Week 9: What Can Go Wrong and How To Fix It, Diagnostics and Solutions

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Princeton

November 14, 16 and 21, 2016

1These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller and Kevin Quinn.
Where We’ve Been and Where We’re Going...

Last Week
- regression in the social sciences

This “Week”
- Monday (14):
  ⋆ unusual and influential data → robust estimation
- Wednesday (16):
  ⋆ non-linearity → generalized additive models
- Monday (21):
  ⋆ unusual errors → sandwich SEs and block bootstrap

After Thanksgiving
- causality with measured confounding

Long Run
- regression → diagnostics → causal inference

Questions?

Stewart (Princeton)
Week 9: Diagnostics and Solutions
November 14-21, 2016
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Questions?
1. Assumptions and Violations
2. Non-normality
3. Outliers
4. Robust Regression Methods
5. Optional: Measurement Error
6. Conclusion and Appendix
7. Detecting Nonlinearity
8. Linear Basis Function Models
9. Generalized Additive Models
10. Fun With Kernels
11. Heteroskedasticity
12. Clustering
13. Optional: Serial Correlation
14. A Contrarian View of Robust Standard Errors
15. Fun with Neighbors
16. Fun with Kittens
Assumptions and Violations

Non-normality

Outliers

Robust Regression Methods

Optional: Measurement Error

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Linear Basis Function Models

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Fun with Neighbors

Fun with Kittens
Argument for Next Three Classes

Residuals are important. Look at them.
Review of the OLS assumptions

1. Linearity: \( y = X\beta + u \)
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1-5 are the Gauss-Markov, allow for large-sample inference
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Review of the OLS Assumptions

- **Identification**
  - Data Description

- **Unbiasedness**
  - Consistency

- **Gauss-Markov (BLUE)**
  - Asymptotic Inference (z and χ²)

- **Classical LM (BUE)**
  - Small-Sample Inference (t and F)

**Branches**:
- **No Perfect Collinearity**
- **Random Sampling**
- **Linearity in Parameters**
- **Zero Conditional Mean**
- **Homoskedasticity**
- **Normality of Errors**
Violations of the assumptions

1. Nonlinearity

   ▶ Result: biased/inconsistent estimates
   ▶ Diagnose: scatterplots, added variable plots, component-plus-residual plots
   ▶ Correct: transformations, polynomials, different model (next class)

2. iid/random sample

   ▶ Result: no bias with appropriate alternative assumptions (structured dependence)
   ▶ Result (ii): violations imply heteroskedasticity
   ▶ Result (iii): outliers from different distributions can cause inefficiency/bias
   ▶ Diagnose/Correct: various, this lecture and next week

3. Perfect collinearity

   ▶ Result: can't run OLS
   ▶ Diagnose/correct: drop one collinear term
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- Result: biased/inconsistent estimates
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Example: Buchanan votes in Florida, 2000
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Wand et al. show that the ballot caused 2,000 Democratic voters to vote by mistake for Buchanan, a number more than enough to have tipped the vote in FL from Bush to Gore, thus giving him FL’s 25 electoral votes and the presidency.

FIGURE 1. The Palm Beach County Bufferfly Ballot
Example: Buchanan votes in Florida, 2000

![Graph showing Buchanan votes vs. Total Votes]
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- In matrix notation:
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- Equivalent to:
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- Equivalent to:
  \[ u_i | \mathbf{x}_i' \sim \mathcal{N}(0, \sigma_u^2) \]

- Fix \( \mathbf{x}_i' \) and the distribution of errors should be Normal
Consequences of non-Normal errors?

- In **small** samples:

  ▶ Sampling distribution of $\hat{\beta}$ will not be Normal
  ▶ Test statistics will not have $t$ or $F$ distributions
  ▶ Probability of Type I error will not be $\alpha$
  ▶ $1 - \alpha$ confidence interval will not have $1 - \alpha$ coverage

- In large samples:

  ▶ Sampling distribution of $\hat{\beta} \approx$ Normal by the CLT
  ▶ Test statistics will be $\approx t$ or $F$ by the CLT
  ▶ Probability of Type I error $\approx \alpha$
  ▶ $1 - \alpha$ confidence interval will have $\approx 1 - \alpha$ coverage

The sample size ($n$) needed for approximation to hold depends on how far the errors are from Normal.
Consequences of non-Normal errors?

- In small samples:
  - Sampling distribution of $\hat{\beta}$ will not be Normal

In large samples:
- Sampling distribution of $\hat{\beta}$ is approximately Normal by the CLT
- Test statistics are approximately $t$ or $F$ by the CLT
- Probability of Type I error is approximately $\alpha$
- $1 - \alpha$ confidence interval has approximately $1 - \alpha$ coverage

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- The sample size ($n$) needed for approximation to hold depends on how far the errors are from Normal.
Marginal versus conditional

- Be careful with this assumption: distribution of the error \((u = y - X\beta)\), not the distribution of the outcome \(y\) is the key assumption.
Marginal versus conditional

- Be careful with this assumption: distribution of the error $(u = y - X\beta)$, not the distribution of the outcome $y$ is the key assumption.
- The marginal distribution of $y$ can be non-Normal even if the conditional distribution is Normal!
Marginal versus conditional

- Be careful with this assumption: distribution of the error ($u = y - X\beta$), not the distribution of the outcome $y$ is the key assumption.
- The marginal distribution of $y$ can be non-Normal even if the conditional distribution is Normal!
- The plausibility depends on the $X$ chosen by the researcher.
Example: Is this a violation?
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How to diagnose?

Assumption is about unobserved \( u = y - X\beta \)
How to diagnose?

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Solution: Carefully investigate the residuals numerically and graphically.
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Solution: Carefully investigate the residuals numerically and graphically.

To understand the relationship between residuals and errors, we need to derive the distribution of the residuals.
Hat matrix

\[
H = X(X'X)^{-1}X' \hat{u} = y - X\hat{\beta} = y - X(X'X)^{-1}X'y \equiv y - Hy = (I - H)y
\]

$H$ is the hat matrix because it puts the "hat" on $y$:

$\hat{y} = Hy$

$H$ is an $n \times n$ symmetric matrix

$H$ is idempotent:

$HH = H$
Hat matrix

- Define matrix $H = X (X'X)^{-1} X'$
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\hat{y} = Hy
$$

- $H$ is an $n \times n$ symmetric matrix
- $H$ is idempotent: $HH = H$
Relating the residuals to the errors

\[ \hat{u} = (I - H)(y) \]
Relating the residuals to the errors

\[ \hat{u} = (I - H)(y) \]
\[ = (I - H)(X\beta + u) \]
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- Residuals \( \hat{u} \) are a linear function of the errors, \( u \)
- For instance,
  \[ \hat{u}_1 = (1 - h_{11})u_1 - \sum_{i=2}^{n} h_{1i}u_i \]
Relating the residuals to the errors

\[ \hat{u} = (I - H) (y) \]
\[ = (I - H) (X\beta + u) \]
\[ = (I - H)X\beta + (I - H)u \]
\[ = I X\beta - X (X'X)^{-1} X' X\beta + (I - H)u \]
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- For instance,
  
  \[ \hat{u}_1 = (1 - h_{11}) u_1 - \sum_{i=2}^{n} h_{1i} u_i \]

- Note that the residual is a function of all of the errors
Distribution of the residuals

\[ \hat{u} = (I - H) E \hat{u} = 0 \]

\[ \text{Var}[\hat{u}] = \sigma^2 \]

The variance of the \( i \)th residual \( \hat{u}_i \) is \( \text{Var}[\hat{u}_i] = \sigma^2 (1 - h_{ii}) \), where \( h_{ii} \) is the \( i \)th diagonal element of the matrix \( H \) (called the hat value).
Distribution of the residuals

\[ E[\hat{u}] = (I - H)E[u] = 0 \]
\[ \text{Var}[\hat{u}] = \sigma_u^2(I - H) \]
Distribution of the residuals

\[ \mathbb{E}[\hat{u}] = (I - H)\mathbb{E}[u] = 0 \]

\[ \text{Var}[\hat{u}] = \sigma^2_u (I - H) \]

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Distribution of the Residuals

Notice in contrast to the unobserved errors, the estimated residuals $\hat{u}_i$ are not independent (because they must satisfy the two constraints $\sum_{i=1}^{n} \hat{u}_i = 0$ and $\sum_{i=1}^{n} \hat{u}_i x_i = 0$) do not have the same variance. The variance of the residuals varies across data points $V[\hat{u}_i] = \sigma^2 (1 - h_{ii})$, even though the unobserved errors all have the same variance $\sigma^2$. These properties can obscure the true patterns in the error distribution, and thus are inconvenient for our diagnostics.
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These properties can obscure the true patterns in the error distribution,
and thus are inconvenient for our diagnostics.
Let's address the second problem (unequal variances) by standardizing $\hat{u}_i$, i.e., dividing by their estimated standard deviations. This produces standardized (or "internally studentized") residuals:

$$\hat{u}_i' = \frac{\hat{u}_i}{\hat{\sigma} \sqrt{1 - h_{ii}}}$$

where $\hat{\sigma}^2$ is our usual estimate of the error variance.

The standardized residuals are still not ideal, since the numerator and denominator of $\hat{u}_i'$ are not independent. This makes the distribution of $\hat{u}_i'$ nonstandard.
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Studentized residuals

If we remove observation \(i\) from the estimation of \(\sigma\), then we can eliminate the dependence and the result will have a standard distribution.

Estimate residual variance without residual \(i\):

\[
\hat{\sigma}^2 - i = u' u - u_i^2 / (1 - h_{ii})
\]

Use this \(i\)-free estimate to standardize, which creates the studentized residuals:

\[
\hat{u}^* i = \hat{u} i \hat{\sigma} - i \sqrt{1 - h_{ii}}
\]

If the errors are Normal, the studentized residuals follow a \(t\) distribution with \((n - k - 2)\) degrees of freedom.

Deviations from \(t\) ⇒ violation of Normality.
Studentized residuals

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- estimate residual variance without residual \( i \):

\[
\hat{\sigma}^2_{-i} = \frac{\mathbf{u}'\mathbf{u} - u^2_i/(1 - h_{ii})}{n - k - 2}
\]

If the errors are Normal, the studentized residuals follow a \( t \) distribution with \( (n - k - 2) \) degrees of freedom.
Studentized residuals

If we remove observation $i$ from the estimation of $\sigma$, then we can eliminate the dependence and the result will have a standard distribution.

- estimate residual variance without residual $i$:
  \[
  \hat{\sigma}^2_{-i} = \frac{u'u - u_i^2}{(1 - h_{ii})} \frac{1}{n - k - 2}
  \]

- Use this $i$-free estimate to standardize, which creates the studentized residuals:
  \[
  \hat{u}_i^* = \frac{\hat{u}_i}{\hat{\sigma}_{-i}\sqrt{1 - h_{ii}}}
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- Deviations from $t \implies$ violation of Normality
Example: Buchanan Votes in Florida

Now that our studentized residuals follow a known standard distribution, we can proceed with diagnostic analysis for the nonnormal errors.

We examine data from the 2000 presidential election in Florida used in Wand et al. (2001).

Our analysis takes place at the county level and we will regress the number of Buchanan votes in each county on the total number of votes in each county.
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R Code

```r
> mod1 <- lm(buchanan00~TotalVotes00, data=dta)
> summary(mod1)

Residuals:
   Min 1Q Median 3Q Max
-947.05 -41.74 -19.47 20.20 2350.54

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   5.423e+01  4.914e+01  1.104   0.274
TotalVotes00  2.323e-03  3.104e-04  7.483  2.42e-10 ***
---
Residual standard error: 332.7 on 65 degrees of freedom
Multiple R-squared: 0.4628, Adjusted R-squared: 0.4545
F-statistic: 56 on 1 and 65 DF, p-value: 2.417e-10

> residuals <- resid(mod1)
> standardized_residuals <- rstandard(mod1)
> studentized_residuals <- rstudent(mod1)
> dotchart(residuals, dta$name, cex=.7, xlab="Residuals")
```
Plotting the residuals
Plotting the residuals

Histogram of resids

Histogram of stand.resids

Histogram of student.resids
Plotting the residuals

![Studentized Residuals vs. Theoretical t distribution](chart)

- **Studentized Residuals**
- **Theoretical t distribution**
Quantile-Quantile plots

- Quantile-quantile plot or QQ-plot is useful for comparing distributions
Quantile-Quantile plots

- **Quantile-quantile plot** or **QQ-plot** is useful for comparing distributions
- Plots the quantiles of one distribution against those of another distribution
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- Plots the quantiles of one distribution against those of another distribution
- For example, one point is the \((m_x, m_y)\) where \(m_x\) is the median of the \(x\) distribution and \(m_y\) is the median for the \(y\) distribution
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- For example, one point is the \((m_x, m_y)\) where \(m_x\) is the median of the \(x\) distribution and \(m_y\) is the median for the \(y\) distribution
- If distributions are equal \(\iff\) 45 degree line
Good QQ-plot
Buchanan QQ-plot

stewart.resids

t quantiles
How can we deal with nonnormal errors?

Drop or change problematic observations (could be a bad idea unless you have some reason to believe the data are wrong or corrupted)

Add variables to $X$ (remember that the errors are defined in terms of explanatory variables)

Use transformations (this may work, but a transformation affects all the assumptions of the model)

Use estimators other than OLS that are robust to nonnormality (later this class)

Consider other causes (next two classes)
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- Consider other causes (next two classes)
Let’s delete Palm Beach and also use log transformations for both variables

```r
##
## Coefficients:
## Estimate Std. Error t value Pr(>|t|)
## (Intercept) -2.48597 0.37889 -6.561 1.09e-08 ***
## log(edaytotal) 0.70311 0.03621 19.417 < 2e-16 ***
## ---
## Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
##
## Residual standard error: 0.4362 on 64 degrees of freedom
## Multiple R-squared:  0.8549, Adjusted R-squared:  0.8526
## F-statistic: 377 on 1 and 64 DF,  p-value: < 2.2e-16
```
Buchanan revisited

Histogram of resids.nopb

Histogram of stand.resids.nopb

Histogram of student.resids.nopb

Studentized Residuals

Residuals

Standardized Residuals

Weekly residuals

Studentized residuals
Buchanan revisited

Studentized Residuals

Density

Studentized Residuals

Theoretical t distribution

-2 -1 0 1 2

-2

-1

0

1

2

t quantiles

student.resids nopb

Stewart (Princeton)
A Note of Caution About Log Transformations

- Log transformations are a standard approach in the literature and intro regression classes

Log transformations are extremely helpful for data that is skewed (e.g. a few very large positive values). Generally, you want to convert these findings back to the original scale for interpretation. You should know though that estimates of marginal effects in the untransformed states are not necessarily unbiased. Jensen's inequality gives us information on this relation:

\[ f\left(E\left[X\right]\right) \leq E\left[f\left(X\right)\right] \]

The results will in general be consistent, which ensures that the bias decreases in sample size.
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Stewart (Princeton)

Week 9: Diagnostics and Solutions

November 14-21, 2016
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1 Assumptions and Violations
2 Non-normality
3 Outliers
4 Robust Regression Methods
5 Optional: Measurement Error
6 Conclusion and Appendix
7 Detecting Nonlinearity
8 Linear Basis Function Models
9 Generalized Additive Models
10 Fun With Kernels
11 Heteroskedasticity
12 Clustering
13 Optional: Serial Correlation
14 A Contrarian View of Robust Standard Errors
15 Fun with Neighbors
16 Fun with Kittens
Assumptions and Violations

1. Non-normality

3. Outliers

4. Robust Regression Methods

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6. Conclusion and Appendix

7. Detecting Nonlinearity

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16. Fun with Kittens
The trouble with Norway

Lange and Garrett (1985): organizational and political power of labor interact to improve economic growth

Jackman (1987): relationship just due to North Sea Oil?

