Evaluation of long-dated assets:
The role of parameter uncertainty

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December 2015

Abstract
We examine the term structures of efficient risk-adjusted discount rates when the random walk of economic growth is affected by parametric uncertainty. Our results are generic in the sense that they do not rely on any structural assumption underlying this uncertainty. We show that parametric uncertainty does not affect the discount rates to be used to value very short zero-coupon bonds and equity. Moreover, it makes the term structure of the risk-free rates decreasing. The term structure of aggregate risk premia is increasing when the uncertain cumulants of log consumption are independent. Under some conditions, the term structure of risk-adjusted discount rates is increasing if and only if the asset's beta is larger than two times the relative risk aversion. We apply these generic results to the case of an uncertain probability of macroeconomic catastrophes à la Barro (2006), and to the case of an uncertain trend or volatility of growth à la Veronesi (200) and Weitzman (2007). We show that these sources of uncertainty have a strong effect on efficient long-term discount rates.

Keywords: term structure, risk premium, decreasing discount rates, uncertain growth, rare events, long-run risk.

JEL Codes: G11, G12, E43, Q54.

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[1] This paper was previously entitled "Evaluation of long-dated investments under uncertain growth trend, volatility and catastrophes". I thank Geir Asheim, Simon Dietz, Stéphane Gallon, Lars Hansen, Bard Harstad, Michael Hoel, Martin Lettau, Hanno Lustig, Bob Pindyck, Bernard Salanié, David Sraer, Thomasz Strzalecki, Marty Weitzman and the seminar participants at Columbia, Berkeley, Stanford, the Netspar International Pension Workshop (Amsterdam), the Third Annual Christmas Meeting of Belgian Economists (Leuven), the University of Oslo, the World Bank-BIS Joint Fourth Public Investors' Conference at Washington, the TIGER Forum at the Toulouse School of Economics, and the joint Harvard-MIT theory seminar for useful comments on earlier versions of this paper. The research leading to these results has received funding from the Chairs "Risk Markets and Value Creation" and "Sustainable Finance and Responsible Investments" at TSE, and from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) Grant Agreement no. 230589.

1. Introduction

Do we do enough for the distant future? This question is implicit in many policy debates, from the fight against climate change to the speed of reduction of public deficits, investments in research and education, or the protection of the environment and of natural resources for example. The dual question is whether markets value assets with extra long-term cash-flows at the right level, thereby providing the efficient price signals to economic agents to invest for the long term. The discount rate used to evaluate investments is the key determinant of our individual and collective efforts in favor of the future. Since Weitzman (1998), an intense debate has emerged among environmental and climate economists about whether one should use different discount rates for different time horizons t. In a completely disconnected literature on financial economics, a similar trend emerged (see e.g. Bansal and Yaron (2004) and Hansen and Scheinkman (2009)) in which it was demonstrated that long-run consumption risks differ markedly from the much studied short-run ones. This empirical observation justifies adapting the discount rates to value zero-coupon assets as a function of their maturity.

A critical dimension surrounding the distant future is the deep uncertainty that affects the dynamics of economic growth. For example, a relatively small change in the trend of growth has an immense impact on future consumption when projected over many decades or centuries. Similarly, the uncertainty about the true volatility of growth magnifies the long-run risk, as does the ambiguity surrounding the probability of rare macroeconomic catastrophes. There has been a recent surge of papers written on the role of such parameter uncertainties on asset prices. But as recalled recently by Collin-Dufresne, Johannes and Lochstoer (2015), conventional wisdom suggests that parameter uncertainty has negligible asset pricing implications, at least for short-lived assets. However, parameter uncertainty and beliefs updating generate long-run risks since shocks to beliefs are permanent.[2] More precisely, because posterior moments are martingales under rational expectations, shocks to rationale beliefs are permanent. This magnifies the long-run consumption risk. That has two important consequences in the framework of the consumption CAPM. First, it raises the precautionary motive of prudent agents to invest in long-dated assets that are safe.[3] In

[2] Halevy and Feltkamp (2005) show that this learning process generates uncertainty aversion for expected-utility maximizers. Gollier (2011a) examines the impact of ambiguity aversion on asset demand.

[3] Following Kimball (1990), an agent is prudent if the first derivative of her utility function is convex. All familiar utility functions used in the finance literature, such as the CRRA function, satisfy this condition.

other words, this reduces the efficient long-term risk-free discount rate. Second, it raises the risk of any asset whose payoff is positively correlated with aggregate consumption. This raises the long-term equity premium.

Our aim in this paper is to isolate the role of parameter uncertainty for the determination of the term structure of risk-adjusted discount rates, in particular the risk-free rates and the aggregate risk premia.[4] In order to do this, we focus on the case in which these term structures are known to be flat in the absence of parameter uncertainty. Under the discounted expected utility model with constant relative risk aversion, we know that this requires the growth of log consumption to be governed by a random walk – meaning that increments are stationary and serially independent. Under this assumption, the growth process is characterized by the distribution of increments in log consumption. We assume that this distribution is subject to some parametric uncertainty. Our generic results hold without making any restriction on this distribution or on the nature of the parameter uncertainty. It can include ambiguity related to the trend, the volatility, the probability of catastrophe for example. This generic approach thus departs from the current trend of the literature on Bayesian learning and asset pricing, where specific forms of parameter uncertainty are examined in the more general Epstein-Zin framework.

The mathematical methodology used in this paper is based on the Cumulant-Generating Function (CGF) associated to the distribution of log consumption. Martin (2013) recently used the properties of CGF to characterize asset prices when the growth of log consumption is not Gaussian. This paper provides various illustrations of the power of this method which we improve in the context of parameter uncertainty. Under our assumptions, we show that the risk-free rate and the risk premium can be obtained by performing sequences of two CGF computations, the first on the conditional increment of log consumption, and the second on the conjugate distribution of the uncertain parameters. We limit our analysis to the impact of the parameter uncertainty on the current term structures of risk-adjusted discount rates. Obviously, these spot rates will be affected by learning in the future, yielding time-varying interest rates and risk premia. We will mostly not examine the dynamics of this learning process and its impact on the dynamics of equilibrium prices and returns.

[4] Brennan (1997) was the first to examine the term structure of risk premia. Ang and Liu (2004) refer to the notion of "spot discount rates", which is parallel to the "prices of zero-coupon equity" in Lettau and Waechter (2007).

As is well-known (see e.g. Billingsley (1995)), a probability distribution can be represented by its vector of cumulants, so that the parameter uncertainty affecting the distribution of increments can be characterized by the joint distribution of its cumulants. We first show that parameter uncertainty has *no* effect on short-term risk-adjusted discount rates. To be more precise, we demonstrate that the efficient instantaneous risk-adjusted discount rates are obtained by applying the standard CAPM pricing formulas using the expected cumulants as if they would be the true values. In words, this implies that mean-preserving spreads in the distribution of cumulants have no impact on the price of very short-term zero-coupon bonds and equity in our framework. This generalizes a result by Veronesi (2000) who has demonstrated that the short-term risk-free interest rate is not affected by parameter uncertainty when this uncertainty affects the drift rate of aggregate consumption, i.e. when only the first cumulants is uncertain.[5] This is due to the fact that the uncertainty affecting the expected growth has no impact on the conditional posterior volatility of consumption in the short run. This result is also reminiscent of a result by Hansen and Sargent (2010) who showed that parameter uncertainty does not contribute to local uncertainty prices in a Bayesian analysis.

We also show that parameter uncertainty makes the term structure of risk-free discount rates decreasing, because of the precautionary effect induced by long-run risk. This is related to Weitzman (1998, 2001) who obtained the same result in a different framework. Various interpretations of Weitzman's work exist, so let us consider the one in which the future growth rate of consumption is subject to some parametric uncertainty. This implies that the future long interest rate is uncertain. Because the present value of a sure future payoff is a convex function of the interest rate, its expected value is increased by this uncertainty. This reduces the "certainty equivalent" discount rate. Because the present value is increasingly convex with maturity, this implies a decreasing term structure of the certainty equivalent discount rate.[6] This is another way to express the fact that parameter uncertainty magnifies long-run risk, so that long-term risk-free discount rates should be reduced. This argument has been frequently used over the last decade to argue in favor of a relatively large social cost of carbon, which is defined as the present value of the marginal climate damage generated by the emission of carbon dioxide today.

[5] Veronesi (2000) considers a Markov multiple-regime switch process for the growth trend. We refer here to the special case in which the growth trend is permanent.
[6] Pastor and Veronesi (2009) provide a similar analysis to price an asset whose dividend growth is uncertain.

However, this argument holds only if long-term climate damages are uncorrelated with long-term aggregate consumption, which is obviously unrealistic.[7] Adjusting the discount rates to risk requires estimating the maturity-specific risk premia. Because the persistence of learning shocks magnifies the long-run aggregate risk compared to this benchmark case, it raises the aggregate long-term risk premium. This provides an intuition to our result that the term structure of the aggregate risk premia is increasing in the generic context of a random walk.[8] This property holds if the set of uncertain cumulants are independently distributed. However, this property does not hold in general. This is because the risk premium is also affected by the higher cumulants of log consumption outside the Gaussian world. For example, if the trend and the volatility of growth are negatively correlated, then the term structure of the annualized skewness of log consumption is locally decreasing. This decreasing skewness effect can be stronger than the increasing variance effect to make the term structure of risk premia decreasing.

