2$). Let $X_t$ be the set of all such vectors that satisfy the inequalities in (B4), and for $X_t \in X_t$, let $X_t(i)$ denote the $i$th component of this vector, and $T_t$ be the dimension of vectors in the set $X_t$ (i.e., $T_t = (N(N + 1))^{2t} + 2$). $X_t$ is a compact metric space with the usual Euclidean distance metric, $d_t(X_t, X') = \left( \sum_{i=1}^{T_t} (X_t(i) - X'(i))^2 \right)^{1/2}$

Let us now construct the product space of the $X_t$’s

$$X = \prod_{t=1}^{\infty} X_t$$

Clearly the sequence $\{c_t(\theta^t), l(\theta^t), K_{t+1}, x_t\}_{t=0}^{\infty}$ must belong to $X$. In fact, it must belong to the subset of $X$, which satisfies (3.11), (3.12), and (3.14), denoted by $\tilde{X}$.

Now by Tychonoff’s theorem (e.g., Dudley, 2002, Theorem 2.2.8), $X$ is compact in the product topology. Since (3.11), (3.12), and (3.14) are (weak) inequalities, $\tilde{X}$ is a closed subset of $X$, and therefore it is also compact in the product topology. Moreover, $X$ with the product topology is meterizable, with the metric

$$d(X, X') = \sum_{t=1}^{\infty} \phi^t d_t(X_t, X'_t)$$

(B5)

for some $\phi \in (0, 1)$ and $X \equiv \{X_t\}_{t=0}^{\infty} \in X$. This shows that $X$ endowed with the product topology is a metric space, and so is $\tilde{X}$. 

From Theorem B2, the set of probability measures defined over a compact metric space is compact in the weak topology. This establishes that the set of probability measures \( \mathcal{P}^\infty \) defined over \( \mathcal{X} \) is compact in the weak topology.

We are concerned not with all probability measures, but those that condition at \( t \) on information revealed up to \( t \). Let \( \mathcal{C} = \{(c, l) \in \mathbb{R}^2 : 0 \leq c \leq \bar{c}, \ 0 \leq l \leq \bar{l}\} \) be the set of possible consumption-labor allocations for agents, so that \( \mathcal{P}^\infty \) defined above is the set of all probability measures over \( \mathcal{C}^\infty \). Now, for each \( t \in \mathbb{N} \) and \( \theta^{t-1} \in \Theta^{t-1} \), let \( \mathcal{P}^t[\theta^{t-1}] \) be the space of \( N + 1 \)-tuples of probability measures on Borel subsets of \( \mathcal{C} \) for an individual with history of reports \( \theta^{t-1} \). Thus each element \( \zeta(\cdot | \theta^{t-1}) = [\zeta(\theta_0 | \theta^{t-1}), \ldots, \zeta(\theta_N | \theta^{t-1})] \) in a \( \mathcal{P}^t[\theta^{t-1}] \) consists of \( N + 1 \) probability measures for each type \( \theta_i \), given their past reports, \( \theta^{t-1} \), and is thus closed. Consider \( \mathcal{P} = \bigcup_{t \in \mathbb{N}} \bigcup_{\theta^{t-1} \in \Theta^{t-1}} \mathcal{P}^t[\theta^{t-1}] \), which is a closed subset of \( \mathcal{P}^\infty \). Since a closed subset of a compact space is compact (e.g., Dudley, 2002, Theorem 2.2.2), \( \mathcal{P} \) is compact in the weak topology.

Finally, choosing \( \phi \leq \beta \) in (B5) shows that the objective function is continuous in the weak topology. This establishes that including randomizations, we have a maximization problem over probability measures in which the objective function is continuous in the weak topology, and the constraint set is compact in the weak topology, and thus there exists a probability measure that reaches the maximum.