Table guide:

$\begin{array}{c}
x_1 = \text{organizational power of labor} \\
x_2 = \text{political power of labor} \\
\text{Parentheses contain} \ t \text{-statistics}
\end{array}$

$\begin{array}{c|c|c|c|c}
\text{Constant} & x_1 & x_2 & x_1 \cdot x_2 & \text{Obs} \\
\hline
\text{Norway Obs Included} & .814 & -.192 & -.278 & .137 & (4.7) & (2.0) & (2.4) & (2.9) \\
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Creative curve fitting with Norway
Creative curve fitting with Norway

Corporate Taxes and Revenue, 2004

Left scale represents tax revenues as a percentage of GDP. Bottom scale represents central government corporate tax rates.

Sources: OECD Revenue Statistics, Kevin Hassett, American Enterprise Institute
The Most Important Lesson: Check Your Data

“Do not attempt to build a model on a set of poor data! In human surveys, one often finds 14-inch men, 1000-pound women, students with ‘no’ lungs, and so on. In manufacturing data, one can find 10,000 pounds of material in a 100 pound capacity barrel, and similar obvious errors. All the planning, and training in the world will not eliminate these sorts of problems. In our decades of experience with ‘messy data,’ we have yet to find a large data set completely free of such quality problems.”

Draper and Smith (1981, p. 418)

Always Carefully Examine the Data First!!

1 Examine summary statistics:

```
summary(data)
```

2 Scatterplot matrix for densities and bivariate relationships:

```
E.g. scatterplotMatrix(data)
```

from `car` library.

3 Further conditional plots for multivariate data:

```
E.g. use the `lattice` library or `ggplot2`
```

Stewart (Princeton)

Week 9: Diagnostics and Solutions

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The Most Important Lesson: Check Your Data

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Always Carefully Examine the Data First!!

1. Examine summary statistics: `summary(data)`
2. Scatterplot matrix for densities and bivariate relationships: E.g. `scatterplotMatrix(data)` from `car` library.
3. Further conditional plots for multivariate data: E.g. use the `lattice` library or `ggplot2`
Three types of extreme values

1. Outlier: extreme in the y direction
2. Leverage point: extreme in one x direction
3. Influence point: extreme in both directions

Not all of these are problematic. If the data are truly “contaminated” (come from a different distribution), can cause inefficiency and possibly bias. Can be a violation of iid (not identically distributed).
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2. Leverage point: extreme in one $x$ direction
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- Can be a violation of iid (not identically distributed)
Outlier definition

- An **outlier** is a data point with very large regression errors, $u_i$
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Very distant from the rest of the data in the $y$-dimension.
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Very distant from the rest of the data in the $y$-dimension

Increases standard errors (by increasing $\hat{\sigma}^2$)
An outlier is a data point with very large regression errors, \( u_i \)

- Very distant from the rest of the data in the \( y \)-dimension

- Increases standard errors (by increasing \( \hat{\sigma}^2 \))

- No bias if typical in the \( x \)'s
Detecting outliers

- Look at standardized residuals, $\hat{u}_i$?

$\hat{u}_i' > \hat{u}_i \hat{\sigma} - i \sqrt{1 - h_i}$ because we drop the large residual from the outlier, and so $\hat{u}_i' < \hat{u}_i \hat{\sigma} - i \hat{\sigma}$ could be biased upwards by the large residual from the outlier.
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- Possible solution: use studentized residuals

$$\hat{u}_i^* = \frac{\hat{u}_i}{\hat{\sigma}_{-i} \sqrt{1 - h_i}}$$
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- $\hat{\sigma} > \hat{\sigma}_{-i}$ because we drop the large residual from the outlier, and so $\hat{u}_i' < \hat{u}_i^*$
Cutoff rules for outliers

The studentized residuals follow a t distribution, \( u_i^* \sim t_{n-k-2} \), when \( u_i \sim N(0, \sigma^2) \).
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- People usually adjust cutoff for multiple testing
What to do about outliers

- Is the data corrupted?

  - Fix the observation (obvious data entry errors)
  - Remove the observation
  - Be transparent either way

- Is the outlier part of the data generating process?

  - Transform the dependent variable ($\log(y)$)
  - Use a method that is robust to outliers (robust regression)
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In the late 70s, NASA used an automated data processing program on satellite measurements of atmospheric data to track changes in atmospheric variables such as ozone. This data "quality control" algorithm rejected abnormally low readings of ozone over the Antarctic as unreasonable. This delayed the detection of the ozone hole by several years until British Antarctic Survey scientists discovered it based on analysis of their own observations (Nature, May 1985).

The ozone hole was detected in satellite data only when the raw data was reprocessed. When the software was rerun without the pre-processing flags, the ozone hole was seen as far back as 1976.
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- Decrease SEs (more $X$ variation)
Values that are extreme in the $x$ direction
That is, values far from the center of the covariate distribution
Decrease SEs (more $X$ variation)
No bias if typical in $y$ dimension
Leverage Points: Hat values

To measure leverage in multivariate data we will go back to the hat matrix $H$:

$$\hat{y} = X\hat{\beta} = X (X'X)^{-1} X'y = Hy$$

$H$ is $n \times n$, symmetric, and idempotent. It generates fitted values as follows:

$$\hat{y}_i = h'_i y = \begin{bmatrix} h_{i,1} & h_{i,2} & \cdots & h_{i,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{j=1}^{n} h_{i,j}y_j$$
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- The diagonal entries $h_{ii} = \sum_{j=1}^n h_{ij}^2$, so they summarize how important $y_i$ is for all the fitted values. We call them the hat values or leverages and a single subscript notation is used: $h_i = h_{ii}$
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- Rule of thumb: examine hat values greater than $2(k + 1)/n$
Influence points

An influence point is one that is both an outlier (extreme in $X$) and a leverage point (extreme in $Y$). Causes the regression line to move toward it (bias?)

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Detecting Influence Points/Bad Leverage Points

- Influence Points:
  
  Influence on coefficients = Leverage × Outlyingness
Detecting Influence Points/Bad Leverage Points

- **Influence Points:**
  Influence on coefficients = Leverage $\times$ Outlyingness

- More formally: Measure the change that occurs in the slope estimates when an observation is removed from the data set. Let

  \[ D_{ij} = \hat{\beta}_j - \hat{\beta}_{j(-i)}, \quad i = 1, \ldots, n, \quad j = 0, \ldots, k \]

  where $\hat{\beta}_{j(-i)}$ is the estimate of the $j$th coefficient from the same regression once observation $i$ has been removed from the data set.
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- $D_{ij}$ is called the DFbeta, which measures the influence of observation $i$ on the estimated coefficient for the $j$th explanatory variable.
Standardized Influence

To make comparisons across coefficients, it is helpful to scale $D_{ij}$ by the estimated standard error of the coefficients:

$$D_{ij}^* = \frac{\hat{\beta}_j - \hat{\beta}_j(-i)}{\hat{SE}_{-i}(\hat{\beta}_j)}$$

where $D_{ij}^*$ is called DFbetaS.
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- $D_{ij}^* > 0$ implies that removing observation $i$ decreases the estimate of $\beta_j \rightarrow \text{obs } i$ has a positive influence on $\beta_j$.

- $D_{ij}^* < 0$ implies that removing observation $i$ increases the estimate of $\beta_j \rightarrow \text{obs } i$ has a negative influence on $\beta_j$.

- Values of $|D_{ij}^*| > 2/\sqrt{n}$ are an indication of high influence.
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- In R: dfbetas(model)
### Coefficients:

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|----------|
| (Intercept) | -2.935e+01 | 5.520e+01 | -0.532 | 0.59686 |
| edaytotal  | 1.100e-03  | 4.797e-04 | 2.293   | 0.02529 * |
| absnbuchanan | 6.895e+00 | 2.129e+00 | 3.238   | 0.00195 ** |

---

Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1

Residual standard error: 317.2 on 61 degrees of freedom
(3 observations deleted due to missingness)

Multiple R-squared:  0.5361, Adjusted R-squared:  0.5209
F-statistic:  35.24 on 2 and 61 DF,  p-value:  6.711e-11
Buchanan influence

<table>
<thead>
<tr>
<th></th>
<th>(Intercept)</th>
<th>edaytotal</th>
<th>absnbuchanan</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3454475146</td>
<td>0.4050504921</td>
<td>-0.7505222758</td>
</tr>
<tr>
<td>2</td>
<td>-0.0234266617</td>
<td>-0.0241000045</td>
<td>-0.0131672181</td>
</tr>
<tr>
<td>3</td>
<td>0.0650795039</td>
<td>-0.7319311820</td>
<td>0.3401669862</td>
</tr>
<tr>
<td>4</td>
<td>-0.0333980968</td>
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<td>-0.0087505576</td>
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<tr>
<td>5</td>
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<td>-0.0073746223</td>
<td>0.0096551713</td>
</tr>
<tr>
<td>6</td>
<td>-0.0009277798</td>
<td>0.0001505476</td>
<td>0.0002210247</td>
</tr>
</tbody>
</table>
Palm Beach county moves each of the coefficients by more than 3 standard errors!
Summarizing Influence across All Coefficients

- Leverage tells us how much one data point affects a single coefficient.
Summarizing Influence across All Coefficients

- Leverage tells us how much one data point affects a single coefficient.
- A number of summary measures exist for influence of data points across all coefficients, all involving both leverage and outlyingness.

\[ D_i = \hat{u}_i'^2 \times k + 1 \times h_i - h_i \]

It can be shown that \( D_i \) is a weighted sum of \( k + 1 \) DFbetaS's for observation \( i \).

In R, `cooks.distance(model)`

\( D_i > 4 / (n - k - 1) \) is commonly considered large.

The influence plot: the studentized residuals plotted against the hat values, size of points proportional to Cook's distance.
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Influence Plot Buchanan

Hat Values
Studentized Residuals
Duval
Broward
Pinellas
Miami-Dade
Palm Beach
Hamilton
Liberty
Stewart (Princeton)
Week 9: Diagnostics and Solutions
November 14-21, 2016
Code for Influence Plot

```r
mod3 <- lm(edaybuchanan ~ edaytotal + absnbuchanan, data = flvote)
symbols(y = rstudent(mod3), x = hatvalues(mod3),
   circles = sqrt(cooks.distance(mod3)),
   ylab = "Studentized Residuals",
   xlab = "Hat Values", xlim = c(-0.05, 1),
   ylim = c(-10, 50), las = 1, bty = "n")
cutoffstud <- 2
cutoffhat <- 2 * (3)/nrow(flvote)
abline(v = cutoffhat, col = "indianred")
abline(h = cutoffstud, col = "dodgerblue")
filter <- rstudent(mod3) > cutoffstud | hatvalues(mod3) > cutoffhat
text(y = rstudent(mod3)[filter],
   x = hatvalues(mod3)[filter],
   flvote$county[filter], pos = 1)
```
A Quick Function for Standard Diagnostic Plots

R Code

```r
> par(mfrow=c(2,2))
> plot(mod1)
```
The Improved Model

R Code

```r
> par(mfrow=c(2,2))
> plot(mod2)
```

Residuals vs Fitted

Normal Q–Q

Scale–Location

Residuals vs Leverage
1. Assumptions and Violations
2. Non-normality
3. Outliers
4. Robust Regression Methods
5. Optional: Measurement Error
6. Conclusion and Appendix
7. Detecting Nonlinearity
8. Linear Basis Function Models
9. Generalized Additive Models
10. Fun With Kernels
11. Heteroskedasticity
12. Clustering
13. Optional: Serial Correlation
14. A Contrarian View of Robust Standard Errors
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8. Linear Basis Function Models
9. Generalized Additive Models
10. Fun With Kernels
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14. A Contrarian View of Robust Standard Errors
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Limitations of the standard tools

What happens when there are two influence points?

Red line drops the red influence point
Blue line drops the blue influence point
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- How comforting should this be? **Not very.**
- The **Linear** point is an artificial restriction. It means the estimator has to be of the form $\hat{\beta} = Wy$ but why only use those?
- With normality assumption we get **Best Unbiased Estimator (BUE)** which is quite comforting when $n \gg p$ (number of observations much larger than number of variables).
This Point is Not Obvious

This flies in the face of most conventional wisdom in textbooks.
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”[Even without normally distributed errors] OLS coefficient estimators remain unbiased and efficient.” - Berry (1993)

Quotes from Rainey and Baissa (2015) presentation
This flies in the face of most conventional wisdom in textbooks.

"[The Gauss-Markov theorem] justifies the use of the OLS method rather than using a variety of competing estimators" - Wooldridge (2013)

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This flies in the face of most conventional wisdom in textbooks.

”We need not look for another linear unbiased estimator, for we will not find such an estimator whose variance is smaller than the OLS estimator”

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This Point is Not Obvious

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"The Gauss-Markov theorem allows us to have considerable confidence in the least squares estimators." - Berry and Feldman (1993)

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Robustly Estimating a Location

- Let's simplify- what if we want to estimate the center of a symmetric distribution.

\[ y_i \sim N(\mu, \sigma^2) \]

\[ \text{Median is } \pi^2 \text{ times larger (i.e. less efficient)} \]

We can measure sensitivity with the influence function which measures change in estimator based on corruption in one datapoint.
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- For normal data $y_i \sim \mathcal{N}(\mu, \sigma^2)$, median is less efficient:
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Imagine that we had a sample $Y$ from a standard normal: -0.068, -1.282, 0.013, 0.141, -0.980, 1.63. \( \bar{Y} = -1.52 \)
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Example from Fox
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The median has a breakdown point of 50% because half the data can be bad without causing the median to become completely unstuck.
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- Other objectives include the Huber objective and Tukey’s biweight objective which have different properties.
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- The shape of the influence function is determined by the derivative of the objective function with respect to \(E\).
- Other objectives include the Huber objective and Tukey’s biweight objective which have different properties.
- Calculating robust \(M\) estimators often requires an iterative procedure and a careful initialization.
$M$-estimation for Regression

- We can apply this to regression fairly straightforwardly. In robust $M$-estimators we choose $\rho()$ so that observations with large residuals get less weight.

- 

- **Least Median Squares:** choose $\hat{\beta}$ to minimize $\text{median}_{i=1}^{n} \left( (y_i - x_i'\hat{\beta}_{LMS})^2 \right)$. Very high breakdown point, but very inefficient.

- **Least Trimmed Squares:** choose $\hat{\beta}$ to minimize the sum of the $p$ smallest elements of $\left\{ (y_i - x_i'\hat{\beta}_{LTS})^2 \right\}_{i=1}^{n}$. High breakdown point and more efficient, still not as efficient as some.

- **MM-estimator:** with Huber’s loss is what I recommend in practice (more in appendix). You can find an asymptotic covariance matrix for $M$-estimators but I would bootstrap it if possible as the asymptotics kick in slowly.
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Some options:

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  - *MM*-estimator: with Huber’s loss is what I recommend in practice (more in appendix)
- You can find an asymptotic covariance matrix for *M*-estimators but I would bootstrap it if possible as the asymptotics kick in slowly.
library(MASS)
set.seed(588)
n <- 50
x <- rnorm(n)
y <- 10 - 2*x + rnorm(n)
x[1:5] <- rnorm(5, mean=5)
y[1:5] <- 10 + rnorm(5)
ols.out <- lm(y~x)
m.out <- rlm(y~x, method="M")
lms.out <- lqs(y~x, method="lms")
lts.out <- lqs(y~x, method="lts")
s.out <- lqs(y~x, method="S")
mm.out <- rlm(y~x, method="MM")
Simulation Results

- OLS
- M
- LMS
- LTS
- S
- MM
Thoughts on Robust Estimators

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I highly recommend Baissa and Rainey (2016) “When BLUE is Not Best: Non-Normal Errors and the Linear Model” for more on this topic.

