Instrumental Variable Identification of Dynamic Variance Decompositions

Mikkel Plagborg-Møller
Princeton University

Christian K. Wolf
University of Chicago

February 25, 2021

Slides: `https://scholar.princeton.edu/mikkelpm`

Macro identification

- Key questions in empirical macro:
  1. What is the effect of a certain shock?
  2. How important is a certain shock?
  3. How did a certain shock contribute to particular historical episodes?
- Impulse response function:

$$\Theta_{i,j,\ell} \equiv E(y_{i,t+\ell} \mid \varepsilon_{j,t} = 1) - E(y_{i,t+\ell} \mid \varepsilon_{j,t} = 0), \quad \ell = 0, 1, 2, \ldots$$

- One empirical strategy: Estimate fully-specified Dynamic Stochastic General Equilibrium model.

Macro identification using SVARs

- Can we avoid fully specifying all aspects of the model?
- Structural Vector Autoregression: Sims (1980)

$$y_t = \sum_{\ell=1}^{p} A_\ell y_{t-\ell} + B\varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} N(0, I), \quad \det(B) \neq 0.$$

- Assumes invertibility: $\varepsilon_t \in \overline{\text{span}}(\{y_\tau\}_{-\infty<\tau\leq t})$.
  - Econometrician shares agents' info set. Nakamura & Steinsson (2018)
  - Known to fail in interesting applications, e.g., news/noise shocks. Blanchard, L'Huillier & Lorenzoni (2013); Leeper, Walker & Yang (2013)
- Need *a priori* restrictions to identify impact impulse responses B.

Macro identification using LP-IV

- Recent push in applied structural macro towards transparent and credible identification. Two strands:
  - Local projections: Unrestricted shock transmission.

$$y_{i,t+\ell} = \hat{\Theta}_{i,1,\ell}\varepsilon_{1,t} + \hat{e}_{t+\ell|t}, \quad \ell = 0, 1, 2, \ldots$$

  - External IV (proxy): Interpretable exclusion restrictions.

$$E(z_t\varepsilon_{1,t}) \neq 0, \quad E(z_t\varepsilon_{j,t}) = 0, \; j \geq 2.$$

- Recently popular combination: LP-IV. Consistently estimate impulse responses through simple 2SLS regressions.
- Unlike SVARs, no need to assume invertibility, known number of shocks, known list of endogenous variables y_t, etc.

Identification of variance/historical decompositions

- Care not just about *effects* of shocks, but also about *importance*.
- SVAR/structural literature quantifies importance using variance decompositions. Key objects for distinguishing between competing business cycle theories.
- No general methods exist for identifying them w/o invertibility.
  Stock & Watson (2018); Gorodnichenko & Lee (2020)
- Also unknown how to identify historical decompositions.
- Unfortunate trade-off for applied people: Must give up on quantifying importance if robust inference desired.

Our contributions

1. Derive identified set of all parameters in LP-IV model.
2. Three different variance decomp. concepts are interval-identified.
   - Sharp, informative bounds.
   - Depend on IV strength and informativeness of macro var's about shock.
3. Degree of invertibility of shock set-identified. Invertibility testable.
4. Provide various sufficient conditions for point identification, if desired; weaker than invertibility. Give conditions to identify hist. decomp.
5. Easily computable partial identification robust confidence intervals.

Literature

- Local projections: Jordà (2005); Angrist, Jordà & Kuersteiner (2018)
- External IV: Stock (2008); Stock & Watson (2012); Mertens & Ravn (2013); Gertler & Karadi (2015); Stock & Watson (2016); Caldara & Kamps (2017)
- LP-IV: Mertens (2015); Ramey (2016); Barnichon & Brownlees (2018); Jordà, Schularick & Taylor (2018); Ramey & Zubairy (2018); Stock & Watson (2018)
- Invertibility: Lippi & Reichlin (1994); Sims & Zha (2006); Forni & Gambetti (2014); Forni, Gambetti & Sala (2019); Plagborg-Møller (2019); Chahrour & Jurado (2020); Miranda-Agrippino & Ricco (2020); Wolf (2020)
- Inference for interval ID: Imbens & Manski (2004); Stoye (2009)
- Partial ID in SVARs: Gafarov, Meier & Montiel Olea (2018); Granziera, Moon & Schorfheide (2018); Giacomini & Kitagawa (2020)