The persistence of shocks to beliefs has an ambiguous effect on the long-term risk-adjusted discount rate because it reduces the risk-free rate and it potentially raises the risk premium. If the asset's beta is large enough, the net effect may be positive, yielding an increasing term structure of the risk-adjusted discount rates. If the parameter uncertainty affects cumulants in a statistically independent way, we show that the risk-adjusted discount rate is reduced by the uncertainty if and only if the consumption CAPM beta of the asset is smaller than half the degree of relative risk aversion. This suggests that parameter uncertainty should induce us to invest more for the distant future if the investment opportunity set contains enough projects with a small beta.

These generic results are presented in Section 3. Following the current trend of the literature, we explore in sections 4 and 5 some special cases to quantify these effects. In Section 4, we assume that the economy may face macroeconomic catastrophes at low frequency. In normal time, the growth of log consumption is Gaussian, but a large drop in aggregate consumption strikes the economy at infrequent dates. Our modeling duplicates the one proposed by Barro (2006, 2009), at the notable exception that we take seriously a critique formulated by Martin (2013). Martin

[7] Dietz, Gollier and Kessler (2015) show that the standard climate models imply a positive "climate beta" because a larger economic growth yields more emissions in the business-as-usual, which raises in turn the marginal climate damage. They estimate a climate beta close to unity.

[8] More recent researches have been aimed at characterizing long-run risk prices in other contexts. See for example Hansen, Heaton and Li (2008), Hansen and Scheinkman (2009), Hansen (2012) and Martin (2012).

convincingly demonstrates that it is very complex to estimate the true probability of infrequent catastrophes, and that a small modification in the choice of parameters values has a huge effect on asset prices. We take this into account by explicitly introducing ambiguity about this probability into the model. As explained earlier, this addition does not affect the instantaneous risk-free rate and the instantaneous aggregate risk premium obtained by Barro. But it dramatically affects the term structures of risk-adjusted discount rates. In a calibration exercise based on Barro (2009) and using a Bayesian approach to calibrate our degree of ignorance on the probability of catastrophes, we show that the risk free rate and the risk premium for long maturities are strongly impacted by the uncertainty affecting the probability of catastrophes, leading to an impressive overshooting of the analysis to explain the standard asset pricing puzzles. For example, our calibration features an asymptotic value of the risk premium associated to zero-coupon equity of 63%. These results are in line with a recent paper by Collin-Dufresne, Johannes and Lochstoer (2015) who consider a two-regime Markov model, with one rare regime yielding a catastrophically negative trend. They show that among the 6 parameters that govern this stochastic process, the uncertainty about the probability of a catastrophe has a much bigger impact on asset prices than the uncertainty affecting the conditional growth trends and the conditional volatilities. Their calibration leads to a conditional equity premium of 22%.

In Section 5, we assume that log consumption follows an arithmetic Brownian motion whose trend or volatility is uncertainty. Weitzman (2007) showed that the standard asset pricing puzzles can easily be reversed once it is recognized that the volatility of the growth of aggregate consumption is uncertain. Weitzman calibrated this uncertainty in such a way that risk-adjusted discount rates are unbounded at any maturity. We reexamine the same model without making any structural assumption about the collective beliefs on this volatility. We also reexamine the model first studied by Veronesi (2000) who considered the case of an uncertain trend of growth. Our aim in that framework is also to characterize the term structures for bonds and equity without restricting the model to some specific beliefs about the trend of growth. This analysis is generalized in Section 6 to the case of mean-reversion, thereby relaxing the random walk restriction that prevails everywhere else in this paper.

Our goal in this paper is normative. We show that the same ingredient, i.e., parametric uncertainty, justifies using a downward sloping risk free discount rate and at the same time an upward sloping risk premium to value investment projects. One should however recognize that markets do not follow these recommendations. The yield curve is indeed usually increasing for maturities below 20 years. And it has recently been discovered that the term structure of the equity premium is downward sloping (e.g. van Binsbergen, Brandt and Koijen (2012)).[9] Moreover, Giglio, Maggiori and Stroebel (2015) showed that the rates at which real estate cash-flows are discounted have a decreasing term structure for maturities ranging from 50 to 999 years. They estimated discount rates by comparing real estate prices of freeholds (with infinite property rights) to those of leaseholds (with property rights of fixed maturity from 50 years to 999 years), both in the UK and in Singapore. This suggests that the risk-adjusted discount rate for real estate assets is low for very long maturities, implying for example discount rates below 2.6% for 100-year benefits.

1. *The model*

We evaluate at date 0 a marginal investment project whose cash flow of net benefits (or dividend) is represented by a continuous-time stochastic process $\left\{D_t : t \geq 0\right\}$. In order to evaluate the social desirability of such a project, we measure its impact on the intertemporal social welfare

$$W_0 = E\left[\int_0 e^{-\delta t} u(c_t)dt \left| I_0 \right.\right] \tag{0.1}$$

where u is the increasing and concave utility function of the representative agent, δ is her rate of pure preference of the present, $\left\{c_t \left| t \geq 0 \right.\right\}$ is the continuous-time process of consumption of the representative agent, and I_0 is the information set available at $t = 0$. We assume that this integral exists. Because we limit the analysis to the pricing of the asset at date 0, we hereafter make the contingency of the expectation operator to I_0 implicit. Because the investment project is marginal, its implementation increases intertemporal social welfare if and only if

[9] We show in the Appendix that our model may generate a locally decreasing term structure for risk premia in some unrealistic cases in which the conditional trend and the conditional volatility are negatively correlated. In a different vein, Croce, Lettau, and Ludvigson (2015) showed that a decreasing term structure can be obtained as an equilibrium in a long-run risk model with bounded rationality and limited information.

$$\int_0 E\left[\frac{e^{-\delta t}D_t u'(c_t)}{u'(c_0)}\right]dt \geq 0. \tag{2}$$

We assume that this integral is bounded. This inequality can be rewritten as a standard NPV formula:

$$P_0 = \int_0 e^{-\rho_t t} ED_t dt \geq 0, \tag{3}$$

where ρ_t is the rate at which the expected cash flow occurring in t years should be discounted. In a frictionless economy, the NPV P_0 would be the equilibrium price at date 0 of the asset that would deliver the cash flow $\{D_t : t \geq 0\}$. It is important to observe at this stage that the asset is decomposed into a bundle of horizon-specific dividends, and that a specific discount rate ρ_t is used for each of them. These discount rates are characterized by the following equation:

$$\rho_t = \delta - \frac{1}{t}\ln\frac{ED_t u'(c_t)}{u'(c_0)ED_t} = r_{ft} + \pi_t. \tag{4}$$

It is traditional in the consumption CAPM to decompose the project-specific discount rate ρ_t into a risk free discount rate r_{ft} and a project-specific risk premium π_t. From (4), we define these two components of the discount rate as follows:

$$r_{ft} = \delta - \frac{1}{t}\ln\frac{Eu'(c_t)}{u'(c_0)}, \tag{5}$$

$$\pi_t = -\frac{1}{t}\ln\frac{ED_t u'(c_t)}{ED_t Eu'(c_t)}. \tag{6}$$

Whereas the risk-free rate r_{ft} only depends upon the distribution of c_t, the estimation of the project-specific risk premium π_t requires knowing the joint probability distribution of (c_t, D_t). Because the risk premium π_t is zero when the project is safe and more generally when its future cash flow is independent of future aggregate consumption, r_{ft} is indeed the rate at which safe projects should be discounted. Throughout the paper, we assume that $u'(c) = c^{-\gamma}$ and that the beta of cash-flow at date *t* is a known $\beta_t \in \mathbb{R}$. Brennan (1997) showed that this implies that for all *t*, there exists $k_t \in \mathbb{R}$ such that

$$E\left[D_t \middle| c_t\right] = k_t c_t^{\beta_t}, \tag{7}$$

where β_t is the consumption CAPM beta of the cash-flow at date *t*.[10] In the Gaussian world, this implies that the expected growth of log dividend is linear in the growth of log consumption, as is the case in most models in consumption-based asset pricing theory.

We characterize the properties of asset prices by using a standard tool in statistics. Let $\chi(a,x) = \ln E \exp(ax)$ denote the Cumulant-Generating Function (CGF) associated to random variable x evaluated at $a \in \mathbb{R}$. The CGF function, if it exists, is the log of the better known moment-generating function. Following Martin (2013), asset pricing formulas (5) and (6) can now be rewritten using CGF functions:

$$r_{ft} = \delta - t^{-1}\chi\left(-\gamma, G_t\right), \tag{8}$$

and $\pi_t = \pi_t(\beta_t)$ with

$$\pi_t(\beta) = t^{-1}\left(\chi\left(\beta, G_t\right) - \chi\left(\beta - \gamma, G_t\right) + \chi\left(-\gamma, G_t\right)\right), \tag{9}$$

where $G_t = \ln c_t / c_0$ is log consumption growth. Notice that, ignoring δ, $\chi\left(\beta, G_t\right)$, $\chi\left(\beta-\gamma, G_t\right)$ and $-\chi\left(-\gamma, G_t\right)$ can be interpreted respectively as the log of expected payoff, the log of price, and the log of the risk free return. This means that the right side of equation (9) is the different between the expected annualized return $t^{-1}\left(\chi\left(\beta, G_t\right) - \chi\left(\beta-\gamma, G_t\right)\right)$ of F_t over the risk free rate. Until recently, the modern theory of finance used to focus the analysis mostly on the short-term aggregate

[10] See also Campbell (1986) and Martin (2013). Weitzman (2013) considers an alternative risk profile $D_t = a + b_t c_t$ which can be interpreted as a portfolio containing *a* units of the risk free asset and an exponentially-decreasing equity share b_t. In Section 7, we explore a more general specification in which the cash-flow D_t is a convex combination of exponential functions of c_t.

risk premium $\pi_0(1) = \lim_{t\to 0} \pi_t(1)$. We are interested in characterizing the full term structure of risk premia $\pi_t(\beta)$ in parallel to the term structure of risk free rates r_{ft}.