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### Appendix: Characterizing Estimator Robustness (formally)

#### Definition (Breakdown Point)

The **breakdown point** of an estimator is the smallest fraction of the data that can be changed an arbitrary amount to produce an arbitrarily large change in the estimate (Seber and Lee 2003, pg 82).

#### Definition (Influence Function)

Let \( F_p = (1 - p)F + p\delta_{z_0} \) where \( F \) is a probability measure, \( \delta_{z_0} \) is the point mass at \( z_0 \in \mathbb{R}^k \), and \( p \in (0, 1) \). Let \( T(\cdot) \) be a statistical functional. The **influence function** of \( T \) is

\[
IF(z_0; T, F) = \lim_{p \downarrow 0} \frac{T(F_p) - T(F)}{p}
\]

The influence function is a function of \( z_0 \) given \( T \) and \( F \). It describes how \( T \) changes with small amounts of contamination at \( z_0 \) (Hampel, Rousseeuw, Ronchetti, and Stahel, (1986), p. 84).
Appendix: S Estimators

To talk about \( MM \)--estimators we need a type of estimator called an \( S \)-estimator.

\[ S \text{-estimators work somewhat differently in that the goal is to minimize the scale estimate subject to a constraint.} \]

An \( S \)-estimator for the regression model is defined as the values of \( \hat{\beta}_S \) and \( s \) that minimize \( s \) subject to the constraint:

\[ \frac{1}{n} \sum_{i=1}^{n} \rho \left( y_i - x_i' \hat{\beta}_S s \right) \geq K \]

where \( K \) is user-defined constant (typically set to 0.5) and \( \rho : \mathbb{R} \to [0, 1] \) is a function with the following properties (Davies, 1990, p. 1653):

1. \( \rho (0) = 1 \)
2. \( \rho (u) = \rho (-u) \), \( u \in \mathbb{R} \)
3. \( \rho : \mathbb{R}^+ \to [0, 1] \) is nonincreasing, continuous at 0, and continuous on the left
4. for some \( c > 0 \), \( \rho (u) > 0 \) if \( |u| < c \) and \( \rho (u) = 0 \) if \( |u| > c \)
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Good properties, but costly to compute (usually impossible to compute exactly).
Assumptions and Violations

Non-normality

Outliers

Robust Regression Methods

Optional: Measurement Error

Conclusion and Appendix

Detecting Nonlinearity

Linear Basis Function Models

Generalized Additive Models

Fun With Kernels

Heteroskedasticity

Clustering

Optional: Serial Correlation

A Contrarian View of Robust Standard Errors

Fun with Neighbors

Fun with Kittens
Optional: Measurement Error
Measurement Error

“It seems as if measurement error has been pushed into the role of the unwanted child whose existence we would rather deny. Maybe because measurement error is common, insipid, and unsophisticated. Unlike the hidden confounder challenging our intellect, to discover measurement error is a ‘no-brainer’ - it simply lurks everywhere. Our epidemiological fingerprints are contaminated with measurement error. Everything we observe, we observe with error. Since observation is our business, we would probably rather deny that what we observe is imprecise and maybe even inaccurate, but time has come to unveil the secret: measurement error is threatening our profession.”

Karen Michals (2001)
Measuring Variables

Often, the variables that we use in our regression analysis are measured with error. For example:

- In cross-country data, variables are often measured by surveys within each country (e.g. perceived corruption)
- In individual level data, individuals may misreport information (sensitive questions about preferences, politics, income, etc.)
- The Bradley effect (1982)
- Shy Tory Factor (1992)
- The Silent Trump Voter?
- In recall studies, people often can't remember (e.g. how many vegetables did you eat last week?)
- Many concepts are hard to measure (e.g. ability, IQ, religiosity, income, etc.)
- Other variables, like gender, number of children, may be measured with less error
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Often, the variables that we use in our regression analysis are measured with error. For example:

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- In individual level data, individuals may misreport information (sensitive questions about preferences, politics, income, etc.)
  - The Bradley effect (1982)
  - Shy Tory Factor (1992)
  - The Silent Trump Voter?

- In recall studies, people often can’t remember (e.g. how many vegetables did you eat last week?)

- Many concepts are hard to measure (e.g. ability, IQ, religiosity, income, etc.)

- Other variables, like gender, number of children, may be measured with less error
US Survey Data

**Sex and drugs**

Men are more likely to use illegal drugs and have more sexual partners than women, according to a 1999-2002 survey.

**Number of sexual partners, ages 20-59**

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>16.6%</td>
<td>25.0</td>
</tr>
<tr>
<td>2-6</td>
<td>33.8</td>
<td>44.3</td>
</tr>
<tr>
<td>7-14</td>
<td>20.7</td>
<td>21.3</td>
</tr>
<tr>
<td>15 or more</td>
<td>28.9</td>
<td>9.4</td>
</tr>
</tbody>
</table>

**Ever used cocaine or street drugs, ages 20-59**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>26.0%</td>
</tr>
<tr>
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<td>16.8%</td>
</tr>
</tbody>
</table>

*Source: Centers for Disease Control and Prevention*
Measurement Error in the Dependent Variable

Consider the simple linear regression model where we observe $Y_i^*$ instead of $Y_i$ and the following relationships hold

\[ Y_i = \beta_0 + \beta_1 X_i + U_i \]

\[ U_i \sim \text{i.i.d.} N(0, \sigma^2) \]

\[ Y_i^* = Y_i + \epsilon_i + U_i \]

Let's assume that the Gauss-Markov assumptions hold for the model with the true (but unobserved) variables so that OLS would be unbiased and consistent if we observed $Y_i$. Does the measurement error in $Y_i^*$ cause any problems when fitting OLS to the observed data?
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Consider the simple linear regression model where we observe $Y_{i}^{*}$ instead of $Y_{i}$ and the following relationships hold

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Y_{i} = \beta_{0} + \beta_{1}X_{i} + U_{i}
\]

\[
U_{i} \sim_{i.i.d} N(0, \sigma^{2})
\]

\[
Y_{i}^{*} = Y_{i} + E_{i}
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Therefore, the model for our observed data is

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Y_{i}^{*} = \beta_{0} + \beta_{1}X_{i} + E_{i} + U_{i}
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- If \( E \) is correlated with \( X \) then OLS is inconsistent and biased.
Measurement Error in the Independent Variable

Consider the simple linear regression model where all the traditional assumptions hold but we observe $X_i^*$ instead of $X_i$

$$Y_i = \beta_0 + \beta_1 X_i + U_i$$
$$X_i^* = X_i + E_i$$

Therefore, the model for our observed data is

$$Y_i = \beta_0 + \beta_1 (X_i^* - E_i) + U_i = \beta_0 + \beta_1 X_i^* + (U_i - \beta_1 E_i)$$

and we must state assumptions in terms of the new error term, $U_i - \beta_1 E_i$.

Let's assume $E[ E_i ] = 0$ (the average error is zero) and the first four GM-assumptions hold (including $\text{Cov}[U, X] = 0$ and $\text{Cov}[U, X^*] = 0$) so that OLS would be unbiased and consistent if we observed $X_i$. Does the measurement error cause any problems when fitting OLS to the observed data (i.e. using $X_i^*$ instead of $X_i$)?
Measurement Error in the Independent Variable

Consider the simple linear regression model where all the traditional assumptions hold but we observe $X_i^{*}$ instead of $X_i$

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Does the measurement error cause any problems when fitting OLS to the observed data (i.e. using $X^*$ instead of $X$)?
Measurement Error in the Independent Variable

The consequences depend on the correlation between $E$ and $X^*$ and or $X$. 

\begin{align*}
Y_i &= \beta_0 + \beta_1 (X^*_i - E_i) + U_i \\
Y_i &= \beta_0 + \beta_1 X^*_i + (U_i - \beta_1 E_i)
\end{align*}

If the measurement error is uncorrelated with the observed variable $\text{Cov}[X^*, E] = 0$, then: since $(U_i - \beta_1 E_i)$ is uncorrelated with $X^*_i$ if both $E$ and $U$ are uncorrelated with $X^*_i$, our estimator $\hat{\beta}_1$ is unbiased and consistent.

If $E$ and $U$ are uncorrelated, the overall error variance is $V[(U_i - \beta_1 E_i)] = \sigma^2_U + \beta_1^2 \sigma^2_E$ which is $> \sigma^2_U$ unless $\beta_1 = 0$.

Note: This has nothing to do with assumptions about errors $U$, we always maintain that $\text{Cov}[X^*, U] = 0$ and $\text{Cov}[X, U] = 0$. 

Stewart (Princeton)
Measurement Error in the Independent Variable

The consequences depend on the correlation between $E$ and $X^*$ and or $X$.

Our model for our observed data is:

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If the measurement error is uncorrelated with the unobserved variable, \( \text{Cov}[X, E] = 0 \), then:

\[ \text{Cov}[X^*, E] = \beta_1 \sigma^2_E \]

This correlation causes problems since now:

\[ \text{Cov}[X^*, U - \beta_1 E] = -\beta_1 \text{Cov}[X^*, E] = -\beta_1 \sigma^2_E \]

and since the composite error is correlated with the observed measure, OLS will be biased and inconsistent.

Can we know the direction of the (asymptotic) bias?
Measurement Error in the Independent Variable

Our model for the observed data is:

\[ Y_i = \beta_0 + \beta_1 (X^*_i - E_i) + U_i \]

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If the measurement error is uncorrelated with the unobserved variable, \( \text{Cov}[X, E] = 0 \), then:

- the observed variable \( X^* = X + E \) and the measurement error \( E \) must be correlated:

\[ \text{Cov}[X^*, E] = \text{E}[X^*E] = \text{E}[(X + E)E] = \text{E}[XE + E^2] = 0 + \sigma^2_E = \sigma^2_E \]

so the covariance between the observed measure \( X^* \) and the measurement error \( E \) is equal to the variance of \( E \) (classical error-in-variables (CEV) assumption).
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- Can we know the direction of the (asymptotic) bias?
Classical Errors in Variables (Appendix)

\[
\hat{\beta}_1 \rightarrow \beta_1 + \text{Cov}\left[ X^*, U \right] E = \beta_1 - \beta_1 \sigma^2 E \sigma^2 X = \beta_1 \left( 1 - \sigma^2 E \sigma^2 X \right)
\]

Notice that given our CEV assumption \(\sigma^2 X \sigma^2 X + \sigma^2 E < 1\) so that the probability limit of \(\beta_1\) is always closer to zero than \(\beta_1\) (attenuation bias). Bias is small if variance of observed measure \(\sigma^2 X\) is large relative to variance of error term \(\sigma^2 E\) (high signal to noise ratio).
Classical Errors in Variables (Appendix)

\[ \hat{\beta}_1 \xrightarrow{p} \beta_1 + \frac{\text{Cov}[X^*, U - \beta_1 E]}{V[X^*]} \]

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Notice that given our CEV assumption \( \left( \frac{\sigma_X^2}{\sigma_X^2 + \sigma_E^2} \right) < 1 \) so that the probability limit of \( \beta_1 \) is always closer to zero than \( \beta_1 \) (attenuation bias).

Bias is small if variance of observed measure \( \sigma_X^2 \) is large relative to variance of error term \( \sigma_E^2 \) (high signal to noise ratio).
Measurement Error in Multiple Independent Variables

\[ Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + u \]

- If only \( X_1 \) is measured with CEV type error (but \( X_2 \) and \( X_3 \) are correct), then \( \beta_1 \) exhibits attenuation bias, and \( \beta_2 \) and \( \beta_3 \) will be inconsistent and biased unless \( X_1 \) is uncorrelated with \( X_2 \) and \( X_3 \).
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- The direction and magnitude of bias in \( X_2 \) and \( X_3 \) are not easy to derive and often unclear.
- If we have CEV measurement error in multiple \( X \)s then the size and direction of biases are unclear.
What can we do about measurement error?

- Improve our measures (e.g. pilot tests for surveys)

- Triangulate several measures (e.g. a battery of survey questions about the same issue)

- Triangulate several studies (replicate experiments several times with different subjects and at different times)

- Rely on variables that are less prone to bias

- Randomized response, list experiments etc.

- Modeling based approaches (next semester)
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Summary for Measurement Error

Measurement error in the outcome ▶ does not cause bias unless measurement error correlated with \( X \) variables ▶ does reduce efficiency

Measurement error in the explanatory variable ▶ does not cause bias if measurement error is uncorrelated with observed, mis-measured \( X \) (but increases variance) ▶ does lead to attenuation bias even if measurement error is uncorrelated with unobserved true \( X \)

Note: This is true only under fairly strong assumptions including mean zero measurement error.
Summary for Measurement Error

- Measurement error in the outcome

  ▶ Measurement error in the outcome does not cause bias unless measurement error correlated with X variables.

  ▶ Measurement error in the outcome does reduce efficiency.

- Measurement error in the explanatory variable

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- Regression rests on a number of assumptions
- Easy to test some of these and hard to test others.
- Always check your data!
- Don’t let regression be a magic black box for you- understand why it is giving the answers it gives.
References


Where We’ve Been and Where We’re Going...

Last Week

- regression in the social sciences

This "Week"

- Monday (14): unusual and influential data → robust estimation
- Wednesday (16): non-linearity → generalized additive models
- Monday (21): unusual errors → sandwich SEs and block bootstrap

After Thanksgiving

- causality with measured confounding

Long Run

- regression → diagnostics → causal inference

Questions?
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**Questions?**
Residuals are still important. Look at them.
1 Assumptions and Violations
2 Non-normality
3 Outliers
4 Robust Regression Methods
5 Optional: Measurement Error
6 Conclusion and Appendix
7 Detecting Nonlinearity
8 Linear Basis Function Models
9 Generalized Additive Models
10 Fun With Kernels
11 Heteroskedasticity
12 Clustering
13 Optional: Serial Correlation
14 A Contrarian View of Robust Standard Errors
15 Fun with Neighbors
16 Fun with Kittens
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Nonlinearity

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  - Possible Exceptions: Returns to scale, constant elasticities, interactive effects, cyclical patterns in time series data, etc.

- Usually we employ “linearity by default” but we should try to make sure this is appropriate: detect non-linearities and model them accurately.
Diagnosing Nonlinearity

- For marginal relationships $Y$ and $X$
  - Scatterplots with loess lines

- Added variables plots and component residual plots
- Semi-parametric regression techniques like Generalized Additive Models (GAMs)
- Non-parametric multiple regression techniques (beyond the scope of this course)
Diagnosing Nonlinearity

- For **marginal** relationships $Y$ and $X$
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- For **partial** relationships $Y$ and $X_1$, controlling for $X_2, X_3, ..., X_k$ the regression surface is high-dimensional.