Outline

SVMA-IV model

$$\underset{}{\overset{n_y \times 1}{y_t}} = \Theta(L) \overset{n_\varepsilon \times 1}{\varepsilon_t}, \quad \Theta(L) = \sum_{\ell=0}^{\infty} \overset{n_y \times n_\varepsilon}{\Theta_\ell} L^\ell,$$

$$\overset{1 \times 1}{z_t} = \sum_{\ell=1}^{\infty} (\Psi_\ell z_{t-\ell} + \Lambda_\ell y_{t-\ell}) + \underbrace{\overset{1 \times 1}{\alpha} \varepsilon_{1,t}}_{\text{exclusion}} + \sigma_v \overset{1 \times 1}{v_t}.$$

- Consistent with DSGE or SVAR structure, but more general. ▸ SVAR
- Impulse responses: $\Theta_{i,j,\ell}$, the (i,j) element of Θ_ℓ.
- Normality for notational ease:

$$(\varepsilon_t', v_t)' \overset{i.i.d.}{\sim} N(0, I_{n_\varepsilon + 1}).$$

- Allow $n_\varepsilon \geq n_y$ and unknown.
- In paper: extensions to multiple IVs, multiple included shocks. ▸

Invertibility and recoverability

- Degree of invertibility using data up to time $t+\ell$: Sims & Zha (2006)

$$R^2_\ell \equiv 1 - \frac{\text{Var}(\varepsilon_{1,t} \mid \{y_\tau\}_{-\infty<\tau\le t+\ell})}{\text{Var}(\varepsilon_{1,t})}, \quad \ell \ge 0.$$

  Shock-specific concept. Special cases: Chahrour & Jurado (2020)

  - Invertibility: $R^2_0 = 1$, i.e., $E(\varepsilon_{1,t} \mid \{y_\tau\}_{-\infty<\tau\le t}) = \varepsilon_{1,t}$.
  - Recoverability: $R^2_\infty = 1$, i.e., $E(\varepsilon_{1,t} \mid \{y_\tau\}_{-\infty<\tau<\infty}) = \varepsilon_{1,t}$.

- SVMA-IV model does not assume invertibility/recoverability *a priori*.

Forecast variance ratio

- Forecast variance ratio (FVR):

$$FVR_{i,\ell} \equiv 1 - \frac{\text{Var}(y_{i,t+\ell} \mid \{y_\tau\}_{-\infty<\tau\leq t}, \{\varepsilon_{1,\tau}\}_{t<\tau<\infty})}{\text{Var}(y_{i,t+\ell} \mid \{y_\tau\}_{-\infty<\tau\leq t})}$$
$$= \frac{\sum_{m=0}^{\ell-1} \Theta_{i,1,m}^2}{\text{Var}(y_{i,t+\ell} \mid \{y_\tau\}_{-\infty<\tau\leq t})}.$$

- Alternative concepts in paper:
  - Forecast var. decomp.: condition on $\{\varepsilon_\tau\}_{\tau\leq t}$ instead of $\{y_\tau\}_{\tau\leq t}$.
  - Unconditional variance decomposition.

Historical decomposition

- Recall moving average model:

$$y_{i,t} = \sum_{j=1}^{n_\varepsilon} \sum_{\ell=0}^{\infty} \Theta_{i,j,\ell} \varepsilon_{j,t-\ell}.$$

- Historical decomposition of variable i attributable to first shock:

$$E(y_{i,t} \mid \{\varepsilon_{1,\tau}\}_{\tau \leq t}) = \sum_{\ell=0}^{\infty} \Theta_{i,1,\ell} \varepsilon_{1,t-\ell}.$$

Outline

1. Model and parameters of interest
2. Identification
3. Illustration using structural macro model
4. Inference
5. Application: monetary shocks
6. Summary

Intuition: static model

- Static SVMA is a classical measurement error model:

$$\underset{n_y\times 1}{y_t} = \underset{n_y\times 1}{\Theta_{\bullet,1,0}}\,\underset{1\times 1}{\varepsilon_{1,t}} + \underset{n_y\times 1}{\xi_t}\;,$$

$$\underset{1\times 1}{z_t} = \underset{1\times 1}{\alpha}\;\varepsilon_{1,t} + \sigma_v\,\underset{1\times 1}{v_t}\;,$$

$$\varepsilon_{1,t} \perp\!\!\!\perp \xi_t \perp\!\!\!\perp v_t.$$

- Unknown signal-to-noise ratio α^2/σ_v^2 in IV equation.
- Intuition for bounds on importance of $\varepsilon_{1,t}$: Klepper & Leamer (1984)
  - Attenuation bias in regression of $y_{i,t}$ on z_t
    $\implies$ Lower bound on importance of $\varepsilon_{1,t}$.
  - $\alpha^2 = \text{Var}(E(z_t \mid \varepsilon_{1,t})) \geq \text{Var}(E(z_t \mid y_t)) \implies$ Lower bound on α^2/σ_v^2
    $\implies$ Upper bound on importance of $\varepsilon_{1,t}$.