In this paper, we use the following properties of CGF (see Billingsley (1995)).

Lemma 1 : *If it exists, the CGF function* $\chi(a,x) = \ln E\exp(ax)$ *has the following properties:*

i. $\chi(a,x) = \sum_{n=1}^{\infty} \kappa_n^x a^n / n!$ *where* κ_n^x *is the nth cumulant of random variable x. If* m_n^x *denotes the centered moment of x, we have that* $\kappa_1^x = Ex$, $\kappa_2^x = m_2^x$, $\kappa_3^x = m_3^x$, $\kappa_4^x = m_4^x - 3(m_2^x)^2$,...

ii. *The most well-known special case is when x is* $N(\mu,\sigma^2)$, *so that* $\chi(a,x) = a\mu + 0.5a^2\sigma^2$.

iii. $\chi(a,x+y) = \chi(a,x) + \chi(a,y)$ *when x and y are independent random variables.*

iv. $\chi(0,x) = 0$ *and* $\chi(a,x)$ *is infinitely differentiable and convex in* a.

v. $a^{-1}\chi(a,x)$ *is increasing in a, from Ex to the supremum of the support of x when a goes from zero to infinity.*

Property *i* explains why χ is called the cumulant-generating function, and it links the sequence of cumulants to those of the centered moments. The first cumulant is the mean. The second, the third and the fourth cumulants are respectively the variance, the skewness and the excess kurtosis of the random variable. Because the cumulants of the normal distribution are all zero for orders *n* larger than 2, the CGF of a normally distributed *x* is a quadratic function of *a*, as expressed by property *ii*. This property also implies that the CGF of a Dirac distribution degenerated at $x = x_0 \in \mathbb{R}$ is equal to ax_0. Property *v* will play a crucial role in this paper. It is a consequence of property *iv*, which is itself an illustration of the Cauchy-Schwarz inequality.

As noticed by Hansen (2012) for example, the simplest configuration to calibrate equations (8) and (9) is when G_t is normally distributed, so that we can use property *ii* of Lemma 1 to rewrite equation (9) as follows:

$$\pi_t(\beta) = \beta\gamma \frac{Var(G_t)}{t}. \tag{10}$$

In this specific Gaussian specification, the term structure of risk premia is proportional to the term structure of the annualized variance of log consumption contingent to current information. Hansen, Heaton, and Li (2008) pursued the same goal of describing the term structure of risk prices in a Gaussian world with persistent shocks to growth.

In this paper, we apply this generic consumption pricing model to the case of parameter uncertainty. Namely, we assume that the stochastic process governing the evolution of G_t is a function of an unknown parameter $\theta \in \Theta$. The current beliefs about θ are represented by some distribution function with support in Θ.[11] Parameter uncertainty usually implies that the unconditional distribution of log consumption is not normal. This is an important source of complexity because, contrary to the Gaussian case examined in the previous section, equation (9) will entail cumulants of order larger than 2.

Let us discretize time by periods of equal duration $\Delta > 0$, which can be arbitrary small. We hereafter crucially assume that, conditional to any $\theta \in \Theta$, $G_t = \ln c_t / c_0$ is a stationary random walk. This means that, conditional to θ, temporal increments in G_t are serially independent and stationary.[12] In this paper, we examine the properties of asset prices under this general assumption. This is a clear difference with respect to recent papers in mathematical finance (see e.g. Hansen, Heaton, and Li (2008), Hansen and Scheinkman (2009), Hansen (2012)) in which the focus is on the effect of unambiguous auto-regressive components of the growth process on the long-term risk prices. Our findings rely on the following theorem.

Theorem 1: *Suppose that, conditional to any $\theta \in \Theta$, log consumption is a random walk whose increment is distributed as $g|\theta$. Then, for all t>0:*

$$r_{ft} = \delta - t^{-1}\chi\left(t, \chi\left(-\gamma, g|\theta\right)\right). \tag{11}$$

and for all $\beta \in \mathbb{R}$:

[11] The observation of future changes in consumption will allow for a Bayesian updates of beliefs which will impact prices. In this framework with exponential Discount Expected Utility, agents' decisions (and so asset prices) will be time consistent. In this paper, we are interested in characterizing asset prices *today*. We do not examine the dynamic of prices, so we do not determine how beliefs are updated. We just do not need to do this for our purpose.

[12] If we require this to assumption to hold independent of the duration Δ, log consumption must be governed by a Lévy process.

$$\pi_t(\beta) = t^{-1}\left(\chi\left(t,\chi\left(\beta,g\middle|\theta\right)\right) - \chi\left(t,\chi\left(\beta-\gamma,g\middle|\theta\right)\right) + \chi\left(t,\chi\left(-\gamma,g\middle|\theta\right)\right)\right). \tag{12}$$

Proof: Let $g_{t,\Delta} = G_{t+\Delta} - G_t = \ln c_{t+\Delta} / c_t$ denote the growth of log consumption during the interval of time $[t, t+\Delta]$. We can then rewrite equation (8) as follows:

$$\begin{aligned}
r_{ft} &= \delta - t^{-1} \ln E\left[E\left[e^{-\gamma\sum_{i=1}^{t/\Delta} g_{(i-1)\Delta,\Delta}} \middle|\theta\right]\right] \\
&= \delta - t^{-1} \ln E\left[E\left[e^{-\gamma g}\middle|\theta\right]^{t/\Delta}\right] \\
&= \delta - t^{-1} \ln E e^{t\chi(-\gamma,g|\theta)/\Delta} \\
&= \delta - t^{-1}\chi\left(t,\chi\left(-\gamma,g\middle|\theta\right)/\Delta\right),
\end{aligned}$$

with $g_\Delta = \ln c / c_0$. The second equality above is a consequence of the assumption that $G_t\,|\theta$ is a random walk, so that the $g_{j\Delta,\Delta}$ are i.i.d. variables. The last two equalities are direct consequences of the definition of the CGF. A similar exercise can be performed on equation (9) for the risk premium, which implies equation (12). □

Equation (11) means that the risk free rate is determined by a sequence of two CGF operations. One must first compute $x|\theta = \chi(-\gamma, g|\theta)$, which is the CGF of consumption growth g conditional to θ. One must then compute $\chi(t,x)$ by using the distribution of θ that characterizes our current beliefs about this unknown parameter. The comparison of equations (8) and (11) indicates the very specific structure that conditional random walk and parameter uncertainty bring to the asset pricing model.

In the benchmark case where log consumption follows an arithmetic Brownian motion with trend μ and volatility σ, G_t is normally distributed with mean μt and variance $\sigma^2 t$. Using property *ii* in Lemma 1 implies that equations (11) and (12) can be rewritten as follows

$$r_{ft} = \delta + \gamma\mu - 0.5\gamma^2\sigma^2, \tag{13}$$

and

$$\pi_t(\beta) = \beta\gamma\sigma^2. \tag{14}$$

Equation (13), which is often referred to as the extended Ramsey rule, holds independent of the maturity of the cash flow. In other words, the term structure of the safe discount rate is flat under this benchmark specification. Its level is determined by three elements: impatience, a wealth effect and a precautionary effect. The wealth effect comes from the observation that investing for the future in a growing economy does increase intertemporal inequality. Because of inequality aversion (which is equivalent to risk aversion under the veil of ignorance), this is desirable only if the return of the project is large enough to compensate for this adverse effect on welfare. From (13), this wealth effect is equal to the product of the expected growth of log consumption by the degree γ of concavity of the utility function which measures inequality aversion. The precautionary effect comes from the observation that consumers want to invest more for the future when this future is more uncertain (Drèze and Modigliani (1972), Kimball (1990)). This tends to reduce the discount rate. The precautionary effect is proportional to the volatility of the growth of log consumption. Equation (14) tells us that the project-specific risk premium $\pi_t(\beta)$ is just equal to the product of the project-specific beta by the aggregate risk premium $\pi = \gamma\sigma^2$. Compared to historical market returns, the standard calibration of these two equations yields a too large risk free rate (risk free rate puzzle, Weil (1989)) and a too small risk premium (equity premium puzzle, Grossman and Shiller (1981), Hansen and Singleton (1983), Mehra and Prescott (1985)). Notice that the risk-adjusted discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ is increasing in the aggregate uncertainty measured by σ^2 if β is larger than $\gamma/2$, in which case the risk premium effect dominates the precautionary effect.

2. *Generic properties of the term structures of risk-adjusted discount rates*

In this section, we derive the generic properties of r_{ft} and $\pi_t(\beta)$ as defined in Theorem 1. We first examine the role of parameter uncertainty on instantaneous risk-adjusted discount rates. This requires us to go to the limit case of continuous-time ($t = \Delta \to 0$). Using property *v* of Lemma 1 we immediately obtain that

$$
\begin{aligned}
r_{f0} &= \delta - E\chi(-\gamma, g|\theta) \\
&= \delta - E\left[\sum_{n=1}^{\infty} \frac{(-\gamma)^n}{n!} \kappa_n^{g|\theta}\right] = \delta + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\gamma^n}{n!} E\left[\kappa_n^{g|\theta}\right].
\end{aligned} \tag{15}
$$

This tells us that the instantaneous discount rate is not affected by parametric uncertainty in the sense that only the *expected* cumulants of the change in log consumption matter to compute it. From equation (12), the same result holds for risk premia, with

$$
\begin{aligned}
\pi_0 &= E\chi(\beta, g|\theta) - E\chi(\beta-\gamma, g|\theta) + E\chi(-\gamma, g|\theta) \\
&= \sum_{n=2}^{\infty} \frac{\beta^n - (\beta-\gamma)^n + (-\gamma)^n}{n!} E\left[\kappa_n^{g|\theta}\right].
\end{aligned} \tag{16}
$$

Only the expected variance, skewness, kurtosis and higher cumulants matter to compute the instantaneous risk premium when these cumulants are uncertain.