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- How to construct an added variable plot:

1. Get residuals from regression of $Y$ on all covariates except $X_j$
2. Get residuals from regression of $X_j$ on all other covariates
3. Plot residuals from (1) against residuals from (2)

In R:
```
avPlots(model)
```
from the *car* package

OLS fit to this plot will have exactly $\hat{\beta}_j$ and 0 intercept (drawing on the partialing out interpretation we discussed before)

Use local smoother (loess) to detect any non-linearity
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Buchanan AV plot

R Code

```r
par(mfrow = c(1,2))
out <- avPlots(mod3, "edaytotal")
lines(loess.smooth(x = out$edaytotal[,1],
  y= out$edaytotal[,2]), col = "dodgerblue", lwd = 2)
out2 <- avPlots(mod3, "absnbuchanan")
lines(loess.smooth(x = out2$absnbuchanan[,1],
  y= out2$absnbuchanan[,2]), col = "dodgerblue", lwd = 2)
```
Component-Residual plots

- CR plots are a refinement of AV plots:
  1. Compute residuals from full regression: 
     \[ \hat{u}_i = Y_i - \hat{Y}_i \]
  2. Compute "linear component" of the partial relationship: 
     \[ C_i = \hat{\beta}_j X_{ij} \]
  3. Add linear component to residual: 
     \[ \hat{u}_j = \hat{u}_i + C_i \]
  4. Plot partial residual \( \hat{u}_j \) against \( X_{ij} \)
     - Same slope as AV plots
     - X-axis is the original scale of \( X_j \), so slightly easier for diagnostics
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Buchanan CR plot

R Code

```r
crPlots(mod3, las = 1)
```

Component + Residual Plots
Limitations of CR Plots

AV plots and CR plots can only reveal partial relationships. Oftentimes, these two-dimensional displays fail to uncover structure in a higher-dimensional problem. We may detect an interaction between $X_1$ and $X_2$ in a 3D scatterplot that we could miss in two scatterplots of $Y$ on each $X$.

Cook (1993) shows that CR plots only work when either:

1) The relationship between $Y$ and $X_j$ is linear
2) Other explanatory variables ($X_1, \ldots, X_{j-1}$) are linearly related to $X_j$

This suggests that linearizing the relationship between the $X$s through transformations can be helpful. Experience suggests weak non-linearities among $X$s do not invalidate CR plots.
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How should we deal with nonlinearity?

Given we have a linear regression model, our options are somewhat limited.
How should we deal with nonlinearity?

Given we have a linear regression model, our options are somewhat limited. However we can partially address nonlinearity by:

- Breaking categorical or continuous variables into dummy variables (e.g. education levels)
- Including interactions
- Including polynomial terms
- Transformations such as logs
- Generalized Additive Models (GAM)

Many more flexible, nonlinear regression models exist beyond the scope of this course.
How should we deal with nonlinearity?

Given we have a linear regression model, our options are somewhat limited. However, we can partially address nonlinearity by:

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- Including interactions
- Including polynomial terms
- Transformations such as logs
- Generalized Additive Models (GAM)

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Transformations such as logs, Generalized Additive Models (GAM) and many more flexible, nonlinear regression models exist beyond the scope of this course.
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- Generalized Additive Models (GAM)
- Many more flexible, nonlinear regression models exist beyond the scope of this course.
Transformed Buchanan regression

**R Code**

```r
mod.nopb2 <- lm(log(edaybuchanan) ~ log(edaytotal) + log(absnbuchanan),
               data = flvote, subset = county != "Palm Beach")
crPlots(mod.nopb2, las = 1)
```

**Component + Residual Plots**
1 Assumptions and Violations
2 Non-normality
3 Outliers
4 Robust Regression Methods
5 Optional: Measurement Error
6 Conclusion and Appendix
7 Detecting Nonlinearity
8 Linear Basis Function Models
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10 Fun With Kernels
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Fun with Kittens
Bias-Variance Tradeoff
Bias-Variance Tradeoff

![Graph showing the bias-variance tradeoff. The graph illustrates the relationship between model complexity and prediction error. As model complexity increases, the prediction error decreases for both training and test samples, but there is a tradeoff between high bias and low variance on one hand, and low bias and high variance on the other. The graph highlights the need for a balance between these two factors to achieve optimal performance.](image-url)
Example Synthetic Problem

\[ y = \sin(1 + x^2) + \epsilon \]

This section adapted from slides by Radford Neal.
Linear Basis Function Models

We talked before about polynomials $x^2, x^3, x^4$ for modeling non-linearities, this is a linear basis function model.

In general the idea is to do a linear regression of $y$ on $\phi_1(x), \phi_2(x), ..., \phi_{m-1}(x)$ where $\phi_j(x)$ are basis functions.

The model is now:

$$y = f(x, \beta) + \epsilon = \beta_0 + \sum_{j=1}^{m-1} \beta_j \phi_j(x) = \beta^T \phi(x)$$
Linear Basis Function Models

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Polynomial Basis Functions

We can look at OLS fits with polynomial basis functions of increasing order.
Polynomial Basis Functions

We can look at OLS fits with polynomial basis functions of increasing order.

- Second-order polynomial model
- Fourth-order polynomial model
- Sixth-order polynomial model

It appears that the last model is too complex and is overfitting a bit.
Polynomial Basis Functions

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It appears that the last model is too complex and is overfitting a bit.
Local Basis Functions

Polynomials are global basis functions, each affecting the prediction over the whole input space. Often local basis functions are more appropriate. One choice is a Gaussian basis function

$$\phi_j(x) = \exp\left(-\frac{(x - \mu_j)^2}{2 \sigma^2}\right)$$
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Polynomials are **global** basis functions, each affecting the prediction over the whole input space. Often **local** basis functions are more appropriate.

One choice is a Gaussian basis function

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Local Basis Functions

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\[
\phi_j(x) = \exp\left(-\frac{(x - \mu_j)^2}{2s^2}\right)
\]
Gaussian Basis Fits

Gaussian basis function fit, $s = 0.1$

Gaussian basis function fit, $s = 0.5$

Gaussian basis function fit, $s = 2.5$
Regularization

We've seen that flexible models can lead to overfitting. Two ways to address: limit model flexibility or use a flexible model and regularize. Regularization is a way of expressing a preference for smoothness in our function by adding a penalty term to our optimization function. Here we will consider a penalty of the form

$$\lambda \sum_{m=1}^{M} \beta_j^2$$

where $\lambda$ controls the strength of the penalty. The penalty trades off some bias for an improvement in variance. The trick in general is how to set $\lambda$. 
Regularization

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- Here we will consider a penalty of the form $\lambda \sum_{j=1}^{m-1} \beta_j^2$ where $\lambda$ controls the strength of the penalty.
- The penalty trades off some **bias** for an improvement in **variance**.
- The trick in general is how to set $\lambda$. 
Results

Here are the results with $\lambda = 0.1$:

1. Gaussian basis function fit, $s = 0.1$, $\lambda = 0.1$
2. Gaussian basis function fit, $s = 0.5$, $\lambda = 0.1$
3. Gaussian basis function fit, $s = 2.5$, $\lambda = 0.1$
Here are the results with $\lambda = 1$: 
Here are the results with $\lambda = 10$: 

- Gaussian basis function fit, $s = 0.1$, lambda = 10
- Gaussian basis function fit, $s = 0.5$, lambda = 10
- Gaussian basis function fit, $s = 2.5$, lambda = 10
Here are the results with $\lambda = 0.01$:
Conclusions from This Example

we can control overfitting by modifying the width of the basis functions or with penalty. In general, we will need some way to tune these. We will also need some way to handle multivariate functions.

Next up: Generalized Additive Models.

Stewart (Princeton)

Week 9: Diagnostics and Solutions

November 14-21, 2016
Conclusions from This Example

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- next up, Generalized Additive Models
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Generalized Additive Models (GAM)

Recall the linear model,

$$y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + x_{3i}\beta_3 + u_i$$
Generalized Additive Models (GAM)

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\[ y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + x_{3i}\beta_3 + u_i \]

For GAMs, we maintain additivity, but instead of imposing linearity we allow flexible functional forms for each explanatory variable, where \( s_1(\cdot), s_2(\cdot), \) and \( s_3(\cdot) \) are smooth functions that are estimated from the data:
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$$y_i = \beta_0 + s_1(x_{1i}) + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$
Generalized Additive Models (GAM)

\[ y_i = \beta_0 + s_1(x_{1i}) + s_2(x_{2i}) + s_3(x_{3i}) + u_i \]

- GAMS are semi-parametric, they strike a compromise between nonparametric methods and parametric regression
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- \( s_j(\cdot) \) are usually estimated with locally weighted regression smoothers or cubic smoothing splines (but many approaches are possible)
- They do NOT give you a set of regression parameters \( \hat{\beta} \). Instead one obtains a graphical summary of how \( E[Y|X, X_2, \ldots, X_k] \) varies with \( X_1 \) (estimates of \( s_j(\cdot) \) at every value of \( X_{i,j} \))
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- Theory and estimation are somewhat involved, but they are easy to use:
  - `gam.out <- gam(y~s(x1)+s(x2)+x3)`
  - `plot(gam.out)`
  - Multiple functions but I recommend `mgcv` package.
Generalized Additive Models (GAM)

The GAM approach can be extended to allow interactions ($s_{12}(\cdot)$) between explanatory variables, but this eats up degrees of freedom so you need a lot of data.

$$y_i = \beta_0 + s_{12}(x_{1i}, x_{2i}) + s_3(x_{3i}) + u_i$$
Generalized Additive Models (GAM)

The GAM approach can be extended to allow interactions ($s_{12}(\cdot)$) between explanatory variables, but this eats up degrees of freedom so you need a lot of data.

$$y_i = \beta_0 + s_{12}(x_{1i}, x_{2i}) + s_3(x_{3i}) + u_i$$

It can also be used for hybrid models where we model some variables as parametrically and other with a flexible function:

$$y_i = \beta_0 + \beta_1 x_{1i} + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$
GAM Fit to Attitudes Toward Immigration

![Graph showing the relationship between educational attainment and linear predictor.](image)

- Title: GAM Fit to Attitudes Toward Immigration
- Description: The graph illustrates the relationship between educational attainment and linear predictor. The x-axis represents educational attainment, while the y-axis represents the linear predictor. The data points show a trend indicating that higher educational attainment is associated with a higher linear predictor value.
GAM Fit to Attitudes Toward Immigration

![Graph showing a relationship between age and linear predictor s(age).]
GAM Fit to Attitudes Toward Immigration
GAM Fit to Attitudes Toward Immigration

red/green are ± 2 s.e.
GAM Fit to Attitudes Toward Immigration

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GAM Fit to Dyadic Democracy and Militarized Disputes

Figure 9: Local linear regression (loess) estimate of joint effects of each partners' democracy score on militarized disputes. The estimated effects are defined over the two-dimensional space of the predictors, forming a surface. The plots are drawn from the perspective of (a) the least democratic dyads and (b) the most democratic dyads.
Concluding Thoughts

Non-linearity is pretty easy to detect and can substantially change our inferences. GAMs are a great way to model/detect non-linearity, but transformations are often simpler. However, be wary of the global properties of transformations and polynomials. Non-linearity concerns are most relevant for continuous covariates with a large range (age).
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Fun With Kernels


---

2I thank Chad Hazlett for sharing many of the slides that follow
Motivation: Misspecification Bias
Consider a data generating process such as:

```r
> # Predictors
> GDP = runif(500)
> Polity = .5*GDP^2 + .2*runif(200)
>
> # True Model
> Stability = log(GDP)+rnorm(500)
```
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> # True Model
> Stability = log(GDP)+rnorm(500)
```

Regressing Stability on polity and GDP:

```r
> # OLS
> lm(Stability ~ Polity + GDP)
```

|            | Estimate | Std. Error | t value | Pr(>|t|) |
|------------|----------|------------|---------|----------|
| (Intercept)| -2.3000  | 0.1039     | -22.145 | < 2e-16  *** |
| Polity     | -3.1983  | 0.7613     | -4.201  | 3.15e-05 *** |
| GDP        | 4.3443   | 0.4237     | 10.252  | < 2e-16  *** |
Motivation: Misspecification Bias

Consider a data generating process such as:

\[
\begin{align*}
\text{GDP} &= \text{runif}(500) \\
\text{Polity} &= 0.5\times \text{GDP}^2 + 0.2 \times \text{runif}(200) \\
\text{Stability} &= \log(\text{GDP}) + \text{rnorm}(500)
\end{align*}
\]

Regressing Stability on Polity and GDP:

\[
\text{lm(Stability} \sim \text{Polity} + \text{GDP})
\]

| Estimate  | Std. Error | t value | Pr(>|t|)    |
|-----------|------------|---------|-------------|
| (Intercept) | -2.3000    | 0.1039  | -22.145     | < 2e-16 *** |
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| GDP       | 4.3443     | 0.4237  | 10.252      | < 2e-16 *** |

Entirely wrong conclusions!
Misspecification Bias

Try more flexible method that still reports marginal effects:

> krls(y=Stability,X=cbind(GDP,Polity))

Average Marginal Effects:

|        | Est   | Std. Error | t value  | Pr(>|t|)    |
|--------|-------|------------|----------|-------------|
| GDP    | 3.386 | 0.522      | 6.489    | 2.084441e-10|
| Polity | -0.414| 0.783      | -0.529   | 5.967968e-01|
## Misspecification Bias

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### Average Marginal Effects:

|        | Est    | Std. Error | t value | Pr(>|t|) |
|--------|--------|------------|---------|----------|
| GDP    | 3.3855912 | 0.5217110  | 6.4893996 | 2.084441e-10 |
| Polity | -0.4143114 | 0.7826758  | -0.5293525 | 5.967968e-01 |

\[ E[\text{stability}|\text{gdp, mean(polity)}] \]

\[ E[\text{stability} | \text{mean(gdp), polity}] \]
Kernel Basics

Kernel

For now, a kernel is a function $\mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}$

$$k(x_i, x_j) \rightarrow \mathbb{R}$$
Kernel Basics

Kernel

For now, a kernel is a function $\mathbb{R}^P \times \mathbb{R}^P \to \mathbb{R}$

$$k(x_i, x_j) \to \mathbb{R}$$

Some kernels are naturally interpretable as a distance metric, e.g. the Gaussian:

Gaussian Kernel

$$k(\cdot, \cdot) : \mathbb{R}^D \times \mathbb{R}^P \mapsto \mathbb{R}$$

$$k(x_j, x_i) = e^{-\frac{||X_j - X_i||^2}{\sigma^2}}$$

where $||X_j - X_i||$ is the Euclidean distance between $X_j$ and $X_i$
Using the Kernel Trick for Regression

- A feature map, $\phi : \mathbb{R}^P \mapsto \mathbb{R}^{P'}$, such that: $k(X_i, X_j) = \langle \phi(X_i), \phi(X_j) \rangle$
Using the Kernel Trick for Regression

- A feature map, \( \phi : \mathbb{R}^P \mapsto \mathbb{R}^{P'} \), such that: \( k(X_i, X_j) = \langle \phi(X_i), \phi(X_j) \rangle \)
- A linear model in the new features: \( f(X_i) = \phi(X_i)^T \theta, \theta \in \mathbb{R}^{P'} \)
Using the Kernel Trick for Regression

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- A linear model in the new features: $f(X_i) = \phi(X_i)^T \theta$, $\theta \in \mathbb{R}^{P'}$
- Regularized (ridge) regression:

$$\arg\min_{\theta \in \mathbb{R}^{P'}} \sum_{i=1}^{N} (Y_i - \phi(X_i)^T \theta)^2 + \lambda \langle \theta, \theta \rangle$$
Using the Kernel Trick for Regression

- A feature map, $\phi : \mathbb{R}^P \mapsto \mathbb{R}^{P'}$, such that: $k(X_i, X_j) = \langle \phi(X_i), \phi(X_j) \rangle$
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$$\arg\min_{\theta \in \mathbb{R}^{P'}} \sum_{i=1}^{N} (Y_i - \phi(X_i)^T \theta)^2 + \lambda \langle \theta, \theta \rangle$$

- Solve the F.O.C.s:

$$R(\theta, \lambda) = \sum_{i=1}^{N} (Y_i - \phi(X_i)^T \theta)^2 + \lambda \theta^\top \theta$$

$$\frac{\partial R(\theta, \lambda)}{\partial \theta} = -2 \sum_{i=1}^{N} \phi(X_i)(Y_i - \phi(X_i)^T \theta) + 2\lambda \theta = 0$$
How would humans learn this?

Linear regression:

\[ E[\text{alt} | \text{lat}, \text{long}] = \beta_0 + \beta_1 \text{lat} + \beta_2 \text{long} + \beta_3 \text{lat} \times \text{long} + \ldots \]

Similarity model:

\[ E[\text{alt} | \text{lat}, \text{long}] = c_1 (\text{similarity to obs1}) + \ldots + c_5 (\text{similarity to obs5}) \]
How would humans learn this?

Linear regression?

\[ E[alt|lat, long] = \beta_0 + \beta_1 lat + \beta_2 long + \beta_3 lat \times long + \ldots \]
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Linear regression?

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Similarity model:

\[ E[alt|lat, long] = c_1(\text{similarity to obs1}) + \ldots + c_5(\text{similarity to obs5}) \]
Intuition: Similarity

Think of this function space as built on similarity:

\[ f(X^*) = \sum_{i=1}^{N} c_i k(X^*, X_i) \]

\[ = c_1 \text{(similarity of } X^* \text{ to } X_1) + \ldots + c_N \text{(similarity of } X^* \text{ to } X_N) \]
Intuition: Similarity

Think of this function space as built on similarity:

\[ f(X^*) = \sum_{i=1}^{N} c_i k(X^*, X_i) \]

\[ = c_1 \text{(similarity of } X^* \text{ to } X_1) + \ldots + c_N \text{(similarity of } X^* \text{ to } X_N) \]

Some random functions from this space:
A real example: Harff 2003

From summary(krls(y,X))

<table>
<thead>
<tr>
<th>DV: Genocide onset</th>
<th>$\beta_{OLS}$</th>
<th>$E[\frac{dy}{dx_i}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prior upheaval</td>
<td>0.009*</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(0.004)</td>
<td>0.00</td>
</tr>
<tr>
<td>Prior genocide</td>
<td>0.26*</td>
<td>0.19*</td>
</tr>
<tr>
<td></td>
<td>(0.12)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>Ideological char. elite</td>
<td>0.15*</td>
<td>0.13*</td>
</tr>
<tr>
<td></td>
<td>(0.084)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>Autocracy</td>
<td>0.16*</td>
<td>0.12*</td>
</tr>
<tr>
<td></td>
<td>(0.077)</td>
<td>(0.07)</td>
</tr>
<tr>
<td>Ethnic char. elite</td>
<td>0.12</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>(0.084)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>log(trade openness)</td>
<td>-0.17*</td>
<td>-0.09*</td>
</tr>
<tr>
<td></td>
<td>(0.057)</td>
<td>(0.03)</td>
</tr>
</tbody>
</table>
Behind the averages

plot(krls(X,y))

Distributions of pointwise marginal effects
Efficiency Comparison

\[ y = 2x + \epsilon, \ x \sim N(0, 1), \epsilon \sim N(0, 1) \]
High-dimensional data with non-linearities

\[ y = (x_1 x_2) - 2(x_3 x_4) + 3(x_5 x_6 x_7) - (x_1 x_8) + 2(x_8 x_9 x_{10}) + x_{10} + \epsilon \]

where all \( X \) are i.i.d. \( \text{Bernoulli}(p) \) at varying \( p \), \( \epsilon \sim N(0, .5) \). 1,000 test points.
Linear model with bad leverage points

- $y = 0.5x + \varepsilon$ where $\varepsilon \sim N(0, 0.3)$
- One bad point, $(y_i = -5, x_i = 5)$. 

![Graph showing the comparison of OLS and KRLS estimates with the true average derivative.]
Interaction or non-linearity?

**Truth:** \( y = 5x_1^2 + \varepsilon, \quad \rho(x_1, x_2) = .72 \)
\( \varepsilon \sim (0, .44). \quad x_1 \sim \text{Uniform}(0, 2) \)
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OLS Model: \( y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 \cdot x_2 \)
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<table>
<thead>
<tr>
<th>Estimator</th>
<th>OLS</th>
<th></th>
<th>KRLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial y}{\partial x_{ij}} )</td>
<td>Average</td>
<td>Average</td>
<td>1st Qu.</td>
</tr>
<tr>
<td>const</td>
<td>-1.50 ( (0.34) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_1 )</td>
<td>7.51 ( (0.40) )</td>
<td>9.22 ( (0.52) )</td>
<td>5.22 ( (0.82) )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>-1.28 ( (0.21) )</td>
<td>0.02 ( (0.13) )</td>
<td>-0.08 ( (0.19) )</td>
</tr>
<tr>
<td>( x_1 \cdot x_2 )</td>
<td>1.24 ( (0.15) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>250</td>
<td></td>
<td></td>
</tr>
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Concluding Thoughts

Strengths

▶ extremely powerful at detecting interactions
▶ captures increasingly complex functions as data increases
▶ great as a robustness check

Difficulties/Future Work

▶ computation scales in number of datapoints \( O(N^3) \) which means it doesn't work for more than about 5000 datapoints
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Stewart (Princeton)
Week 9: Diagnostics and Solutions
November 14-21, 2016
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▶ causality with measured confounding

Long Run

▶ regression → diagnostics → causal inference

Questions?

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- Questions?
Assumptions and Violations

Non-normality

Outliers

Robust Regression Methods

Optional: Measurement Error

Conclusion and Appendix

Detecting Nonlinearity

Linear Basis Function Models

Generalized Additive Models

Fun With Kernels

Heteroskedasticity

Clustering

Optional: Serial Correlation

A Contrarian View of Robust Standard Errors

Fun with Neighbors

Fun with Kittens
A Quick Note of Thanks
Review of the OLS Assumptions

- **Identification**
  - Data Description
  - No Perfect Collinearity

- **Unbiasedness**
  - Consistency
  - No Perfect Collinearity
  - Random Sampling
  - Linearity in Parameters
  - Zero Conditional Mean

- **Gauss-Markov (BLUE)**
  - Asymptotic Inference (z and $\chi^2$)
  - No Perfect Collinearity
  - Random Sampling
  - Linearity in Parameters
  - Zero Conditional Mean
  - Homoskedasticity

- **Classical LM (BUE)**
  - Small-Sample Inference (t and F)
  - No Perfect Collinearity
  - Random Sampling
  - Linearity in Parameters
  - Zero Conditional Mean
  - Homoskedasticity
  - Normality of Errors
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1. Linearity: \( y = X\beta + u \)
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- 1-5 are the Gauss-Markov, allow for large-sample inference
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How Do We Deal With This?

A scatter plot shows a linear relationship between two variables, X and Y, with a diagonal line indicating the trend. The data points are scattered around the line, suggesting a strong correlation between the variables.
Plan for Today

Talk about different forms of error variance problems
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1. Heteroskedasticity
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1. Heteroskedasticity
2. Clustering
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Each is a violation of **homoskedasticity**, but each has its own diagnostics and corrections.
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Each is a violation of homoskedasticity, but each has its own diagnostics and corrections.

Then we will discuss a contrarian view
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- Remember:

\[ \hat{\beta} = (X'X)^{-1} X'y \]
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  \[ = \sigma^2 (X'X)^{-1} \]

- Replace \( \sigma^2 \) with estimate \( \hat{\sigma}^2 \) will give us our estimate of the covariance matrix
Non-constant Error Variance

- Homoskedastic:

\[ V[u|X] = \sigma^2 I = \begin{bmatrix}
\sigma^2 & 0 & 0 & \ldots & 0 \\
0 & \sigma^2 & 0 & \ldots & 0 \\
0 & 0 & \sigma^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \sigma^2
\end{bmatrix} \]
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\]

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\[
V[u|X] = \begin{bmatrix}
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0 & \sigma_2^2 & 0 & \ldots & 0 \\
0 & 0 & \sigma_2^2 & \ldots & 0 \\
\vdots & & & \ddots & \sigma_n^2 \\
0 & 0 & 0 & \ldots & \sigma_n^2
\end{bmatrix}
\]

Independent, not identical

\[
\text{Cov}(u_i, u_j|X) = 0
\]

\[
\text{Var}(u_i|X) = \sigma_i^2
\]
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  \vdots & & & & \\
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  \vdots & & & & \\
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  0 & \sigma_2^2 & 0 & \ldots & 0 \\
  \vdots \\
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- Independent, not identical
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Example: $V[u|X] = \sigma^2$ Homoskedasticity
Example: \( V[u|X] = \sigma_i^2 \) Heteroskedasticity
Consequences of Heteroskedasticity

Standard $\hat{\sigma}^2$ is biased and inconsistent for $\sigma^2$
Standard error estimates incorrect: $\hat{SE}[\hat{\beta}_1] = \hat{\sigma}^2 \sum_i (X_i - \bar{X})^2$
Test statistics won't have $t$ or $F$ distributions
$\alpha$-level tests, the probability of Type I error $\neq \alpha$
Coverage of $1-\alpha$ CIs $\neq 1-\alpha$
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▶ degree of the problem depends on how serious the heteroskedasticity is
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Visual diagnostics

1. Plot of residuals versus fitted values

   In R, `plot(mod, which = 1)`

   - y-axis: Square-root of the absolute value of the residuals (folds the plot in half)
   - x-axis: Fitted values
   - Usually has loess trend curve to check if variance varies with fitted values

   In R, `plot(mod, which = 3)`
Visual diagnostics

1 Plot of residuals versus fitted values
   ▶ In R, `plot(mod, which = 1)`
Visual diagnostics

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   - y-axis: Square-root of the absolute value of the residuals
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Example: Buchanan votes

```r
flvote <- foreign::read.dta("flbuchan.dta")
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```r
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    5.423e+01  4.914e+01  1.104   0.274
## edaytotal     2.323e-03  3.104e-04  7.483  2.42e-10 ***
## ---
## Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
## Residual standard error: 332.7 on 65 degrees of freedom
## Multiple R-squared:  0.4628, Adjusted R-squared:  0.4545
## F-statistic:  56 on 1 and 65 DF,  p-value: 2.417e-10
```
Diagnostics

par(mfrow = c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")
plot(mod, which = 1, lwd = 3)
plot(mod, which = 3, lwd = 3)
Formal Tests for Non-constant Error Variances

Plots are usually sufficient, but formal tests for heteroskedasticity exist (e.g. Breusch-Pagan, Cook-Weisberg, White, etc.). They are all roughly based on the same idea:

$H_0: V[u_i|X] = \sigma^2$

Under the zero conditional mean assumption, this is equivalent to

$H_0: E[u_i^2|X] = E[u_i^2] = \sigma^2$, a constant unrelated to $X$

This implies that, under $H_0$, the squared residuals should also be unrelated to the explanatory variables.

The Breusch-Pagan test:

1. Regression $y_i$ on $x_i'$ and store residuals, $\hat{u}_i$
2. Regress $\hat{u}_i^2$ on $x_i'$
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In R, `bptest` in the `lmtest` package
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  3. Run $F$-test against null that all slope coefficients are 0

- In R, `bptest` in the `lmtest` package
library(lmtest)
bptest(mod)

##
## studentized Breusch-Pagan test
##
## data:  mod
## BP = 12.59, df = 1, p-value = 0.0003878
Dealing with Non-Constant Error Variance

1. Transform the dependent variable (this will affect other model assumptions)

2. Adjust for the heteroskedasticity using known weights and Weighted Least Squares (WLS)

3. Use an estimator of \( \text{Var}[\hat{\beta}] \) that is robust to heteroskedasticity

4. Admit we have the wrong model and use a different approach
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Variance Stabilizing Transformations

If the variance for each error ($\sigma^2_i$) is proportional to some function of the mean ($x_i \beta$), then a variance stabilizing transformation may be appropriate. Note: Transformations will affect the other regression assumptions, as well as interpretation of the regression coefficients.

Examples:

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Mean/Variance Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{Y}$</td>
<td>$\sigma^2_i \propto x_i \beta$</td>
</tr>
<tr>
<td>$\log Y$</td>
<td>$\sigma^2_i \propto (x_i \beta)^2$</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>log ( Y )</td>
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</tr>
<tr>
<td>( 1/Y )</td>
<td>( \sigma_i^2 \propto (x_i / \beta)^4 )</td>
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</tbody>
</table>
Example: Transforming Buchanan Votes

mod2 <- lm(log(edaybuchanan) ~ log(edaytotal), data = flvote)
summary(mod2)

##
## Coefficients:
##      Estimate Std. Error  t value Pr(>|t|)
## (Intercept)  -2.7279   0.39956   -6.827  3.5e-09 ***
## log(edaytotal)  0.7285   0.0380    19.154  < 2e-16 ***
## ---
## Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
## Residual standard error: 0.4688 on 65 degrees of freedom
## Multiple R-squared:  0.8495, Adjusted R-squared:  0.8472
## F-statistic: 366.9 on 1 and 65 DF,  p-value: < 2.2e-16
Example: Transformed Scale-Location Plot

\[ \text{plot(mod2, which=3)} \]

\[ \text{lm(log(edaybuchanan) ~ log(edaytotal))} \]

\[ \text{Scale-Location} \]

\[ \sqrt{\text{Standardized residuals}} \]

Fitted values

\[ \text{lm(log(edaybuchanan) ~ log(edaytotal))} \]
Example: Transformed

bptest(mod, studentize=FALSE)

##
## Breusch-Pagan test
##
## data:  mod
## BP = 250.07, df = 1, p-value < 2.2e-16

bptest(mod2, studentize=FALSE)

##
## Breusch-Pagan test
##
## data:  mod2
## BP = 0.01105, df = 1, p-value = 0.9163
Appendix: Weighted Least Squares

- Suppose that the heteroskedasticity is known up to a multiplicative constant:
  \[ \text{Var}[u_i | X] = a_i \sigma^2 \]
  where \( a_i = a_i(x'_i) \) is a positive and known function of \( x'_i \)
- WLS: multiply \( y_i \) by \( 1/\sqrt{a_i} \):
  \[ y_i/\sqrt{a_i} = \beta_0/\sqrt{a_i} + \beta_1 x_{i1}/\sqrt{a_i} + \cdots + \beta_k x_{ik}/\sqrt{a_i} + u_i/\sqrt{a_i} \]
Appendix: Weighted Least Squares Intuition

Rescales errors to \( u_i / \sqrt{a_i} \), which maintains zero mean error but makes the error variance constant again:

\[
\text{Var}\left[ \frac{u_i}{\sqrt{a_i}} \right| X] = \frac{1}{a_i} \cdot \text{Var}\left[ u_i \right| X] = \frac{\sigma^2}{a_i}
\]

If you know \( a_i \), then you can use this approach to make the model homoskedastic and, thus, BLUE again.

When do we know \( a_i \)?
Appendix: Weighted Least Squares Intuition

- Rescales errors to $u_i / \sqrt{a_i}$, which maintains zero mean error.
- But makes the error variance constant again:

$$\text{Var} \left[ \frac{1}{\sqrt{a_i}} u_i | X \right] = \frac{1}{a_i} \text{Var} [u_i | X]$$
Appendix: Weighted Least Squares Intuition

- Rescales errors to $u_i / \sqrt{a_i}$, which maintains zero mean error
- But makes the error variance constant again:

$$ \text{Var} \left( \frac{1}{\sqrt{a_i}} u_i \middle| X \right) = \frac{1}{a_i} \text{Var} [u_i \middle| X] $$

$$ = \frac{1}{a_i} a_i \sigma^2 $$

If you know $a_i$, then you can use this approach to make the model homoskedastic and, thus, BLUE again.
Appendix: Weighted Least Squares Intuition

- Rescales errors to $u_i/\sqrt{a_i}$, which maintains zero mean error
- But makes the error variance constant again:
  \[
  \text{Var} \left[ \frac{1}{\sqrt{a_i}} u_i | X \right] = \frac{1}{a_i} \text{Var} [u_i | X]
  = \frac{1}{a_i} a_i \sigma^2
  = \sigma^2
  \]
- If you know $a_i$, then you can use this approach to make the model homoskedastic and, thus, BLUE again
- When do we know $a_i$?
Appendix: Weighted Least Squares procedure

Define the weighting matrix:

\[ W = \begin{pmatrix} 1 / \sqrt{a_1} & 0 & 0 & 0 \\ 0 & 1 / \sqrt{a_2} & 0 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & 1 / \sqrt{a_n} \end{pmatrix} \]

Run the following regression:

\[ Wy = WX \beta + Wu \]

\[ y^* = X^* \beta + u^* \]

Run regression of \( y^* = Wy \) on \( X^* = WX \) and all Gauss-Markov assumptions are satisfied.

Plugging into the usual formula for \( \hat{\beta} \):

\[ \hat{\beta} = \left( X^* W^* X \right)^{-1} X^* W^* Wy \]
Appendix: Weighted Least Squares procedure

- Define the weighting matrix:

\[
W = \begin{bmatrix}
\frac{1}{\sqrt{a_1}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{a_2}} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \frac{1}{\sqrt{a_n}}
\end{bmatrix}
\]

- Run the following regression:

\[
Wy = WX\beta + Wu
\]

\[
y^* = X^*\beta + u^*
\]

- Run regression of \(y^* = Wy\) on \(X^* = WX\) and all Gauss-Markov assumptions are satisfied

- Plugging into the usual formula for \(\hat{\beta}\):

\[
\hat{\beta}_W = (X'W'WX)^{-1}X'W'Wy
\]
Appendix: WLS Example

- In R, use `weights` argument to `lm` and give the weights squared: $1/a_i$
- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