▸ Details

Dynamic model: notation

- Residualized IV:

$$\tilde{z}_t \equiv z_t - E(z_t \mid \{y_\tau, z_\tau\}_{-\infty<\tau<t}) = \alpha\varepsilon_{1,t} + \sigma_v v_t.$$

  Serially uncorrelated by construction.

- Projections of $\tilde{z}_t$ and $\varepsilon_{1,t}$ onto whole time series of macro variables:

$$\tilde{z}_t^\dagger \equiv E(\tilde{z}_t \mid \{y_\tau\}_{-\infty<\tau<\infty}),$$
$$\varepsilon_{1,t}^\dagger \equiv E(\varepsilon_{1,t} \mid \{y_\tau\}_{-\infty<\tau<\infty}).$$

- Spectral density of a time series x_t: $s_x(\omega)$, $\omega \in [0, 2\pi]$.

Identification up to scale

- Impulse responses identified up to scale:

$$\text{Cov}(y_{i,t}, \tilde{z}_{t-\ell}) = \alpha\Theta_{i,1,\ell}, \quad i = 1, \dots, n_y, \; \ell \geq 0.$$

  Relative IRs $\Theta_{i,1,\ell}/\Theta_{1,1,0}$ point-identified. Stock & Watson (2018)

- Degree of invertibility at time $t + \ell$ identified up to scale:

$$R^2_\ell = \frac{1}{\alpha^2} \times \text{Var}(E(\tilde{z}_t \mid \{y_\tau\}_{-\infty<\tau\leq t+\ell})), \quad \ell \geq 0.$$

- FVRs identified up to scale:

$$FVR_{i,\ell} = \frac{1}{\alpha^2} \times \frac{\sum_{m=0}^{\ell-1} \text{Cov}(y_{i,t}, \tilde{z}_{t-m})^2}{\text{Var}(y_{i,t+\ell} \mid \{y_\tau\}_{-\infty<\tau\leq t})}, \quad i = 1, \dots, n_y, \; \ell \geq 1.$$

Bounds for α

- Upper bound:

$$\alpha^2 \leq \text{Var}(\tilde{z}_t) \equiv \alpha_{UB}^2.$$

  Binds when IV is perfectly informative.

- Lower bound: Since $\tilde{z}_t^\dagger = \alpha \varepsilon_{1,t}^\dagger$, we have

$$\forall\, \omega \in [0, 2\pi]: \quad s_{\tilde{z}^\dagger}(\omega) = \alpha^2 s_{\varepsilon_1^\dagger}(\omega) \leq \alpha^2 s_{\varepsilon_1}(\omega) = \alpha^2 \times \tfrac{1}{2\pi}$$

$$\implies \quad \alpha^2 \geq 2\pi \sup_{\omega \in [0,\pi]} s_{\tilde{z}^\dagger}(\omega) \equiv \alpha_{LB}^2.$$

  Binds when macro var's y_t are perfectly informative about shock $\varepsilon_{1,t}$ at *some* frequency $\overline{\omega} \in [0, \pi]$: $s_{\varepsilon_1 - \varepsilon_1^\dagger}(\overline{\omega}) = 0$.

- Note: Closed form for $s_{\tilde{z}^\dagger}(\omega)$ in terms of joint spectrum of data.

Identified set for α: sharpness

Proposition

Let there be given a joint spectral density for $w_t = (y_t', \tilde{z}_t)'$, continuous and positive definite at every frequency, with $\tilde{z}_t$ being unpredictable from $\{w_\tau\}_{-\infty<\tau<t}$. Choose any $\alpha \in (\alpha_{LB}, \alpha_{UB}]$.

Then there exists a model with the given α such that the spectral density of w_t implied by the model matches the given spectral density.

- Proposition does not cover boundary case $\alpha = \alpha_{LB}$ due to economically inessential technicalities.

Identified set for α: interpretation

- Express identified set for $\frac{1}{\alpha^2}$ in terms of underlying model parameters:

$$\left[\underbrace{\frac{\alpha^2}{\alpha^2 + \sigma_v^2}}_{\text{IV strength}} \times \frac{1}{\alpha^2}\ ,\ \underbrace{\frac{1}{1 - 2\pi \inf_{\omega \in [0,\pi]} s_{\varepsilon_1 - \varepsilon_1^\dagger}(\omega)}}_{\text{informativeness of } y_t \text{ for } \varepsilon_{1,t}} \times \frac{1}{\alpha^2} \right].$$

- Identified set is narrower when...
  - IV is stronger.
  - Macro var's are more informative about *some* cycle of the shock.