Proposition 1: *Suppose that log consumption conditional to θ follows a random walk, for all $\theta \in \Theta$. This parametric uncertainty has no effect on the instantaneous risk-free rate and risk premium in the sense that they only depend upon the expected cumulants of the incremental log consumption.*

This generalizes earlier observations that parameter uncertainty does not affect short-lived asset prices (Veronesi (2000), Hansen and Sargent (2010), and Collin-Dufresne, Johannes and Lochstoer (2015)). The intuition of this result can be derived from the observation that the annualized variance of log consumption satisfies the following condition:

$$
t^{-1} Var \ln c_t = EVar(g|\theta) + tE\left[\left(E\left[g|\theta\right] - E\left[g\right]\right)^2\right]. \tag{17}
$$

Because the second term in the RHS of this equation vanishes when *t* tends to zero, it tells us that, for short maturities, the annualized variance of log consumption is unaffected by a mean-preserving transformation of the probability distribution of increments.[13] This result illustrates the fact that when valuing a short-term claim, investors care about the short-term dividend growth, but they do not care about news concerning more distant consumption growth. This is specific to the Discounted Expected Utility à la Lucas (1978). Collin-Dufresne, Johannes and Lochstoer (2015)

[13] This observation is not a proof of Proposition 1 since the other cumulants of log consumption should also matter for the determination of the risk-free rate.

show that Proposition 1 is not robust to the introduction of Epstein-Zin preferences precisely because it makes the representative agent caring about long-run risks when evaluation instantaneous cash flows.

But parameter uncertainty does affect the discount rate associated to longer maturities. Indeed, by application of property *v* of Lemma 1, the risk free discount rate has a decreasing term structure when the parametric uncertainty is such that $\chi(-\gamma, g|\theta)$ is sensitive to θ. The intuition of this result can be derived from equation (17) which shows that the annualized variance of log consumption has an increasing term structure when the trend of growth is uncertain. This observation generalizes a result obtained by Gollier (2008) in the special case of a geometric Brownian motion for aggregate consumption with an uncertain trend.

Proposition 2: *Suppose that log consumption conditional to θ follows a random walk, for all $\theta \in \Theta$. This parametric uncertainty makes the term structure of risk-free discount rates monotonically decreasing.*

One can estimate the slope of this term structure for small maturities by observing from property 1 of Lemma 1 that the derivative of $t^{-1}\chi(t,x)$ at $t=0$ equals $0.5Var(x)$. Applying this to equation (11) implies that

$$\left.\frac{\partial r_{ft}}{\partial t}\right|_{t=0} = -0.5Var\left(\chi\left(-\gamma, g|\theta\right)\right) = -0.5Var\left(\sum_{n=1}^{\infty} \frac{(-\gamma)^n}{n!}\kappa_n^{g|\theta}\right). \tag{18}$$

In the special case in which uncertain cumulants are not correlated, this equation simplifies to

$$\left.\frac{\partial r_{ft}}{\partial t}\right|_{t=0} = -\sum_{n=1} \frac{\gamma^{2n}}{2(n!)^2} Var\left(\kappa_n^{g|\theta}\right). \tag{19}$$

We see that the slope of the term structure is increasing in absolute value in the degree of uncertainty measured by the variance of the unknown cumulants.

The analysis of the term structure of aggregate risk premia is more complex because the risk premium is the sum and difference of three different double-CGF functionals. Observe that property *i* of Lemma 1 applied to equation (12) implies that

$$\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} = 0.5\Big(Var\big(\chi(\beta, g|\theta)\big) + Var\big(\chi(-\gamma, g|\theta)\big) - Var\big(\chi(\beta-\gamma, g|\theta)\big)\Big). \tag{20}$$

In the special case in which the uncertain cumulants are statistically independent, we obtain that

$$\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} = \sum_{n=1} \frac{\beta^{2n} + \gamma^{2n} - (\beta-\gamma)^{2n}}{2(n!)^2} Var\left(\kappa_n^{g|\theta}\right). \tag{21}$$

In the proof of the following proposition, we show that the coefficients of $Var\left(\kappa_n^{g|\theta}\right)$ have the same sign than β, which implies that the term structure of the aggregate risk premia is locally increasing.

Proposition 3: *Suppose that log consumption conditional to θ follows a random walk, for all $\theta \in \Theta$. Suppose also that the uncertain cumulants of its increments are statistically independent. This implies that the risk premium $\pi_t(\beta)$ has an increasing (resp. decreasing) term structure for small maturities when $\beta > 0$ (resp. $\beta < 0$).*

Proof: From equation (21), we would be done if $f_\beta(\gamma) = \beta^{2n} + \gamma^{2n} - (\beta-\gamma)^{2n}$ has the same sign as β. Observe that

$$f_\beta^{'}(\gamma) = 2n\left(\gamma^{2n-1} + (\beta-\gamma)^{2n-1}\right). \tag{22}$$

This equation implies that $f_\beta^{'}(0)$ has the same sign as β. Because $f_\beta^{'}$ never alternate in sign, we can conclude that f_β is monotonically increasing (resp. decreasing) in γ when β is positive (resp. negative). Because $f_\beta(0) = 0$, we obtain that $f_\beta(\gamma)$ has the same sign as β for all $\gamma \in \mathbb{R}^+$. □

However, contrary to what we obtained in Proposition 2 for the risk-free rate, Proposition 3 is not robust to the introduction of correlation among two or more cumulants. To show this, suppose that cumulants of degrees *m* and *n* are the only two uncertain cumulants of *g*. In that case, it is easy to show from equation (20) that

$$\begin{aligned}\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} &= \frac{\beta^{2m} + \gamma^{2m} - (\beta-\gamma)^{2m}}{2(m!)^2} Var\left(\kappa_m^{g|\theta}\right) + \frac{\beta^{2n} + \gamma^{2n} - (\beta-\gamma)^{2n}}{2(n!)^2} Var\left(\kappa_n^{g|\theta}\right) \\ &+ \frac{\beta^{m+n} + (-\gamma)^{m+n} - (\beta-\gamma)^{m+n}}{m!n!} Cov(\kappa_m^{g|\theta}, \kappa_n^{g|\theta}).\end{aligned} \tag{23}$$

This equation is useful to show that the term structure of risk premia is not necessarily increasing. Indeed, a growing literature documents evidence that the term structure of the equity premium is downward sloping for time horizons standard for financial markets.[14] Observe however that equation (23) is compatible with a downward sloping term structure for $\pi_t(1)$ if the covariance term is sufficiently negative, since the two variance terms are positive. Such an example is developed in the Appendix. This example illustrates the fact that parameter uncertainty does not only make the annualized variance of log consumption increasing. It also affects the term structures of its annualized skewness and excess kurtosis. Although that cannot reverse the result that the term structure of risk-free rates is decreasing, it can make the term structure of risk premia decreasing when the uncertain cumulants of log consumption are statistically dependent.

Proposition 3 makes it unclear whether parameter uncertainty actually raises the global willingness to invest for the medium and distant future. If the betas of investment projects are large enough, the parameter uncertainty should reduce the intensity of investments, because it will raise the risk premium more than it will reduce the risk-free rate. It is interesting to determine the critical beta at which the risk-adjusted discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ has a flat term structure in the neighborhood of t=1. Suppose again that the uncertain cumulants are statistically independent. From equations (19) and (21), the risk-adjusted discount rate is such that

$$\left.\frac{\partial \rho_t(\beta)}{\partial t}\right|_{t=0} = \sum_{n=1} \frac{\beta^{2n} - (\beta-\gamma)^{2n}}{2(n!)^2} Var\left(\kappa_n^{g|\theta}\right) \tag{24}$$

This implies that ρ_t is locally increasing in t if and only if β^2 is larger than $(\beta-\gamma)^2$, i.e., if and only if β is larger than $\gamma/2$.

Corollary 1: *Under the assumptions of Proposition 3, the term structure of the risk-adjusted discount rate* $\rho_t(\beta)$ *is increasing (resp. decreasing) for small maturities if* β is larger (resp. smaller) than $\gamma/2$.

There is a simple intuition for this result. It combines the observation that the parametric uncertainty magnifies the uncertainty affecting the future level of development of the economy

[14] See for example van Binsbergen, Brandt and Koijen (2012) and the references mentioned in that paper.

with the observation made at the end of the previous section that long-run risk decreases or increases the discount rate depending upon whether β is smaller or larger than $\gamma/2$. But again this results is not robust to the introduction of correlation among the uncertain cumulants. If the cumulants of degrees *m* and *n* are the only two uncertain cumulants of *g*, we obtain that

$$\left.\frac{\partial\rho_t(\beta)}{\partial t}\right|_{t=0} = \frac{\beta^{2m}-(\beta-\gamma)^{2m}}{2(m!)^2}Var\left(\kappa_m^{g|\theta}\right)+\frac{\beta^{2n}-(\beta-\gamma)^{2n}}{2(n!)^2}Var\left(\kappa_n^{g|\theta}\right) \\ +\frac{\beta^{m+n}-(\beta-\gamma)^{m+n}}{m!n!}Cov(\kappa_m^{g|\theta},\kappa_n^{g|\theta}). \tag{25}$$

In the special case of $\beta=\gamma/2$, the first two terms in the RHS of this equation are zero. But the coefficient of the covariance in the third term is clearly positive if $m+n$ is an odd integer. This implies that the term structure of the risk-adjusted discount rate with $\beta=\gamma/2$ will not be flat when the uncertain cumulants are correlated. In particular, a positive correlation between two subsequent cumulants tends to make this term structure of $\rho_t(\beta=\gamma/2)$ increasing, at least at small maturities.