```r
mod.wls <- lm(edaybuchanan ~ edaytotal, weights = 1/edaytotal, data = flvote)
summary(mod.wls)
```

## Coefficients:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| (Intercept) 2.707e+01 8.507e+00 3.182 0.00225 ** |
| edaytotal 2.628e-03 2.502e-04 10.503 1.22e-15 *** |

---

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.5645 on 65 degrees of freedom
Multiple R-squared: 0.6292, Adjusted R-squared: 0.6235
F-statistic: 110.3 on 1 and 65 DF, p-value: 1.22e-15
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<td>Estimate</td>
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**F-statistic:** 110.3 on 1 and 65 DF, **p-value:** 1.22e-15
Appendix: Comparing WLS to OLS

par(mfrow=c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")
plot(mod, which = 3, main = "OLS", lwd = 2)
plot(mod.wls, which = 3, main = "WLS", lwd = 2)
Heteroskedasticity Consistent Estimator

Under non-constant error variance:

\[ \text{Var}[u] = \Sigma = \begin{bmatrix}
\sigma_1^2 & 0 & 0 & \cdots & 0 \\
0 & \sigma_2^2 & 0 & \cdots & 0 \\
0 & 0 & \sigma_3^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sigma_n^2
\end{bmatrix} \]

When \( \Sigma \neq \sigma^2 I \), we are stuck with this expression:

\[ \text{Var}[\hat{\beta}|X] = (X'X)^{-1}X'\Sigma X(X'X)^{-1} \]

Idea: If we can consistently estimate the components of \( \Sigma \), we could directly use this expression by replacing \( \Sigma \) with its estimate, \( \hat{\Sigma} \).
Heteroskedasticity Consistent Estimator

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\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \sigma_n^2 \\
\end{bmatrix}$$

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Heteroskedasticity Consistent Estimator

- Under non-constant error variance:

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\text{Var}[u] = \Sigma = \begin{bmatrix}
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0 & 0 & \sigma_3^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \sigma_n^2
\end{bmatrix}
\]

- When \( \Sigma \neq \sigma^2 I \), we are stuck with this expression:

\[
\text{Var}[\hat{\beta}|X] = (X'X)^{-1} X' \Sigma X (X'X)^{-1}
\]
Heteroskedasticity Consistent Estimator

- Under non-constant error variance:

\[
\text{Var}[u] = \Sigma = \begin{bmatrix}
\sigma_1^2 & 0 & 0 & \ldots & 0 \\
0 & \sigma_2^2 & 0 & \ldots & 0 \\
0 & 0 & \sigma_3^2 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \ldots & \sigma_n^2
\end{bmatrix}
\]

- When \( \Sigma \neq \sigma^2 I \), we are stuck with this expression:

\[
\text{Var}[\hat{\beta} | X] = (X'X)^{-1} X'\Sigma X (X'X)^{-1}
\]

- Idea: If we can consistently estimate the components of \( \Sigma \), we could directly use this expression by replacing \( \Sigma \) with its estimate, \( \hat{\Sigma} \).
White’s Heteroskedasticity Consistent Estimator

Suppose we have heteroskedasticity of unknown form:

\[ V[u] = \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2^2 & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & 0 & \ldots & \sigma_n^2 \end{bmatrix} \]
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then \( V[\hat{\beta}|X] = (X'X)^{-1} X' \Sigma X (X'X)^{-1} \) and White (1980) shows that
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then \( V[\hat{\beta}|X] = (X'X)^{-1} X' \Sigma X (X'X)^{-1} \) and White (1980) shows that

\[ \overline{V[\hat{\beta}|X]} = (X'X)^{-1} X' \begin{bmatrix}
\hat{u}_1^2 & 0 & 0 & \ldots & 0 \\
0 & \hat{u}_2^2 & 0 & \ldots & 0 \\
& & & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \hat{u}_n^2
\end{bmatrix} X (X'X)^{-1} \]

is a consistent estimator of \( V[\hat{\beta}|X] \) under any form of heteroskedasticity consistent with \( V[u] \) above.
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\end{bmatrix} \]

then \( V[\hat{\beta}|X] = (X'X)^{-1} X' \Sigma X (X'X)^{-1} \) and White (1980) shows that

\[ \sqrt{\hat{\Sigma}} = \begin{bmatrix}
\hat{u}_1^2 & 0 & 0 & \ldots & 0 \\
0 & \hat{u}_2^2 & 0 & \ldots & 0 \\
\vdots & & & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \hat{u}_n^2
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is a consistent estimator of \( V[\hat{\beta}|X] \) under any form of heteroskedasticity consistent with \( V[u] \) above.

The estimate based on the above is called the heteroskedasticity consistent (HC) or robust standard errors.
White’s Heteroskedasticity Consistent Estimator

Robust standard errors are easily computed with the “sandwich” formula:

Fit the regression and obtain the residuals $\hat{u}$. Construct the “meat” matrix $\hat{\Sigma}$ with squared residuals in diagonal:

$$\hat{\Sigma} = \begin{bmatrix}
\hat{u}_1^2 & 0 & 0 & \ldots & 0 \\
0 & \hat{u}_2^2 & 0 & \ldots & 0 \\
0 & 0 & \hat{u}_3^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \hat{u}_n^2
\end{bmatrix}$$

Plug $\hat{\Sigma}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix $V[\hat{\beta}|X] = (X'X)^{-1}X'\hat{\Sigma}X(X'X)^{-1}$.

There are various small sample corrections to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|X] = \frac{n}{n-k-1} \cdot \left( (X'X)^{-1}X'\hat{\Sigma}X(X'X)^{-1} \right)$$

Stewart (Princeton)
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3. Plug $\hat{\Sigma}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix

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V[\hat{\beta}|X] = \frac{n}{n-k-1} \cdot (X^'X)^{-1} X^' \hat{\Sigma} X (X^'X)^{-1}
\]
Regular & Robust Standard Errors in Florida Example

R Code

```r
> library(sandwich)
> library(lmtest)
> coeftest(mod1) # homoskedasticity
  t test of coefficients:

    Estimate  Std. Error   t value     Pr(>|t|)
(Intercept) 5.4231e+01  4.9141e+01   1.1036       0.2738
TotalVotes00 2.3229e-03  3.1041e-04   7.4831  2.417e-10 ***

> coeftest(mod1, vcov = vcovHC(mod1, type = "HC0")) # classic White
  t test of coefficients:

    Estimate  Std. Error   t value     Pr(>|t|)
(Intercept) 5.4231e+01  4.0612e+01   1.3353       0.18642
TotalVotes00 2.3229e-03  8.7047e-04   2.6685  0.00961 **

> coeftest(mod1, vcov = vcovHC(mod1, type = "HC1")) # small sample correction
  t test of coefficients:

    Estimate  Std. Error   t value     Pr(>|t|)
(Intercept) 5.4231e+01  4.1232e+01   1.3153       0.19304
TotalVotes00 2.3229e-03  8.8376e-04   2.6284  0.01069 *
```
WLS vs. White’s Estimator

- **WLS:**

  - With known weights, WLS is efficient
  - \( \hat{\text{SE}} \left[ \hat{\beta}_{\text{WLS}} \right] \) is unbiased and consistent
  - But weights usually aren’t known

- **White’s Estimator:**

  - Doesn’t change estimate
  - Consistent for \( \text{Var} \left[ \hat{\beta} \right] \) under any form of heteroskedasticity
  - Because it relies on consistency, it is a large sample result, best with large \( n \)
  - For small \( n \), performance might be poor (correction factors exist but are often insufficient)
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1 Assumptions and Violations
2 Non-normality
3 Outliers
4 Robust Regression Methods
5 Optional: Measurement Error
6 Conclusion and Appendix
7 Detecting Nonlinearity
8 Linear Basis Function Models
9 Generalized Additive Models
10 Fun With Kernels
11 Heteroskedasticity
12 Clustering
13 Optional: Serial Correlation
14 A Contrarian View of Robust Standard Errors
15 Fun with Neighbors
16 Fun with Kittens
Clustered Dependence: Intuition

Think back to the Gerber, Green, and Larimer (2008) social pressure mailer example. Their design: randomly sample households and randomly assign them to different treatment conditions. But the measurement of turnout is at the individual level. Violation of iid/random sampling:

- Errors of individuals within the same household are correlated.
- Violation of homoskedasticity.

Called clustering or clustered dependence.
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- Clusters: \( j = 1, \ldots, m \)
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- Units: \( i = 1, \ldots, n_j \)

Especially important when outcome varies at the unit-level, \( y_{ij} \) and the main independent variable varies at the cluster level, \( x_j \).

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Units (usually) belong to a single cluster:

- voters in households
- individuals in states
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Clustered Dependence: Example Model

\[ y_{ij} = \beta_0 + \beta_1 x_{ij} + \varepsilon_{ij} \]

\[ \varepsilon_{ij} = \beta_0 + \beta_1 x_{ij} + v_j + u_{ij} \]

\( v_j \) and \( u_{ij} \) are assumed to be independent of each other.

\( \rho \in (0, 1) \) is called the within-cluster correlation.

What if we ignore this structure and just use \( \varepsilon_{ij} \) as the error?

Variance of the composite error is \( \sigma^2 \):

\[ \text{Var}[\varepsilon_{ij}] = \text{Var}[v_j + u_{ij}] = \text{Var}[v_j] + \text{Var}[u_{ij}] = \rho \sigma^2 + (1 - \rho) \sigma^2 = \sigma^2 \]
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- What if we ignore this structure and just use \( \varepsilon_{ij} \) as the error?
- Variance of the composite error is \( \sigma^2 \):

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Lack of Independence

- Covariance between two units $i$ and $s$ in the same cluster is $\rho \sigma^2$:

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sj}] = \rho \sigma^2$$
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- Covariance between two units $i$ and $s$ in the same cluster is $\rho \sigma^2$:
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- Correlation between units in the same group is just $\rho$:
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  \[ \text{Cor}[\varepsilon_{ij}, \varepsilon_{sj}] = \rho \]

- Zero covariance of two units $i$ and $s$ in different clusters $j$ and $k$:
  \[ \text{Cov}[\varepsilon_{ij}, \varepsilon_{sk}] = 0 \]
Example Covariance Matrix

\[ \mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{bmatrix} \]′

\[ \text{Var}[\mathbf{\varepsilon}] = \mathbf{\Sigma} = \begin{bmatrix}
\sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\
\sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\
\sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\
0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\
0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 
\end{bmatrix} \]
### Appendix: Example 6 Units, 2 Clusters

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{bmatrix}'$$

$$V[\boldsymbol{\varepsilon}] = \Sigma = \begin{bmatrix}
V[\varepsilon_{1,1}] & \text{Cov}[\varepsilon_{1,1}, \varepsilon_{2,1}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{1,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{1,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{2,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{3,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{4,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{4,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{6,2}] & V[\varepsilon_{4,2}] & V[\varepsilon_{4,2}] & V[\varepsilon_{5,2}] & V[\varepsilon_{5,2}] & V[\varepsilon_{5,2}] & V[\varepsilon_{5,2}] & V[\varepsilon_{6,2}] & V[\varepsilon_{6,2}] \end{bmatrix}
$$

$$= \begin{bmatrix}
\sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\
\sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\
\sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\
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0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 
\end{bmatrix}$$

which can be verified as follows:

- **V[εij] = V[\nu_j + \mu_{ij}] = V[\nu_j] + V[\mu_{ij}] = \rho \sigma^2 + (1 - \rho) \sigma^2 = \sigma^2**

- **Cov[εij, εlj] = E[εijεlj] - E[εij]E[εlj] = E[εijεlj] = E((\nu_j + \mu_{ij})(\nu_j + \mu_{lj}))**

  $$= E[\nu_j^2] + E[\nu_j \mu_{ij}] + E[\nu_j \mu_{lj}] + E[\mu_{ij} \mu_{lj}]$$

  $$= E[\nu_j^2] + E[\nu_j]E[\mu_{ij}] + E[\nu_j]E[\mu_{lj}] + E[\mu_{ij}]E[\mu_{lj}]$$

  $$= E[\nu_j^2] = V[\nu_j] + (E[\nu_j])^2 = V[\nu_j] = \rho \sigma^2$$

- **Cov[εij, εlk] = E[εijεlk] - E[εij]E[εlk] = E[εijεlk] = E((\nu_j + \mu_{ij})(\nu_k + \mu_{lk}))**

  $$= E[\nu_j \nu_k] + E[\nu_j \mu_{lk}] + E[\nu_k \mu_{ij}] + E[\mu_{ij} \mu_{lk}]$$

  $$= E[\nu_j]E[\nu_k] + E[\nu_j]E[\mu_{lk}] + E[\nu_k]E[\mu_{ij}] + E[\mu_{ij}]E[\mu_{lk}] = 0$$
Error Variance Matrix with Cluster Dependence

The variance-covariance matrix of the error, $\Sigma$, is block diagonal:

$$
\begin{bmatrix}
\Sigma_1 \\
0 \\
\vdots \\
0 \\
\Sigma_M
\end{bmatrix}
$$

But the errors may be correlated for units within the same cluster:

$$
\Sigma_j =
\begin{bmatrix}
\sigma_1^2 & \sigma_1 \sigma_2 \rho & \cdots & \sigma_1 \sigma_M \rho \\
\sigma_1 \sigma_2 \rho & \sigma_2^2 & \cdots & \sigma_2 \sigma_M \rho \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_1 \sigma_M \rho & \sigma_2 \sigma_M \rho & \cdots & \sigma_M^2
\end{bmatrix}
$$
Error Variance Matrix with Cluster Dependence

The variance-covariance matrix of the error, $\Sigma$, is block diagonal:

- By independence, the errors are uncorrelated across clusters:

$$V[\varepsilon] = \Sigma = \begin{bmatrix} \Sigma_1 & 0 & \ldots & 0 \\ 0 & \Sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Sigma_M \end{bmatrix}$$
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Correcting for Clustering

1. Including a dummy variable for each cluster (fixed effects)
2. "Random effects" models (take above model as true and estimate $\rho$ and $\sigma^2$
3. Cluster-robust ("clustered") standard errors
4. Aggregate data to the cluster-level and use OLS

If $n_j$ varies by cluster, then cluster-level errors will have heteroskedasticity

Can use WLS with cluster size as the weights
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- First, let’s write the within-cluster regressions like so:

\[ y_j = X_j \beta + \varepsilon_j \]
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- Remember our sandwich expression:
  
  \[ \text{Var}[\hat{\beta}|X] = (X'X)^{-1} X'\Sigma X (X'X)^{-1} \]
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- Remember our sandwich expression:
  
  \[ \text{Var}[\hat{\beta} | X] = (X'X)^{-1} X' \Sigma X (X'X)^{-1} \]

- Under this clustered dependence, we can write this as:
  
  \[ \text{Var}[\hat{\beta} | X] = (X'X)^{-1} \left( \sum_{j=1}^{m} X'_j \Sigma_j X_j \right) (X'X)^{-1} \]
Estimating the Variance Components: $\rho$ and $\sigma^2$

The overall error variance $\sigma^2$ is easily estimated using our usual estimator based on the regression residuals:

$$\hat{\sigma}^2 = \frac{\hat{\epsilon}'\hat{\epsilon}}{N - k - 1}$$
Estimating the Variance Components: $\rho$ and $\sigma^2$

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The within-cluster correlation can be estimated as follows:

1. Subtract from each residual $\hat{\epsilon}_{ij}$ the mean residual within its cluster. Call this vector of demeaned residuals $\tilde{\epsilon}$, which estimates the unit error component $u$
Estimating the Variance Components: $\rho$ and $\sigma^2$

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$$\tilde{\sigma}^2 = \frac{\tilde{\epsilon}'\tilde{\epsilon}}{N-M-k-1}$$, which estimates $(1 - \rho)\sigma^2$.
Estimating the Variance Components: $\rho$ and $\sigma^2$

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2. Compute the variance of the demeaned residuals as: 

$$\tilde{\sigma}^2 = \frac{\tilde{\epsilon}'\tilde{\epsilon}}{N-M-k-1}, \text{ which estimates } (1-\rho)\sigma^2$$

3. The within cluster correlation is then estimated as: 

$$\hat{\rho} = \frac{\tilde{\sigma}^2 - \hat{\sigma}^2}{\tilde{\sigma}^2}$$
Estimating Cluster Robust Standard Errors

We can now compute the CRSEs using our sandwich formula:

\[ \hat{\Sigma} = \begin{bmatrix} \hat{\Sigma}_1 & 0 & \ldots & 0 \\ 0 & \hat{\Sigma}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \hat{\Sigma}_M \end{bmatrix} \]

\[ \hat{\Sigma}_j = \begin{bmatrix} \hat{\sigma}_2 & \hat{\sigma}_2 \cdot \hat{\rho} & \ldots & \hat{\sigma}_2 \cdot \hat{\rho} \\ \hat{\sigma}_2 \cdot \hat{\rho} & \hat{\sigma}_2 & \ldots & \hat{\sigma}_2 \cdot \hat{\rho} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_2 \cdot \hat{\rho} & \hat{\sigma}_2 \cdot \hat{\rho} & \ldots & \hat{\sigma}_2 \cdot \hat{\rho} \end{bmatrix} \]

Plug \( \hat{\Sigma} \) into the sandwich estimator to obtain the cluster “corrected” estimator of the variance-covariance matrix

\[
\begin{bmatrix} \hat{\beta} | X \end{bmatrix} = \left( X' X \right)^{-1} X' \hat{\Sigma} X \left( X' X \right)^{-1}
\]

No canned function for CRSE in R; use our custom function posted on the course website