**Properties of \( \mathcal{U}((C_t, L_t)_{t=0}^\infty) \)**

We now establish the main properties of \( \mathcal{U}((C_t, L_t)_{t=0}^\infty) \). The only additional restriction is that in all the proofs we assume that the solution to the maximization problem (3.10) is at a regular point. This needs to be imposed as an assumption, since it is not possible to check that the solution is indeed at a regular point. Nevertheless, this assumption is not a strong one, since if the solution is not at their regular point, a perturbation of the utility functions or the production function should ensure that the solution shifts to a regular point (i.e., solutions that are not at regular points in this context are “non-generic,” though we do not present a precise mathematical statement of this property to economize on further notation and space).

**Lemma B2** \( \mathcal{U}((C_t, L_t)_{t=0}^\infty) \) is continuous and concave on \( \Lambda^\infty \), nondecreasing in \( C_s \) and nonincreasing in \( L_s \) for any \( s \) and differentiable in \( (C_t, L_t)_{t=0}^\infty \).

**Proof.** The above argument established that in the problem of maximizing (3.10) subject to (3.11), (3.12), and (3.14) over probability measures, a maximum exists and \( \mathcal{U}((C_t, L_t)_{t=0}^\infty) \) is therefore well defined.
To show concavity, consider \((C^0, L^0)\) and \((C^1, L^1)\) and corresponding \(\zeta^0, \zeta^1\). We have
\[
\int (u(c, l; \theta) - u(c, l; \hat{\theta}))\zeta^\alpha(d(c, l), \theta) = \alpha \int (u(c, l; \theta) - u(c, l; \hat{\theta}))\zeta^0(d(c, l), \theta) + (1 - \alpha) \int (u(c, l; \theta) - u(c, l; \hat{\theta}))\zeta^1(d(c, l), \theta) \geq 0
\]

In a similar way we can show that \(\zeta^\alpha\) satisfies (3.11), (3.12), and (3.14), this convex combination is feasible and it gives the same utility as \(\alpha \zeta^0 \cdot u(\theta) + (1 - \alpha) \zeta^1 \cdot u(\theta)\).

Next, note that the constraint set expands if \(C_s\) increases or \(L_s\) decreases for any \(s\), therefore \(U\) must be weakly increasing in \(C_s\) and weakly decreasing in \(L_s\).

Finally, returning to the original topology, \(U(\{C_t, L_t\}_{t=0}^\infty)\) is defined over a Banach space. Given the assumption that the solution to (3.10) is at a regular point, we can use Theorem B1 to conclude that \(U(\{C_t, L_t\}_{t=0}^\infty)\) is differentiable in \(\{C_t, L_t\}_{t=0}^\infty\), completing the proof.

\textbf{Lemma B3} \(\Lambda^\infty\) is compact and convex.

\textbf{Proof.} \textbf{(Convexity)} Consider \(\{C_t, L_t\}_{t=0}^\infty\) and \(\{C'_t, L'_t\}_{t=0}^\infty\) \(\in \Lambda^\infty\) and some \(\zeta^0, \zeta^1\) feasible for \(\{C_t, L_t\}_{t=0}^\infty\) and \(\{C'_t, L'_t\}_{t=0}^\infty\), respectively. Now for any \(\alpha \in (0, 1)\), \(\zeta^\alpha \equiv \alpha \zeta^0 + (1 - \alpha) \zeta^1\) is feasible for \(\{\alpha \{C_t, L_t\}_{t=0}^\infty + (1 - \alpha) \{C'_t, L'_t\}_{t=0}^\infty\}\), so that this set is non-empty. Moreover, since \(\zeta^0, \zeta^1\) satisfy the incentive compatibility constraints, \(\zeta^\alpha\) satisfies it as well. Similarly, \(\zeta^\alpha\) satisfies the constrains on aggregate \(\{C_t, L_t\}_{t=0}^\infty\).