The methodology based on the CGF proposed by Martin (2013) is also useful to explore the curvature of the term structures. From Lemma 1, we have that

$$\left.\frac{\partial^2\rho_t(\beta)}{\partial t^2}\right|_{t=0} = \frac{1}{3}Skew\big(\chi(\beta,g|\theta)-\chi(\beta-\gamma,g|\theta)\big), \tag{26}$$

where $Skew(x)$ is the skewness of *x*. If the uncertainty on the distribution of log consumption is concentrated on the *n*th cumulant, then this equation simplifies to

$$\left.\frac{\partial^2\rho_t(\beta)}{\partial t^2}\right|_{t=0} = \frac{\beta^{3n}-(\beta-\gamma)^{3n}}{3(n!)^3}Skew\left(\kappa_n^{g|\theta}\right). \tag{27}$$

This means for example that the term structure of the discount rate for the aggregate risk ($\beta=1$) is convex at *t*=0 if the trend (*n=1*) of growth is uncertain and positively skewed. Pursuing in the same vein for larger derivatives of ρ_t would allow us to fully describe the shape of the term structure of discount rates from the sign of the successive cumulants of g.

In the spirit of Hansen and Scheinkman (2009) and Hansen (2012), we now determine the asymptotic values of the discount rates in the context of parameter uncertainty. Property *v* of Lemma 1 immediately yields the following result. The suprema are taken with respect to $\theta \in \Theta$.

Proposition 4: *Suppose that log consumption conditional to θ follows a random walk, for all $\theta \in \Theta$. Assuming that the support of $\chi(k, g|\theta)$ has an upper bound for $k \in \{\beta, -\gamma, \beta-\gamma\}$, this implies that*

$$\lim_{t\to\infty} r_{ft} = \delta - \sup \chi(-\gamma, g|\theta). \tag{28}$$

$$\lim_{t\to\infty} \pi_t(\beta) = \sup \chi\left(\beta, g|\theta\right) + \sup \chi\left(-\gamma, g|\theta\right) - \sup \chi\left(\beta-\gamma, g|\theta\right) \tag{29}$$

$$\lim_{t\to\infty} \rho_t(\beta) = \delta + \sup \chi\left(\beta, g|\theta\right) - \sup \chi\left(\beta-\gamma, g|\theta\right). \tag{30}$$

Equation (28) tells us that the asymptotic risk-free rate corresponds to the smallest possible conditional risk-free rate. This is reminiscent of similar results obtained by Weitzman (1998, 2001) in another context. Notice that we can infer from equation (29) that the symmetric result for the risk premium does not hold, because the sum of suprema is generally not equal to the supremum of the sum. Thus, it is not true in general that the asymptotic risk premium equals the largest possible conditional risk premium. A counterexample is presented in the next section.

3. *Application 1: Unknown probability of a macroeconomic catastrophe*

In this section, in the spirit of Barro (2006, 2009), Backus, Chernov and Martin (2011) and Martin (2013), we examine a discrete version of a mixture of Brownian and Poisson processes. Rare events have recently been recognized for being a crucial determinant of assets prices. The underlying discrete-time model is such that the growth of log consumption follows a stationary random walk, so that this model is a special case of the one studied in Section 3. We assume in this section that the per-period change in log consumption compounds two normal distributions:

$$g \sim (h_1, 1-p; h_2, p) \quad \textit{with} \quad h_i \sim N(\mu_i, \sigma_i^2), \tag{31}$$

The log consumption growth compounds a "business-as-usual" random variable $h_1 \sim N(\mu_1, \sigma_1^2)$ with probability $1-p$, with a catastrophe event $h_2 \sim N(\mu_2, \sigma_2^2)$ with probability p and $\mu_2 << 0 < \mu_1$ and $\sigma_2 \geq \sigma_1$. Barro (2006, 2009) convincingly explains that the risk free puzzle and the equity premium puzzle can be explained by using credible values of the intensity μ_2 of the macro catastrophe and of its frequency *p*. However, Martin (2013) shows that the levels of the r_{f0} and $\pi_0(1)$ are highly sensitive to the frequency *p* of rare events, and that this parameter *p* is extremely difficult to estimate. In this section, in parallel to Collin-Dufresne, Johannes and Lochstoer (2015), we contribute to this emerging literature by integrating this source of parametric uncertainty into the asset pricing model.

In the absence of parametric uncertainty, equations (11) and (12) imply that the term structures of risk-adjusted discount rates are flat. We hereafter assume alternatively that *p* is uncertain. Because a change in *p* affects all cumulants of $g|p$ simultaneously, these cumulants are not independent. We calibrate this model as in the Discounted Expected Utility benchmark version of Barro (2006, 2009) and Martin (2013). We assume that $\delta = 3\%$ and $\gamma = 4$. In the business-as-usual scenario, the trend of growth is $\mu_1 = 2.5\%$ and its volatility is equal to $\sigma_1 = 2\%$. In case of a catastrophe, the trend of growth is $\mu_2 = -39\%$ and the volatility is $\sigma_2 = 25\%$. Finally, we take a Bayesian approach for the estimation of the probability of a catastrophe. Suppose that our prior belief about *p* is uninformative. Barro (2006) observes 60 catastrophes over 3500 country-years. Using Bayes' rule, it is well-known that this yields that the posterior distribution for *p* is *Beta*(61, 3441).[15] The posterior mean of *p* is 1.74%, and its posterior standard deviation is 0.22%.[16]

In Table 1, we report the discount rates for different maturities and asset's beta. As said before, this model is good to fit observed asset prices with short maturities. Moreover, as noticed by Collin-Dufresne et al. (2015), the learning process about *p* is very slow, so that parameter uncertainty has a limited impact for small maturities. However in this calibration, we also obtain an interest rate of

[15] See Leamer (1978) for example.

[16] Collin-Dufresne, Johannes and Lochstoer (2015) also consider the case of an uncertain probability of catastrophic events. Their calibration is different because they use Epstein-Zin preferences, and the conditional log consumption process is not a random walk. They show that the uncertainty affecting the transition probabilities has a large effect on the expected excess return of equity.

-2.6% for a 200-year maturity, and asymptotic rate of $r_{f\infty} = -203\%$ per annum. This is in fact the risk free rate that would prevail in an economy whose annual growth rate is always selected from the catastrophic urn, as predicted by Proposition 4. We also have that the asymptotic risk premium is equal to $\pi_{\infty}(1) = 63.4\%$ per annum. These numbers demonstrates that the calibration proposed by Barro (2006) is useful to explain the price of short-lived assets, but is potentially problematic to predict the interest rate and risk premium for extra-long maturities once the uncertainty affecting the probability of catastrophes is recognized. However, the fact that the risk-adjusted discount rates are very low in this model for large maturities could explain why leaseholds (with maturities between 50 and 999 years) and freeholds on real estate markets differ much in price, as shown by Giglio, Maggiori and Stroebel (2015). Owners value a lot the very distant rental benefits of freeholds, thereby suggesting a low risk-adjusted discount rate for extra-long maturities. This could be explained in this model by the plausibility of a succession of macroeconomic catastrophes with an uncertain probability of occurrence.

	$t = 1$ year	$t = 10$ years	$t = 200$ years	$t \to \infty$
r_{ft}	0.2%	0.1%	-2.6%	-203.0%
$\pi_t(1)$	6.0%	6.1%	8.2%	63.4%
$\pi_t(5)$	12.6%	12.7%	15.4%	216.5%

Table 1: The annual interest rate r_{ft} and risk premia $\pi_t(\beta)$ when the probability of macroeconomic catastrophe is uncertain, with $p \sim Beta(61, 3441)$.

It is interesting to observe that the risk premium $\pi_t(\beta)$ is not proportional to β. There are two reasons that explain this feature of asset prices in this context. Both are linked to the non-Gaussian nature of G_t. The first one comes from the fat tail induced by rare events. The second reason comes from the uncertainty affecting the probability of rare events, which also fatten the tails of the distribution.