```r
> source("vcovCluster.r")
> coeftest(model, vcov = vcovCluster(model, cluster = clusterID))
```
Estimating Cluster Robust Standard Errors

We can now compute the CRSEs using our sandwich formula:

1. Take your estimates of $\hat{\sigma}^2$ and $\hat{\rho}$ and construct the block diagonal variance-covariance matrix $\hat{\Sigma}$:

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2. Plug $\hat{\Sigma}$ into the sandwich estimator to obtain the cluster “corrected” estimator of the variance-covariance matrix

$$V[\hat{\beta}|X] = (X'X)^{-1} X'\hat{\Sigma}X (X'X)^{-1}$$

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V[\hat{\beta}|X] = (X'X)^{-1} X'\hat{\Sigma}X (X'X)^{-1}
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Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY — VOTE!

<table>
<thead>
<tr>
<th>MAPLE DR</th>
<th>Aug 04</th>
<th>Nov 04</th>
<th>Aug 06</th>
</tr>
</thead>
<tbody>
<tr>
<td>9995 JOSEPH JAMES SMITH</td>
<td>Voted</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9995 JENNIFER KAY SMITH</td>
<td>Voted</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9997 RICHARD B JACKSON</td>
<td>Voted</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9999 KATHY MARIE JACKSON</td>
<td>Voted</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Social Pressure Model

load("gerber_green_larimer.RData")
social$voted <- 1 * (social$voted == "Yes")
social$treatment <- factor(social$treatment,
    levels = c("Control", "Hawthorne", "Civic Duty",
               "Neighbors", "Self"))
mod1 <- lm(voted ~ treatment, data = social)
coeftest(mod1)

##
## t test of coefficients:
##
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)       0.296638  0.001061   279.53 < 2.2e-16 ***
## treatmentHawthorne 0.025736  0.002601    9.896 < 2.2e-16 ***
## treatmentCivic Duty 0.017899  0.002600    6.883 5.849e-12 ***
## treatmentNeighbors 0.081309  0.002601   31.263 < 2.2e-16 ***
## treatmentSelf      0.048513  0.002600   18.657 < 2.2e-16 ***
## ---
## Signif. codes: 0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
Social Pressure Model, CRSEs

Again no canned CRSE in R, so we use our own.

```r
source("vcovCluster.R")
coeftest(mod1, vcov = vcovCluster(mod1, "hh_id"))
```

```r
##
## t test of coefficients:
##
##                  Estimate Std. Error  t value Pr(>|t|)
## (Intercept) 0.2966383  0.0013096 226.5172 < 2.2e-16 ***
## treatmentHawthorne 0.0257363  0.0032579   7.8997 2.804e-15 ***
## treatmentCivic Duty 0.0178993  0.0032366   5.5302 3.200e-08 ***
## treatmentNeighbors 0.0813099  0.0033696  24.1308 < 2.2e-16 ***
## treatmentSelf 0.0485132  0.0033000  14.7009 < 2.2e-16 ***
## ---
## Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```
Cluster-Robust Standard Errors

CRSE do not change our estimates $\hat{\beta}$, cannot fix bias
Cluster-Robust Standard Errors

- CRSE do not change our estimates $\hat{\beta}$, cannot fix bias
- CRSE is consistent estimator of $\text{Var}[\hat{\beta}]$ given clustered dependence

- Relies on independence between clusters, dependence within clusters
- Doesn’t depend on the model we present
- CRSEs usually $>\text{conventional SEs}$—use when you suspect clustering

- Consistency of the CRSE are in the number of groups, not the number of individuals
- CRSEs can be incorrect with a small ($<50$ maybe) number of clusters (often biased downward)
- Block bootstrap can be a useful alternative (key idea: bootstrap by resampling the clusters)
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2 Non-normality
3 Outliers
4 Robust Regression Methods
5 Optional: Measurement Error
6 Conclusion and Appendix
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8 Linear Basis Function Models
9 Generalized Additive Models
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Time Dependence: Serial Correlation

Sometimes we deal with data that is measured over time, $t = 1, \ldots, T$. Examples: a country over several years or a person over weeks/months. Often have serially correlated errors, where errors in one time period are correlated with errors in other time periods. Many different ways for this to happen, but we often assume a very limited type of dependence called AR(1).
Time Dependence: Serial Correlation

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Time Dependence: Serial Correlation

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- Examples: a country over several years or a person over weeks/months
- Often have *serially correlated*: errors in one time period are correlated with errors in other time periods
- Many different ways for this to happen, but we often assume a very limited type of dependence called AR(1).
Suppose we observe a unit at multiple times $t = 1, \ldots, T$ (e.g. a country over several years, an individual over several months, etc.). Such observations are often serially correlated (not independent across time). We can model this with the following AR(1) model:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

where the autoregressive error is

$$u_t = \rho u_{t-1} + e_t$$

where $|\rho| < 1$. $e_t$ is assumed to be Gaussian with $N(0, \sigma^2_e)$. $ho$ is an unknown autoregressive coefficient (note if $\rho = 0$ we have classic errors used before).

Typically assume stationarity meaning that $V[u_t]$ and $Cov[u_t, u_{t+h}]$ are independent of $t$. Generalizes to higher order serial correlation (e.g. an AR(2) model is given by

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Stewart (Princeton)
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Stewart (Princeton)
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The Error Structure for the AR(1) Model

We have

\[
\begin{bmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_T
\end{bmatrix}
\]

and the AR(1) model implies the following error structure (derivation in appendix):

\[
\Sigma = \sigma^2 (1 - \rho^2) 
\begin{bmatrix}
    1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\
    \rho & 1 & \rho & \cdots & \rho^{T-2} \\
    \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1
\end{bmatrix}
\]

That is, the covariance between errors in \( t = 1 \) and \( t = 2 \) is \( \sigma^2 (1 - \rho^2) \rho \), between errors in \( t = 1 \) and \( t = 3 \) is \( \sigma^2 (1 - \rho^2) \rho^2 \), etc. This implies that the correlation between the errors decays exponentially with the number of periods separating them. \( \rho \) is usually positive, which implies that we underestimate the variance if we ignore serial correlation.
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How to Detect and Fix Serial Correlated Errors

Detection:
- Plot residuals over time (or more fancy "autocorrelation" plots)
- Formal tests (e.g. Durbin-Watson statistics)

Possible Corrections:
- Use standard errors that are robust to serial correlation (e.g. Newey-West)
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Monthly Presidential Approval Ratings and Gas Prices

---

R Code

```r
> library(Zelig)
> data(approval)
> mod1 <- lm(approve ~ avg.price, data=approval)
> coeftest(mod1)

t test of coefficients:

                  Estimate  Std. Error  t value     Pr(>|t|)  
(Intercept)  100.472076  3.567277   28.165 < 2.2e-16 ***
avg.price    -0.243885   0.019465  -12.529 < 2.2e-16 ***
```

---
Tests for Serial Correlation: Durbin-Watson

Recall our AR(1) model is:

\[ y_t = \beta_0 + \beta_1 x_t + u_t \]

where \( u_t = \rho u_{t-1} + e_t, \) \( e_t \sim N(0,\sigma^2), \) and \( \rho \) is our unknown autoregressive coefficient (with \( |\rho| < 1 \)).
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One common test for serial correlation is the Durbin-Watson statistic:

\[ DW = \frac{\sum_{t=2}^{n} \hat{u}_t - \hat{u}_{t-1}}{\sum_{t=1}^{n} \hat{u}_t^2} \]

where \( DW \approx 2(1 - \hat{\rho}) \)
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where \( DW \approx 2(1 - \hat{\rho}) \)

- If \( DW \approx 2 \) then \( \hat{\rho} \approx 0 \) (Note that \( 0 \leq DW \leq 4 \))
- If \( DW < 1 \) we have serious positive serial correlation
- If \( DW > 3 \) we have serious negative serial correlation
Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(lmtest)
> dwtest(approve ~ avg.price, data=approval)

Durbin-Watson test

data:  approve ~ avg.price
DW = 0.4863, p-value = 1.326e-14
alternative hypothesis: true autocorrelation is greater than 0
```

The test suggests strong positive serial correlation. Standard errors are severely downward biased.
A common way to correct for serial correlation is to use OLS but to estimate the variances using an estimator that is heteroskedasticity and autocorrelation consistent (HAC) (Newey and West (1987)). The theory behind the HAC variance estimator is somewhat complicated, but the interpretation is similar to our usual OLS robust standard errors.

- HAC standard errors leave estimate of $\hat{\beta}$ unchanged and do not fix potential bias in $\hat{\beta}$
- HAC are consistent estimator for $V[\hat{\beta}]$ in the presence of heteroskedasticity and or autocorrelation
- The sandwich package in R implements a variety of HAC estimators
- A common option is NeweyWest

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Monthly Presidential Approval Ratings and Gas Prices

R Code

```r
> mod1 <- lm(approve~avg.price, data=approval)
> coeftest(mod1) # homoskedastic errors

| Estimate  | Std. Error | t value | Pr(>|t|)    |
|-----------|------------|---------|-------------|
| (Intercept) | 100.472076 | 3.567277 | 28.165 < 2.2e-16 *** |
| avg.price  | -0.243885  | 0.019465 | -12.529 < 2.2e-16 *** |
```

> coeftest(mod1, vcov = NeweyWest) # HAC errors