\textbf{(Compactness)} For any sequence \(\{C_t^n, L_t^n\}_{t=0}^\infty \in \Lambda^\infty\), \(\{C_t^n, L_t^n\}_{t=0}^\infty \to \{C_t^\infty, L_t^\infty\}_{t=0}^\infty\), there exists a sequence \(\{\zeta^n\}_{t=0}^\infty\) corresponding to \(\{C_t^n, L_t^n\}_{t=0}^\infty\), such that \(\zeta^n \to \zeta^\infty\), satisfying the incentive compatibility, aggregate constraints and feasibility, therefore \(\{C_t^\infty, L_t^\infty\}_{t=0}^\infty \in \Lambda^\infty\) is closed. Boundedness follows from boundedness of \(C\) and \(L\).

\textbf{PROOF OF THEOREM ??}

\textbf{Proof.} We showed above that, when randomizations are introduced, \(U(\{C_t, L_t\}_{t=0}^\infty)\) is a well-defined functional and is continuous, concave, and differentiable. In this proof, we suppress randomization to simplify notation.

We write the problem of characterizing the best sustainable mechanism non-recursively following Marcet and Marimon (1998) as
\[
\max_{\{C_t, L_t, K_t, x_t\}_{t=0}^\infty} \mathcal{L} = U(\{C_t, L_t\}_{t=0}^\infty) + \sum_{t=0}^\infty \delta^t \left\{ \mu_t v(x_t) - (\mu_t - \mu_{t-1}) v(F(K_t, L_t)) \right\} \quad \text{(B6)}
\]
subject to
\[
C_t + x_t + K_{t+1} \leq F(K_t, L_t), \quad \text{and} \quad \text{(B7)}
\]
for all \( t \), where \( \mu_t = \mu_{t-1} + \psi_t \) with \( \mu_{-1} = 0 \) and \( \delta \psi_t \geq 0 \) is the Lagrange multiplier on the constraint (3.18). The differentiability of \( \Upsilon((C_t, L_t)_{t=0}^{\infty}) \) implies that for \( (C_t, L_t)_{t=0}^{\infty} \in \text{Int}\Lambda^{\infty} \), we have:

\[
\begin{align*}
\Upsilon_{C_t} &= \delta^t (\mu_t - \mu_{t-1}) v'(F(K_t, L_t)) F_{L_t} - \Upsilon_{C_t} \cdot F_{L_t} \\
\Upsilon_{C_t} &= [\Upsilon_{C_{t+1}} - \delta^t (\mu_{t+1} - \mu_t) v'(F(K_{t+1}, L_{t+1}))] F_{K_{t+1}}
\end{align*}
\]  

(B8)

(B9)

Since \( \mu_t \geq \mu_{t-1} \), there will be downward labor and intertemporal distortions whenever \( \mu_t > \mu_{t-1} \) and \( \mu_{t+1} > \mu_t \), respectively, i.e., whenever \( \psi_t > 0 \) and \( \psi_{t+1} > 0 \).

**Part 1:** Suppose to obtain a contradiction that \( \mu_t = 0 \) for all \( t \geq 0 \). Then, no consumption is allocated to the politician, \( x_t = 0 \) for all \( t \). But in this case, if \( L_t > 0 \) for any \( t \), then the politician can improve by expropriating the entire output at \( t \). Thus we must have \( L_t = 0 \) for all \( t \). Since, by hypothesis, \( (C_t, L_t)_{t=0}^{\infty} \in \text{Int}\Lambda^{\infty} \) with \( L_t > 0 \) is feasible and the associated \( (C_t, L_t)_{t=0}^{\infty} \in \text{Int}\Lambda^{\infty} \) necessarily gives higher ex ante utility to citizens than \( L_t = C_t = 0 \), the plan with \( L_t = 0 \) for all \( t \) cannot be optimal. Therefore, the sustainability constraint of politician (3.14) must bind at some \( t \) with \( \psi_t > 0 \). Then (B8) implies that there will be downward labor distortions at that \( t \), and (B9) implies that there will be downward intertemporal distortions at \( t-1 \).