4. *Application 2 : Brownian growth process with an unknown trend or volatility*

In this section, we examine the special case of our general model in which log consumption follows an arithmetic Brownian motion with unknown constant drift μ and volatility σ. Let us first consider the case in which the trend is known, but the volatility is ambiguous. Weitzman (2007) examined this question by assuming in addition that σ^2 has an inverted Gamma distribution. This implies that the unconditional growth rate g has a Student's t-distribution rather than a normal, yielding fat tails, an unboundedly low safe discount rate and an unboundedly large market risk premium. In our terminology, because the Inverse Gamma distribution for σ^2 has no real CGF, equations (11) and (12) offer another proof of Weitzman (2007)'s inexistence result. If we relax Weitzman's assumption governing our beliefs on the unknown volatility of growth, applying Lemma 1 to equations (11) and (12) yields

$$r_{ft} = \delta + \gamma\mu - \frac{1}{2}\gamma^2\kappa_1^{\sigma^2} - \frac{1}{8}\gamma^4\kappa_2^{\sigma^2}t - \frac{1}{48}\gamma^6\kappa_3^{\sigma^2}t^2 - \ldots \tag{32}$$

$$\pi_t(\beta) = \beta\gamma\kappa_1^{\sigma^2} + \frac{1}{2}\beta\gamma\kappa_2^{\sigma^2}\left[\beta^2 - \frac{3}{2}\beta\gamma + \gamma^2\right]t + \ldots \tag{33}$$

In the remainder of this section, we alternatively follow Veronesi (2000) by considering the symmetric case in which only the drift of growth is uncertain.[17] Suppose first that the current beliefs about the true value of the drift can be represented by a normal distribution: $\mu \sim N(m_1^\mu, m_2^\mu)$. Applying property *ii* recursively in each double-CGF operations appearing in equations (11) and (12) implies the following characterization:

$$r_{ft} = r_{f0} - 0.5\gamma^2 m_2^\mu t \tag{34}$$

$$\pi_t(\beta) = \beta\gamma\left(\sigma^2 + m_2^\mu t\right) \tag{35}$$

Equation (35) can be seen as an application of equation (10) since G_t is Gaussian in this case with $Var(G_t) = \sigma^2 t + m_2^\mu t^2$. In this Gaussian world, the standard use of filtering techniques allows for

[17] Collin-Dufresne, Johannes and Lochstoer (2015) also examine this case with a normally distributed prior for the growth trend. Their numerical analysis in the DEU case confirms that the instantaneous risk-free rate is not affected by the parameter uncertainty, and that the slope of the term structure of risk-free rate is decreasing. Similar results are obtained in an Epstein-Zin framework, except that the instantaneous risk-free rate is negatively impacted by parameter uncertainty under that framework.

the analytical characterization of the time-varying term structures of the risk-adjusted discount rates.

In the remainder of this section, we explore the case of non-Gaussian distributions to describe our current beliefs about the trend of growth. Applying Lemma 1 to the equations in Theorem 1 yields:

$$\rho_t(\beta) = \rho_0(\beta) + t^{-1}\chi(t, \beta(\mu - m_1^\mu)) - t^{-1}\chi(t, (\beta-\gamma)(\mu - m_1^\mu)). \tag{36}$$

Observe that if μ has a symmetric distribution, equation (36) implies that $\rho_t(\beta = 0.5\gamma)$ is constant in *t*. Moreover, because $\chi(t, kx) = \chi(tk, x)$ for all $k \in \mathbb{R}$, the above equation implies that

$$\frac{\partial \rho_t(\beta)}{\partial t \partial \gamma} = (\beta - \gamma)\frac{\partial^2 \chi}{\partial t^2}(t, (\beta-\gamma)(\mu - m_1^\mu)). \tag{37}$$

Because χ is convex in its first argument, this implies that $\partial \rho_t(\beta)/\partial t$ is decreasing in γ if and only if β is smaller than γ. Combining these observations, we obtain the following result.

Proposition 5: *Suppose that log consumption follows an arithmetic Brownian motion with a known volatility $\sigma \in \mathbb{R}^+$ and with an unknown trend μ with a bounded support. When μ is symmetrically distributed, the term structure of the discount rates is flat when $\beta = \gamma/2$. The term structure of the risk-adjusted discount rates is decreasing when β is smaller than $\gamma/2$, and it is increasing when β is in interval $[\gamma/2, \gamma]$.*

This result is stronger than the one described in Corollary 1 because it is not restricted to small maturities. We now characterize the asymptotic properties of the term structure of discount rates when the distribution of the trend μ has a bounded support $[\mu_{\min}, \mu_{\max}]$. We first rewrite condition (36) as

$$\rho_t(\beta) = \delta + \gamma\sigma^2(\beta - 0.5\gamma) + t^{-1}\chi(t, \beta\mu) - t^{-1}\chi(t, (\beta-\gamma)\mu). \tag{38}$$

Property *v* of Lemma 1 tells us that $a^{-1}\chi(a, x)$ tends to the supremum of the support of *x* when *a* tends to infinity. If β is negative, the supremum of the support of $\beta\mu$ is $\beta\mu_{\min}$, and the supremum of the support of $(\beta-\gamma)\mu$ is $(\beta-\gamma)\mu_{\min}$. This implies that the sum of the last two terms of the

RHS of the above equality tends to $\gamma\mu_{\min}$. The other two cases are characterized in in the same way to finally obtain the following result:

$$\rho_\infty(\beta) = \begin{cases} \delta + \gamma\sigma^2(\beta - 0.5\gamma) + \gamma\mu_{\min} & if \quad \beta \le 0 \\ \delta + \gamma\sigma^2(\beta - 0.5\gamma) + (\gamma - \beta)\mu_{\min} + \beta\mu_{\max} & if \quad 0 < \beta \le \gamma \\ \delta + \gamma\sigma^2(\beta - 0.5\gamma) + \gamma\mu_{\max} & if \quad \beta > \gamma \end{cases} \tag{39}$$

For distant futures, the ambiguity affecting the trend is crucial for the determination of the discount rate. The long term wealth effect is equal to the product of γ by a growth rate of consumption belonging to its support $[\mu_{\min}, \mu_{\max}]$. Its selection depends here upon the beta of the project. When β is negative, the wealth effect should be computed on the basis of the smallest possible growth rate $\mu_{\min}$ of the economy. On the contrary, when β is larger than γ, the wealth effect should be computed on the basis of the largest possible rate $\mu_{\max}$.

Equation (39) also tells us that the condition of a symmetric distribution for μ in Proposition 4 cannot be relaxed. Indeed, equation (39) implies that $\rho_0(\beta = \gamma/2)$ and $\rho_\infty(\beta = \gamma/2)$ are equal only if $E\mu = m_1^\mu$ and $(\mu_{\min} + \mu_{\max})/2$ coincide. Most asymmetric distributions will not satisfy this condition, which implies that the constancy of $\rho_t(\beta = \gamma/2)$ with respect to *t* will be violated.

In Figure 1, we illustrate some of the above findings in the following numerical example. We assume that $\delta = 0$, $\gamma = 2$, $\sigma = 4\%$ and μ is uniformly distributed on interval $[0\%, 3\%]$. As predicted by Proposition 5, the term structure is decreasing, flat or decreasing depending upon whether β is smaller, equal, or larger than $\gamma/2 = 1$. Observe also that, contrary to the Gaussian case, risk premia are not proportional to β. For example, consider a time horizon of 400 years. For this maturity, the risk premia associated to $\beta = 1$ and $\beta = 4$ are respectively equal to $\pi_{400}(1) = 2.5\%$ and $\pi_{400}(4) = 6.3\% < 4\pi_{400}(1)$.

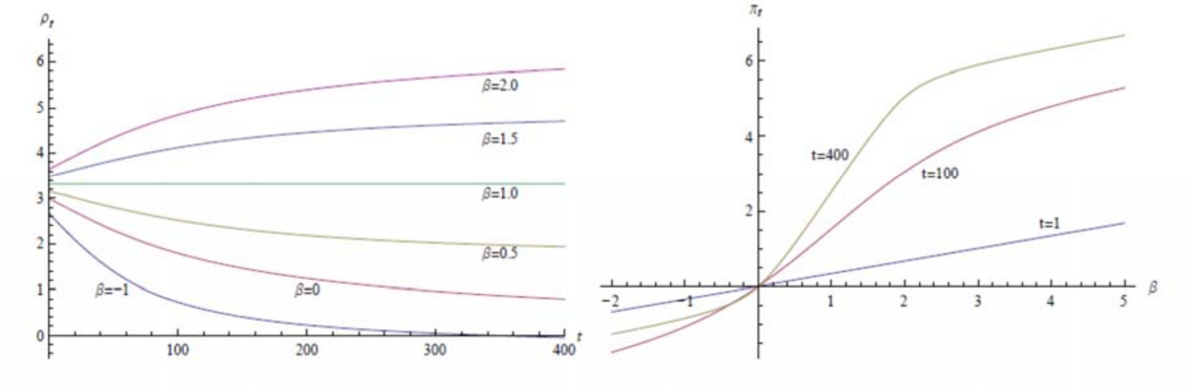

Figure 1: The discount rate as a function of maturity (left) and beta (right). We assume that $\delta = 0$, $\gamma = 2$, $\sigma = 4\%$ and μ is $U[0\%, 3\%]$. Rates are in %.

In this example, the sensitiveness of the discount rate to changes in the beta, which measures the local risk premium, is increasing in the maturity of the cash flows, in line with the idea that the term structure of risk premia is increasing. The aggregate risk premium is increased tenfold from $\pi_0(1) = 0.3\%$ to $\pi_\infty(1) = 3.3\%$ when maturity goes from 0 to infinity. This shows that it is particularly crucial to estimate the beta of projects having long-term impacts on the economy. Projects whose main benefits are to reduce emissions of greenhouse gases illustrate this point. It is thus particularly disappointing that we know virtually nothing about the "climate beta". Second, this observation should reinforce the need to promote the integration of risk measures in the evaluation of public policies with long-term impacts. Up to our knowledge (Gollier (2011b)), France is the only OECD country in which public institutions are requested to estimate the beta of the public investment projects.[18]

[18] In the U.S., the Office of Management and Budget (OMB) recommend to use a flat discount rate of 7% since 1992. It was argued that the "7% is an estimate of the average before-tax rate of return to private capital in the U.S. economy" (OMB (2003)). In 2003, the OMB also recommended the use of a discount rate of *3%*, in addition to the 7% mentioned above as a sensitivity. The 3% corresponds to the average real rate of return of the relatively safe 10-year Treasury notes. Interestingly enough, the recommended use of 3% and 7% is not differentiated by the nature of the underlying risk, and is independent of the time horizon of the project.