```
| Estimate  | Std. Error | t value | Pr(>|t|)    |
|-----------|------------|---------|-------------|
| (Intercept) | 100.472076 | 14.499337 | 6.9294 2.652e-09 *** |
| avg.price  | -0.243885  | 0.071733 | -3.3999 0.001174 ** |
```

Once we correct for autocorrelation, standard errors increase dramatically.
Review of Standard Errors

- Violations of homoskedasticity can come in many forms

- Non-constant error variance
- Clustered dependence
- Serial dependence

Use plots or formal tests to detect heteroskedasticity

“Robust SEs” of various forms are consistent even when these problems are present

- White HC standard errors
- Cluster-robust standard errors
- Newey-West HAC standard errors
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Appendix: Derivation of Error Structure for the AR(1) Model

We have

\[ V[u_t] = V[\rho u_{t-1} + e_t] = \rho^2 V[u_{t-1}] + \sigma^2 \]

with stationarity, \( V[u_t] = V[u_{t-1}] \), and so

\[ V[u_t](1 - \rho^2) = \sigma^2 \implies V[u_t] = \frac{\sigma^2}{(1 - \rho^2)} \]

also

\[ \text{Cov}[u_t, u_{t-1}] = E[u_t u_{t-1}] = E[(\rho u_{t-1} + e_t) e_{t-1}] = \rho V[e_{t-1}] = \rho \frac{\sigma^2}{(1 - \rho^2)} \]

or generally

\[ \text{Cov}[u_t, u_{t-h}] = \rho^h \frac{\sigma^2}{(1 - \rho^2)} \]
1 Assumptions and Violations
2 Non-normality
3 Outliers
4 Robust Regression Methods
5 Optional: Measurement Error
6 Conclusion and Appendix
7 Detecting Nonlinearity
8 Linear Basis Function Models
9 Generalized Additive Models
10 Fun With Kernels
11 Heteroskedasticity
12 Clustering
13 Optional: Serial Correlation
14 A Contrarian View of Robust Standard Errors
15 Fun with Neighbors
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A Contrarian View of Robust Standard Errors
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\(^3\)I thank Gary and Molly for the slides that follow.
Robust Standard Errors: Used Everywhere

Robust standard errors: a widely used technique to fix SEs under model misspecification

▶ In Political Science:
⋆ APSR (2009-2012): 66% of articles using regression
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Instead, they are a bright, reg flag saying:
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In the best case scenario:
- Some coefficients: unbiased but inefficient
- Other quantities of interest: Biased

In the worst case scenario:
- The functional form, variance, or dependence specification is wrong
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RSEs and SEs are the same
Consistent with a correctly specified model

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Their Alternative Procedure

Robust standard errors:

What they are not:
an elixir

What they are:
Extremely sensitive misspecification detectors!

We should use them to:
Test misspecification!

1. Do RSEs and SEs differ?
2. If they do:
   ▶ Use model diagnostics (e.g. residual plots, qq-plots, misspecification tests)
   ▶ Evaluate misspecification
   ▶ Respecify the model

Keeping going, until they don’t differ.
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Goldilocks Region

Model Misspecified
RSEs differ from SEs
Point estimates biased
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Biased just enough to make RSEs useful,

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Goldilocks Region

but not so much as to bias everything else

Model Misspecified

RSEs differ from SEs
Point estimates biased
Respecify!
The Goldilocks Region is not Idyllic

In the Goldilocks region, no fully specified model. Only a few QOIs can be estimated.

- Suppose DV: Democrat proportion of two-party vote.
- We can estimate $\beta$, but not:
  - the probability the Democrat wins,
  - the variation in vote outcome,
  - or vote predictions with confidence intervals.
- We can't check: whether model implications are realistic.

Parts of the model are wrong; why do we think the rest are right?
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<thead>
<tr>
<th>Coefficient on Omitted Variable</th>
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<tbody>
<tr>
<td>0</td>
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Difference Between SE and RSE Exposes Misspecification

![Graph showing the difference between classical SE and robust SE. The x-axis represents the coefficient on the omitted variable, and the y-axis represents the absolute difference between classical SE and robust SE. The graph indicates a trend where the difference increases as the coefficient on the omitted variable increases. The correct model is represented by a yellow arrow pointing towards the upper left corner of the graph, suggesting that the robust SE is closer to zero for the correct model.](image-url)
Difference Between SE and RSE Exposes Misspecification

![Graph showing the difference between classical SE and robust SE vs. coefficient on omitted variable. The graph illustrates how the difference diverges as the coefficient increases, indicating misspecification.]
Example: RSEs Expose Non-normality
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- Replication Neumayer, ISQ 2003
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- Treatment of interest: population

Conclusion: Aid favors less populous countries.

Difference between RSEs and SEs: Large.

$\text{Robust SE: } 0.72, \text{ SE: } 0.37$
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### Problem: Highly Skewed Dependent Variable

<table>
<thead>
<tr>
<th>Original Multilateral Aid Flows</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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</table>

<table>
<thead>
<tr>
<th>Transformed Multilateral Aid Flows</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-4</td>
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<tr>
<td></td>
<td>-2</td>
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<tr>
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</tr>
<tr>
<td></td>
<td>2</td>
</tr>
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<td></td>
<td>4</td>
</tr>
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Stewart (Princeton)
Problem: Highly Skewed Dependent Variable

Original
Multilateral Aid Flows
Frequency
0 5 10 15

Transformed
Multilateral Aid Flows
Frequency
-4 -2 0 2 4
0 5 10 15 20
Problem: Highly Skewed Dependent Variable

Original Multilateral Aid Flows

Transformed Multilateral Aid Flows

Multilateral Aid Flows

Frequency

Multilateral Aid Flows

Frequency
Diagnostics: Reveal Misspecification

Population vs Residuals, Author's Model

Log Population

Residuals
Diagnostics: Reveal Misspecification

Population vs Residuals, Author's Model

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Residuals
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Population vs Residuals, Author's Model

Population vs Residuals, Altered Model

Textbook case of heteroskedasticity

Textbook case of homoskedasticity
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After Fix: Different Conclusion
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Concluding Contrarian Thoughts

Their advice:

RSEs: not an elixir.

Should not be used as a patch.

Instead:
a sensitive detector of misspecification.
Evaluate misspecification,
by conducting diagnostic tests.
Respecify the model,
until robust and classical SE's coincide.

Their Examples:
Robust SEs indicate fundamental modelling problems
Easily identified with diagnostics
Fixing these problems
⇒

hugely different substantive conclusions
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- Fixing these problems ⇒ hugely different substantive conclusions
Concluding Thoughts on Diagnostics

Residuals are important. Look at them.
Next Week

- Causality with Measured Confounding
- Reading:
  - Angrist and Pishke Chapter 2 (The Experimental Ideal) Chapter 3.2 (Regression and Causality)
  - Morgan and Winship Chapters 3-4 (Causal Graphs and Conditioning Estimators)
  - Optional: Morgan and Winship Chapter 11 Repeated Observations and the Estimation of Causal Effects
- As a side note: if you want to read the argument against the contrarian response: Aronow (2016) ”A Note on ‘How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It.’” It is an interesting piece- feel free to come talk to me about this debate!
Appendix: Derivation of Variance under Homoskedasticity

\[ \hat{\beta} = (X'X)^{-1} X'y \]
\[ = (X'X)^{-1} X'(X\beta + u) \]
\[ = \beta + (X'X)^{-1} X'u \]

\[ V[\hat{\beta}|X] = V[\beta|X] + V[(X'X)^{-1} X'u|X] \]
\[ = V[(X'X)^{-1} X'u|X] \]
\[ = (X'X)^{-1} X' V[u|X]((X'X)^{-1} X')' \text{ (note: } X \text{ nonrandom } |X) \]
\[ = (X'X)^{-1} X' V[u|X]X (X'X)^{-1} \]
\[ = (X'X)^{-1} X' \sigma^2 I X (X'X)^{-1} \text{ (by homoskedasticity)} \]
\[ = \sigma^2 (X'X)^{-1} \]

Replacing \( \sigma^2 \) with our estimator \( \hat{\sigma}^2 \) gives us our estimator for the \((k+1) \times (k+1)\) variance-covariance matrix for the vector of regression coefficients:

\[ \widehat{V[\hat{\beta}|X]} = \hat{\sigma}^2 (X'X)^{-1} \]
1 Assumptions and Violations
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9 Generalized Additive Models
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12 Clustering
13 Optional: Serial Correlation
14 A Contrarian View of Robust Standard Errors
15 Fun with Neighbors
16 Fun with Kittens
Fun With Neighbors

We talked about error dependence induced by time and by cluster. An alternative process is spatial dependence. Just as with the other types of models we have to specify what it means to be close to a neighbor, but this choice is often more influential than anticipated.

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Our Main Questions

1. Who are a country’s neighbors?
2. How do neighbors affect each other?
3. How do we make these assumptions?
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2. How do neighbor’s affect each other? **Spatial Weights Assumption**
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How Do We Generally Choose Neighbors?

Contiguity is the most common variable. 75% of articles with spatial variable in the top 6 IR journals of the last 10 years. Athoretical choices, rarely justified. Different types of neighbors tell different stories.
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Visualization of Connections: Contiguity

Figure: Contiguity neighbors with 500 km snap distance
Visualization of Connections: Minimum Distance

Figure: Minimum distance neighbors (capital cities)
Visualization of Connections: K-Nearest Neighbors

Figure: $k = 4$ Nearest Neighbors (capital cities)
Visualization of Connections: Graph-based Neighbors

Figure: Sphere of Influence Neighbors (capital cities)
Application: Democratic Diffusion

**Gleditsch and Ward (2006)**
Changes of political regime modeled as a first-order Markov chain process with the transition matrix

$$K = \begin{bmatrix}
    Pr(y_{i,t} = 0 | y_{i,t-1} = 0) & Pr(y_{i,t} = 1 | y_{i,t-1} = 0) \\
    Pr(y_{i,t} = 0 | y_{i,t-1} = 1) & Pr(y_{i,t} = 1 | y_{i,t-1} = 1)
\end{bmatrix}$$

where $y_{i,t} = 1$ if an (A)utocratic regime exists in country $i$ at time $t$, and $y_{i,t} = 0$ if the regime is (D)emocratic.

... in other words:

$$K = \begin{bmatrix}
    Pr(D \rightarrow D) & Pr(D \rightarrow A) \\
    Pr(A \rightarrow D) & Pr(A \rightarrow A)
\end{bmatrix}$$
If a regime transition takes place in country \( i \), what is the change in predicted probability of a regime transition in country \( j \) (country \( i \)'s neighbor)?

\[
QI = Pr(y_{j,t}|y_{i,t} = y_{i,t-1}) - Pr(y_{j,t}|y_{i,t} \neq y_{i,t-1})
\]

where \( y_{i,t} = 0 \) if country \( i \) is a democracy at time \( t \) and \( y_{i,t} = 1 \) if it is an autocracy. All other covariates are held constant.
Equilibrium Effects of Democratic Transition

If a regime transition takes place in country \( i \), what is the change in predicted probability of a regime transition in country \( j \) (country \( i \)’s neighbor)?

\[
Q_i = Pr(y_{j,t} | y_{i,t} = y_{i,t-1}) - Pr(y_{j,t} | y_{i,t} \neq y_{i,t-1})
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**Illustrative cases**

- Iraq transitions from autocracy to democracy.
- Russia transitions from democracy to autocracy.
Iraq’s democratization and regional regime stability

Contiguity + 500 km

Iraq transitions from autocracy to democracy (1998 data)

Monte Carlo simulation (1,000 runs)
Iraq’s democratization and regional regime stability

Minimum Distance

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$k = 4$ Nearest Neighbors

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Sphere of Influence

Russia transitions from democracy to autocracy (1998 data)

Monte Carlo simulation (1,000 runs)
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Optional: Serial Correlation

A Contrarian View of Robust Standard Errors

Fun with Neighbors

Fun with Kittens
Kitten Wars
Kitten Wars

Alfredo vs. Amy

Click on the cutest to decide the winner!!

Can't decide? Refresh the page for a draw.
Winningest Kittens!

1. **Bitsy** has won 76% of 8642 battles.

2. **Freddie** has won 76% of 3768 battles.
Kitten Wars

Winningest Kittens!

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Losingest Kittens!

1. **Scary Cat**
   has lost 79% of 11211 battles.

2. **BELTSIM**
   has lost 79% of 1919 battles.
Kitten Wars for Ideas (Salganik and Levy)
Wiki Surveys: Open and Quantifiable Social Data Collection

Matthew J. Salganik, Karen E. C. Levy

Published: May 20, 2015 • DOI: 10.1371/journal.pone.0123483

Abstract

In the social sciences, there is a longstanding tension between data collection methods that facilitate quantification and those that are open to unanticipated information. Advances in technology now enable new, hybrid methods that combine some of the benefits of both approaches. Drawing inspiration from online information aggregation systems like Wikipedia and from traditional survey research, we propose a new class of research instruments called wiki surveys. Just as Wikipedia evolves over time based on contributions from participants, we envision an evolving survey driven by contributions from respondents. We develop three general principles that underlie wiki surveys: they should be greedy, collaborative, and adaptive. Building on these principles, we develop methods for data collection and data analysis for one type of wiki survey, a pairwise wiki survey. Using two proof-of-concept case studies involving our free and open-source website www.allourideas.org, we show that pairwise wiki surveys can yield insights that would be difficult to obtain with other methods.

Figures
Kitten Wars for Ideas (Salganik and Levy)

Bringing survey research into the digital age.

Mix core ideas from survey research with new insights from crowdsourcing. Add a heavy dose of statistics. Stir in a bit of fresh thinking. Enjoy.

Try a Wiki Survey
Create a Wiki Survey

HOW A WIKI SURVEY WORKS

Create
Start with a question and some seed ideas, and you can create a wiki survey in moments.

Participate
The participants you invite will enjoy our simple process of voting and adding new ideas.

Discover
The best ideas will bubble to the top using our system that is open, transparent, and powerful.
Which do you think is better for creating a greener, greater New York City?

- Install and maintain drinking-water fountains on busy city streets.
- Create or enhance public plazas in every community

Add your own idea here...

Users can vote by clicking one of these options: 54578 votes on 268 ideas
Which do you think is better for creating a greener, greater New York City?

<table>
<thead>
<tr>
<th>Ideas</th>
<th>Score (0 - 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Require all big buildings to make certain energy efficiency upgrades</td>
<td>67</td>
</tr>
<tr>
<td>Promote cycling by installing safe bike lanes</td>
<td>65</td>
</tr>
<tr>
<td>Promote the use of solar energy using the latest technology on all high-rise buildings.</td>
<td>65</td>
</tr>
<tr>
<td>Invest in multiple modes of transportation and provide both improved infrastructure and improved safety</td>
<td>65</td>
</tr>
<tr>
<td>Continue enhancing bike lane network, to finally connect separated bike lane systems to each other across all five boroughs.</td>
<td>65</td>
</tr>
<tr>
<td>Replace sodium vapor street lights with LED or other energy-saving lights.</td>
<td>64</td>
</tr>
<tr>
<td>Utilize NYC Roftops to install Solar PV panels</td>
<td>63</td>
</tr>
<tr>
<td>Plant more trees</td>
<td>62</td>
</tr>
<tr>
<td>Create a network of protected bike paths throughout the entire city</td>
<td>62</td>
</tr>
<tr>
<td>Add improvements to the bike lanes in the inner city. This will encourage exercise and reduce city’s carbon footprint.</td>
<td>62</td>
</tr>
</tbody>
</table>
The Power of Releasing Software
The Governor asks ... again

Governor Tarso Genro of the state of Rio Grande do Sul, Brazil has done it again. The Governor and his team completed a second round of their amazing open government project called Governador Pergunta (The Governor Asks), which collects public feedback on important policy challenges using a customized version of allourideas.org.
wiki surveys to assess risks of state-led mass killings

As part of their work with the Holocaust Museum’s Center for the Prevention of Genocide, Jay Ulfelder and Ben Valentino launched a wiki survey to help assess the risks of state-led mass killing onsets in 2014. You can read about their results on this interesting blog post.
We are happy to announce that the United Nations Division for Sustainable Development is using allourideas.org to solicit ideas from scientists around the world for the 2013 UN Global Sustainability Report. The report will
Backed by research

All Our Ideas is a research project based at Princeton University that is dedicated to creating new ways of collecting social data. You can learn more about the theory and methods behind our project by reading our paper or watching our talk. Thanks to Google, the National Science Foundation, and Princeton for supporting this research.

$z_i \sim \begin{cases} 
N(\mathbf{x}_i^T \mathbf{\theta}, 1)I(z_i^* > 0) & \text{if } y_i = 1 \\
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\end{cases}$
Bradley–Terry model

From Wikipedia, the free encyclopedia

The Bradley–Terry model is a probability model that can predict the outcome of a comparison. Given a pair of individuals $i$ and $j$ drawn from some population, it estimates the probability that the pairwise comparison $i > j$ turns out true, as

$$P(i > j) = \frac{p_i}{p_i + p_j}$$

where $p_i$ is a positive real-valued score assigned to individual $i$. The comparison $i > j$ can be read as "$i$ is preferred to $j$", "$i$ ranks higher than $j$", or "$i$ beats $j$", depending on the application.

For example, $p_i$ may represent the skill of a team in a sports tournament, estimated from the number of times $i$ has won a match. $P(i > j)$ then represents the probability that $i$ will win a match against $j$.\cite{1}\cite{2} Another example used to explain the model's purpose is that of scoring products in a certain category by quality. While it's hard for a person to draft a direct ranking of (many) brands of wine, it may be feasible to compare a sample of pairs of wines and say, for each pair, which one is better. The Bradley–Terry model can then be used to derive a full ranking.\cite{2}
References