**Part 2:** We start by proving that \( \varphi \equiv \inf \{ \varphi \in [0, 1] : \lim_{t \to \infty} t^{-1} \Upsilon_{C_t}^* = 0 \} \) is well-defined and strictly less than 1. To see this, recall that by hypothesis, a steady state exists, so that \( (C_t, L_t, K_{t+1})_{t=0}^{\infty} \to (C^*, L^*, K^*) \), thus \( (C_t)_{t=0}^{\infty} \) is in the space \( c \) of convergent infinite sequences (rather than simply in the space of all bounded infinite sequences, \( \ell_{\infty} \)). The dual of \( c \) is \( \ell_1 \), that is, the space of sequences \( \{y_t\}_{t=0}^{\infty} \) such that \( \sum_{t=0}^{\infty} |y_t| < \infty \). Since \( \Upsilon_{C_t} \) is equal to the Lagrange multiplier for the constraint (3.18), it lies in the dual space of \( (C_t)_{t=0}^{\infty} \) (see, e.g., Luenberger, 1969, Chapter 9), thus in \( \ell_1 \), which implies that \( \lim_{t \to \infty} \Upsilon_{C_t} = 0 \), hence \( \varphi < 1 \).

Rearranging equations (B8) and (B9) and substituting for \( \Upsilon_{C_t} \) and taking the limit as \( t \to \infty \), we have

\[
\begin{align*}
-\frac{\Upsilon_{C_t}^*}{\Upsilon_{C_t}^* F_{L_t}(K^*, L^*)} &= 1 - \frac{(\mu_t - \mu_{t-1}) v'(F(K_t, L_t))}{\mu_t v'(x^*)} \\
\frac{F_{K_{t+1}}(K^*, L^*) \Upsilon_{C_t}^*}{\Upsilon_{C_t}} &= 1 + \frac{(\mu_{t+1} - \mu_t) v'(F(K^*, L^*)) F_{K_{t+1}}(K^*, L^*)}{\mu_t v'(x^*)},
\end{align*}
\]  

(B10)

(B11)

where all derivatives are evaluated at the limit \( (C^*, L^*, K^*) \).

The first-order condition with respect to \( x_t \) then implies:

\[
\frac{\Upsilon_{C_t}}{\delta^t v'(x_t)} = \mu_t \leq \mu_{t+1} = \frac{\Upsilon_{C_{t+1}}}{\delta^{t+1} v'(x_{t+1})}.
\]  

(B12)

By construction, \( \mu_t \) is an increasing sequence, so it must either converge to some value \( \mu^* \) or go to infinity. Since as \( t \to \infty \) an interior steady state \( (C^*, L^*, K^*, x^*) \) exists by hypothesis and \( \Upsilon_{C_t}^* \) is proportional
to $\varphi^t$, (B12) can be written as

$$\frac{\varphi^t U^*_t}{\delta^t \varphi'(x^*)} = \mu_t \leq \mu_{t+1} = \frac{\varphi^{t+1} U^*_t}{\delta^{t+1} \varphi'(x^*)} \text{ as } t \to \infty. \quad (B13)$$

Since $\varphi = \delta$, we have that (B13) implies that as $t \to \infty$, $|\mu_{t+1} - \mu_t| \to 0$ and $\mu_t \to \mu^* \in (0, \infty)$ (where the fact that $\mu^* > 0$ follows from Part 1, since $\mu_{t+1} \geq \mu_t$ and $\mu_t > 0$ for some $t$). Therefore, $(\mu_t - \mu_{t-1})/\mu_t \to 0$, and distortions disappear asymptotically.

**Part 3:** Suppose that $\varphi > \delta$. In this case, (B12) implies that $U^*_t$ is proportional to $\varphi^t$ as $t \to \infty$. This implies that $(\mu_t - \mu_{t-1})/\mu_t > 0$ as $t \to \infty$, so from (B8) and (B9), aggregate distortions cannot disappear, completing the proof. ■

**REFERENCES FOR APPENDIX B**