As in Veronesi (2000),[19] we now examine the impact of this form of parametric uncertainty on the price of an asset that delivers a flow of dividends $c_t^\beta\big|_{t\geq 0}$ in continuous-time. The price of this asset at date 0 must be equal to

$$P_0 = \int_0^\infty e^{-\rho_t(\beta)t} Ec_t^\beta dt. \tag{40}$$

The discount rates $\rho_t(\beta)$ are characterized by equation (38). Because

$$Ec_t^\beta = c_0^\beta \exp\left(\chi(t,\beta\mu) + 0.5\beta^2\sigma^2 t\right), \tag{41}$$

equation (40) can easily be rewritten as follows:[20]

$$\begin{aligned} P_0 &= c_0^\beta E\int_0^\infty \exp - t\left(\delta + (\gamma-\beta)\mu - 0.5(\gamma-\beta)^2\sigma^2\right)dt \\ &= c_0^\beta E\left(\delta + (\gamma-\beta)\mu - 0.5(\gamma-\beta)^2\sigma^2\right)^{-1}. \end{aligned} \tag{42}$$

This means that the price-dividend ratio is sensitive to information on μ. Assuming $\beta < \gamma$, equation (42) implies that any information that generates a first-degree stochastic dominant shift in the beliefs about the drift μ reduces the price of the asset. So surprisingly, as long as β is smaller than γ, a good news on future economic growth depresses asset prices. This is just because this information has a positive impact on the discount rates that is larger than its positive impact on expected growth rate of dividends.

5. *Extension to mean-reversion*

In the benchmark specification with a CRRA utility function and a Brownian motion for log consumption, the term structures of risk-adjusted discount rates are flat and constant through time.

[19] Veronesi considers a model in which aggregate consumption is generated by a diffusion process whose drift rate is unknown and may change at random time. We examine here the special case in which the drift μ is uncertain but constant. Under his notation, this means that *p*=0. This special case of Veronesi's model corresponds to the special case of our model that we examine in this section. Notice also that Veronesi limits the analysis to β =1.

[20] To guarantee convergence, we assume that $\delta + (\gamma-\beta)\mu - 0.5(\gamma-\beta)^2\sigma^2$ is positive almost surely. In the special case of $\beta = 1$, this result corresponds to Proposition 1 and Lemma 2 in Veronesi (2000).

In the specification with some parametric uncertainty on this Brownian motion that we examined in the previous section, they are monotone with respect to maturity and they move smoothly through time due to the revision of beliefs about the true values of the uncertain parameters. But this specification ignores the cyclicality of the economic activity. The introduction of predictable changes in the trend of growth introduces a new ingredient to the evaluation of investments. When expectations are temporarily diminishing, the discount rate associated to short horizons should be reduced to bias investment decisions toward projects that dampen the forthcoming temporary recession. Long termism is a luxury that should be favored only in periods of economic prosperity with relatively pessimistic expectations for the distant future.

In this section, we propose a simple model in which the economic growth is cyclical, with some uncertainty about the parameter governing this process. Following Bansal and Yaron (2004) and Hansen, Heaton and Li (2008) for example, we assume in this section that the change in log consumption follows an auto-regressive process: [21]

$$\begin{aligned} &\ln c_{t+1} / c_t = x_t \\ &x_t = \mu_\theta + y_t + \varepsilon_{xt}, \\ &y_t = \phi y_{t-1} + \varepsilon_{yt,} \end{aligned} \tag{43}$$

for some initial (potentially ambiguous) state characterized by y_{-1}, where ε_{xt} and ε_{yt} are independent and serially independent with mean zero and variance σ_x^2 and σ_y^2, respectively. Parameter ϕ, which is between 0 and 1, represents the degree of persistence in the expected growth rate process. We hereafter allow the trend of growth μ_θ to be uncertain. By forward induction of (43), it follows that:

$$G_t = \ln c_t / c_0 = \mu_\theta t + y_{-1}\phi \frac{1-\phi^t}{1-\phi} + \sum_{\tau=0}^{t-1} \frac{1-\phi^{t-\tau}}{1-\phi} \varepsilon_{y\tau} + \sum_{\tau=0}^{t-1} \varepsilon_{x\tau}. \tag{44}$$

It implies that, conditional to θ, G_t is normally distributed with annualized variance

[21] A more general model entails a time-varying volatility of growth as in Bansal and Yaron (2004) and Hansen (2012). Mean-reversion in volatility is useful to explain the cyclicality of the market risk premium.

$$t^{-1}Var(\ln c_t / c_0) = \frac{\sigma_y^2}{(1-\phi)^2}\left[1 - 2\phi\frac{\phi^t - 1}{t(\phi-1)} + \phi^2\frac{\phi^{2t}-1}{t(\phi^2-1)}\right] + \sigma_x^2. \tag{45}$$

Using property *ii* of Lemma 1, equations (8) and (9) imply that

$$\begin{aligned}\rho_t(\beta) &= t^{-1}\left(\chi(\beta, G_t) - \chi(\beta-\gamma, G_t)\right)\\ &= \delta + \gamma y_{-1}\phi\frac{1-\phi^t}{t(1-\phi)} + \gamma(\beta - 0.5\gamma)\left(\frac{\sigma_y^2}{(1-\phi)^2}\left[1 - 2\phi\frac{\phi^t - 1}{t(\phi-1)} + \phi^2\frac{\phi^{2t}-1}{t(\phi^2-1)}\right] + \sigma_x^2\right)\\ &\quad + t^{-1}\left(\chi(t, \beta\mu) - \chi(t, (\beta-\gamma)\mu)\right)\end{aligned} \tag{46}$$

One can then treat the last term in the RHS of this equation as in the previous section. Bansal and Yaron (2004) consider the following calibration of the model, using annual growth data for the United States over the period 1929-1998. Taking the month as the time unit, they obtained, $\mu = 0.0015$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, and $\phi = 0.979$. Using this ϕ yields a half-life for macroeconomic shocks of 32 months. Let us assume that $\delta = 0$, and let us introduce some uncertainty about the historical trend of growth from the sure $\mu = 0.0015$ to the uncertain context with two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$. In Figure 2, we draw the term structures of discount rates for three different positions y_{-1} in the business cycle. In the set of curves starting from below, the current growth rate of the economy is 0.6% per year, well below its unconditional expectation of 1.8%. In this recession phase, the short term discount rate is a low 1%, but the expectation of a recovery makes the term structure steeply increasing for low maturities. For low betas, the term structure is non-monotone because of the fact that for very distant maturities, the effect of parametric uncertainty eventually dominates. In the set of curves starting from above, the current growth rate is 1.2% per year above its unconditional expectation. In this expansionary phase of the cycle, the short term discount rate is large at around 6%, but is steeply decreasing for short maturities because of the diminishing expectations. At the middle of the business cycle, when expectations are in line with the historical average ($y_{-1} = 0$), the term structure corresponds to the set of curves starting from the center of the vertical axe. Although short-term discount rates are highly sensitive to the position in the business cycle, the long-term discount rates are not.

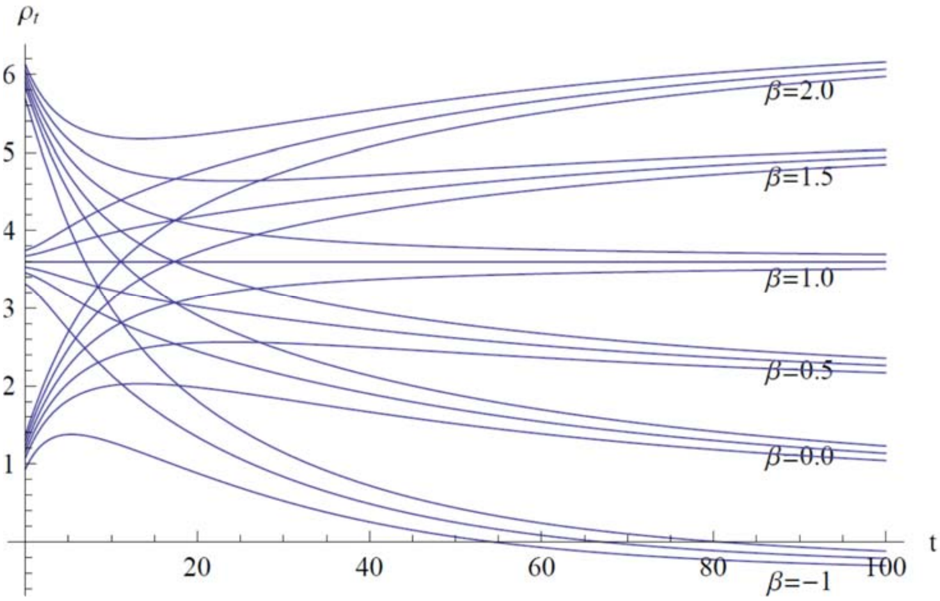

Figure 2: The discount rate (in % per year) as a function of the maturity (in years) in recession for different betas. Equation (46) is calibrated with $\delta = 0$, $\gamma = 2, \sigma_x = 0.0078$, $\sigma_y = 0.00034$, $\phi = 0.979$, two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$, and $y_{-1} = -0.001$ (bottom), $y_{-1} = 0$ (center), and for $y_{-1} = 0.001$ (top).

One can also examine a model in which the current state variable y_{-1} is uncertain. It is easy to generalize equation (46) to examine this ambiguous context. We obtain the following pricing formula:

$$\begin{aligned}\rho_t(\beta) &= \delta + \gamma(\beta - 0.5\gamma)\left(\frac{\sigma_y^2}{(1-\phi)^2}\left[1 - 2\phi\frac{\phi^t - 1}{t(\phi - 1)} + \phi^2\frac{\phi^{2t} - 1}{t(\phi^2 - 1)}\right] + \sigma_x^2\right) \\ &+ t^{-1}\left(\chi\left(t, \beta\mu + \beta y_{-1}\phi\left(1 - \phi^t\right)/t(1-\phi)\right) - \chi\left(t, (\beta - \gamma)\mu + (\beta - \gamma)y_{-1}\phi\left(1 - \phi^t\right)/t(1-\phi)\right)\right).\end{aligned} \tag{47}$$

Observe again that when $\beta = 0.5\gamma$, the term structure of discount rates is flat if both μ and y_{-1} have a symmetric distribution function. We also observe that the ambiguity on y_{-1} plays a role similar to the ambiguity on μ to shape the term structure. Our numerical simulations (available upon request) show that the hidden nature of the state variable does not modify the general characteristics of the term structures described above.

6. Pricing projects with a non-constant beta

Because assets' betas are assumed to be constant in the baseline framework of the consumption CAPM, this model implicitly assume a power relationship between the conditional expectation of the dividend generated by an asset and aggregate consumption, or an affine relationship between their log. Of course, the cases $\beta = 0$ (safe asset) and $\beta = 1$ (equity) are of special interest, and this paper characterizes the term structure of the discount rates of these two assets as special cases.

The power relationship is clearly critical for our results, in particular because it allows us to use the properties of CGF functions that appears in all pricing formulas used in this paper. Although specification (7) is restrictive, exploring the pricing of a project satisfying it opens the path to examining the pricing of a much larger class of projects of the form

$$D_t = \sum_{i=1}^{n} \alpha_{it} D_{it}, \quad \text{with } D_{it} = \xi_{it} c_t^{\beta_i}, \tag{48}$$

where random variables ξ_{it} are independent of c_t. This project can be interpreted as a portfolio of n different projects, each project i having a constant beta. Of course, the resulting projects portfolio has a non-constant beta. By a standard arbitrage argument, the value of this portfolio is the sum of the values of each of its beta-specific components. We can thus rely on our results in this paper for the evaluation of projects with a non-constant beta. If $\rho_t(\beta_i)$ is the efficient discount rate to evaluate the $\beta_i -$component of the project, the global value of the project will be equal to

$$\sum_{i=1}^{n} e^{-\rho_t(\beta_i)t} \alpha_{it} ED_{it}, \tag{49}$$

where we assumed without loss of generality that $E\xi_{it} = 1$. In other words, the discount factors – rather than the discount rates – must be averaged to determine the discount factor to be used to evaluate the cash flows of the global portfolio. In the absence of parametric uncertainty, Weitzman (2013) examines the case in which the project under scrutiny is a time-varying portfolio of a risk free asset ($\beta_1 = 0$) and of a risky project with a unit beta ($\beta_2 = 1$).

7. Concluding remarks

By focusing on the riskiness of future benefits and costs, this paper contributes to the debate on the discount rate for long-dated assets and investment projects, in particular those associated to climate change. Following Weitzman (2007) and a growing literature in finance since Veronesi (2000), we assumed that consumption growth follows a random walk, but its distribution entails parametric uncertainties. Our main contribution to this literature comes from the fact that we delivered generic results on how parameter uncertainty affects the level and the slope of the term structures of the risk-free rate and the risk premium, without relying on any specification of the nature of this uncertainty.

Our main messages in this framework are as follows. First, we showed that investors and policy evaluators should be particularly concerned by parameter uncertainty when they value dividends and benefits that materialize in the distant future. We indeed showed that although parameter uncertainty has *no* effect on the value of very short term zero-coupon bonds and equity, it has an increasingly important impact on risk-adjusted discount rates when contemplating longer maturities.

We also showed that the shape of the term structure of risk-adjusted discount rates for risky projects is determined by the relative intensity of a precautionary effect that pushes towards a decreasing term structure of the risk free discount rate, and of a risk aversion effect that pushes towards an increasing term structure of risk premia. Under some conditions, the term structure is flat, decreasing or increasing depending upon whether the beta of the project is respectively equal, smaller or larger than half the relative risk aversion of the representative investor. This implies that the sensitiveness of the risk-adjusted discount rate to changes in the beta of the asset is increasing in the maturity of the cash flow. As a consequence, the standard recommendation existing in the United States together as in many European countries to use a single discount rate to evaluate public policies independent of their risk profile is particularly problematic for policies having long-lasting socioeconomic impacts.

We also calibrated our general model with various specifications of the parametric uncertainty affecting growth. We have examined in particular a model à la Barro (2006, 2009) in which the growth rate of consumption has fat tails, because of a small probability of a macroeconomic catastrophe that is added to the otherwise Gaussian business-as-usual growth process. The recognition of the intrinsic ambiguity that affects the frequency of catastrophes provides another justification for the risk-free discount rate and the aggregate risk premium to be respectively decreasing and increasing with maturity.

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Appendix: On the possibility of an increasing term structure of aggregate risk premia

We have shown in Proposition 3 that the term structure of aggregate risk premia must be increasing when the set of uncertain cumulants of the increments of log consumption are statistically independent. Equation (23) shows that this result may not be robust to correlated cumulants. In this appendix, we build a counterexample. Take $\beta = 1$ and $\gamma = 0.1$. Suppose that the log consumption process is Brownian with unknown trend and volatility. There are two states of nature. The first state $\theta = 1$ yields a zero trend and a high volatility: $\mu_1 = 0\%$ and $\sigma_1 = 30\%$. The second state $\theta = 2$ yields a higher trend and a lower volatility: $\mu_2 = 5\%$ and $\sigma_2 = 10\%$. The probability of state 1 is $p_1 = 0.95$.

In Figure 3, we draw the term structure of aggregate risk premia in this economy. Observe that the aggregate risk premium is locally decreasing in maturity.

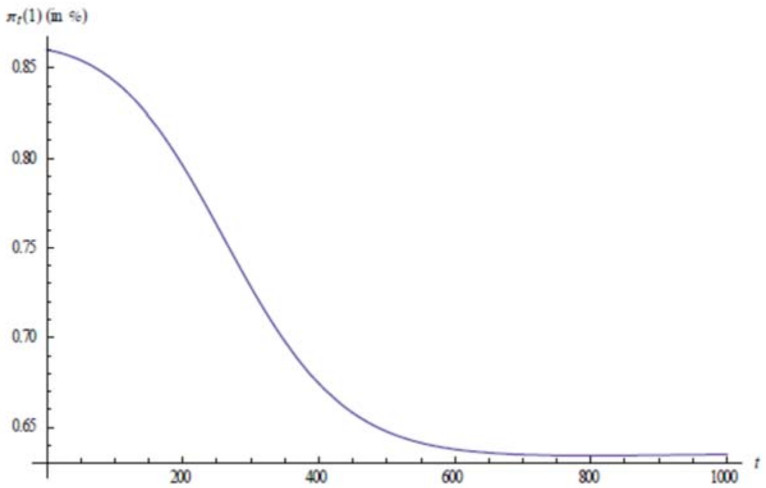

Figure 3: Counterexample in which risk premia are locally decreasing in maturity.

This counterexample illustrates the fact that the cumulants of order larger than 2 also affect the aggregate risk premium. To see this, we can rewrite equation (9) as:

$$\pi_t(\beta) = \sum_{n=2}^{\infty} \frac{\kappa_n^{\ln c_t}}{t} \frac{\beta^n - (\beta - \gamma)^n + (-\gamma)^n}{n!}. \tag{50}$$

For $\beta = 1$ and $\gamma = 0.1$, this simplifies to

$$\pi_t(1) = 0.1\frac{\kappa_2^{\ln c_t}}{t} + 0.045\frac{\kappa_3^{\ln c_t}}{t} + 0.0143\frac{\kappa_4^{\ln c_t}}{t} + ... \qquad (51)$$

Observe that, under these values of the parameters, the risk premium is increasing in the first three annualized cumulants of log consumption. After some tedious computations, we also obtain that

$$\begin{aligned} t^{-1}\kappa_2^{\ln c_t} &= E\kappa_2^{g|\theta} + \kappa_2^{\mu|\theta} t \\ &= 8.6\times 10^{-2} + 1.1875\times 10^{-4} t \end{aligned} \qquad (52)$$

$$\begin{aligned} t^{-1}\kappa_3^{\ln c_t} &= E\kappa_3^{g|\theta} + 3\operatorname{cov}(\kappa_1^{g|\theta}, \kappa_2^{g|\theta})t + \kappa_3^{\mu|\theta} t^2 \\ &= -5.7\times 10^{-4} t + 5.34375\times 10^{-6} t^2 \end{aligned} \qquad (53)$$

$$\begin{aligned} t^{-1}\kappa_4^{\ln c_t} &= 3tVar(\kappa_2^{g|\theta}) + 6\operatorname{cov}(\kappa_2^{g|\theta}, (\mu_\theta - \mu)^2)t^2 + \kappa_4^{\mu|\theta} t^3 \\ &= 9.12\times 10^{-4} t - 5.13\times 10^{-5} t^2 + 2.12\times 10^{-7} t^3 \end{aligned} \qquad (54)$$

Observe from equations (52), (53) and (54) that for small maturities, the term structures of the annualized variance and of the annualized excess kurtosis are increasing in maturity *t*, whereas the term structure of the annualized skewness is decreasing. It happens in this example that the dominating effect for the slope of the term structure of risk premia comes from the decreasing skewness. This means that the RHS of equation (23) is negative.