Degree of invertibility/recoverability

- Degree of invertibility R_0^2 and recoverability R_∞^2 are each interval-identified.
- When are the data consistent with invertibility/recoverability?

Proposition

Assume $\alpha_{LB}^2 > 0$.

The identified set for R_0^2 contains 1 if and only if $\tilde{z}_t$ does not Granger cause y_t.

The identified set for R_∞^2 contains 1 if and only if $\tilde{z}_t^\dagger$ is white noise.

Point identification

- Assumptions yielding point identification of α, R^2_ℓ, FVR:

  1. Macro var's y_t are perfectly informative about $\varepsilon_{1,t}$ at some frequency (untestable).
  2. Shock is recoverable (testable). Economically weaker condition than invertibility. Can also identify shock: $\varepsilon_{1,t} = \frac{1}{\alpha}\tilde{z}_t^\dagger$.
  3. $n_\varepsilon = n_y$ (testable). Then all shocks are recoverable. (But our partial ID bounds obtain even if we know $n_\varepsilon = n_y + 1$.)
  4. IV is perfect, i.e., $\sigma_v = 0$ (untestable).

- Cond's 2–4 point-identify historical decomposition $\sum_{\ell=0}^{\infty} \Theta_{i,1,\ell}\varepsilon_{1,t-\ell}$.

- Auxiliary assumptions are not needed for informative *partial* ID.

Outline

1. Model and parameters of interest
2. Identification
3. Illustration using structural macro model
4. Inference
5. Application: monetary shocks
6. Summary

Informative bounds in Smets-Wouters (2007) model

- Smets & Wouters (2007) model, posterior mode estimates.
- Data series y_t: output, inflation, nominal interest rate. Known spectrum. SVMA-IV analysis does not exploit DSGE structure.
- Econometrician observes IV:

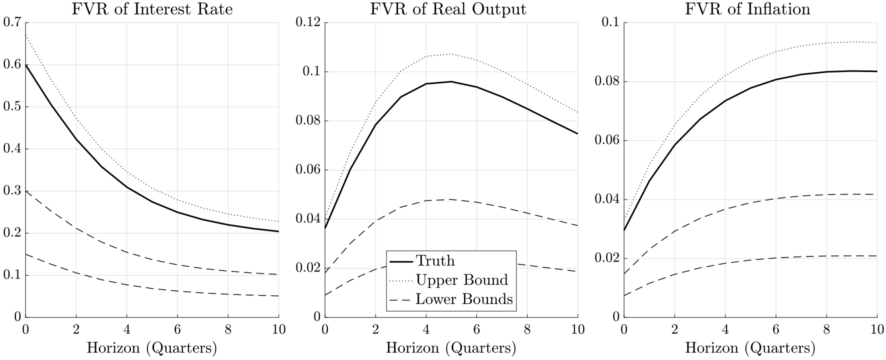
$$z_t = \alpha\varepsilon_{1,t} + \sigma_v v_t, \quad v_t \overset{i.i.d.}{\sim} N(0,1).$$

  Set true $\alpha = 1$. Consider IV strengths $\frac{1}{1+\sigma_v^2} \in \{0.25, 0.5\}$.

- Three shocks of interest $\varepsilon_{1,t}$ (seven total in model):
  1. Monetary shock. Nearly invertible.
  2. Forward guidance shock. Highly noninvertible, nearly recoverable.
  3. Technology shock. Data only informative about longest cycles.

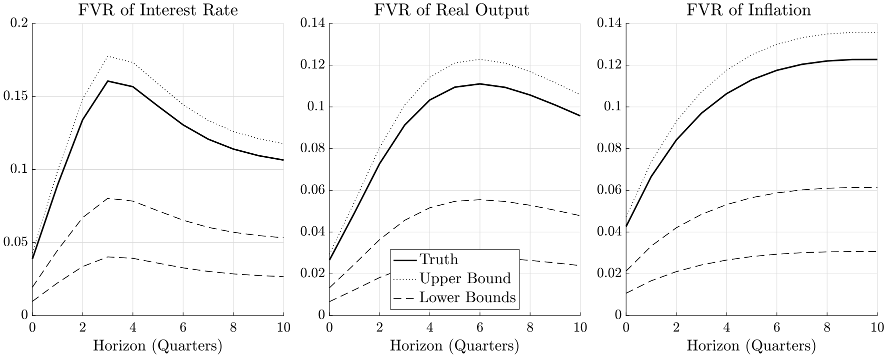
Monetary shock

- Shock nearly invertible: $R_0^2 = 0.87$, $R_\infty^2 = 0.88$.
- Tight lower bound on α: $\alpha_{LB}^2 = 0.90$.



Forward guidance shock

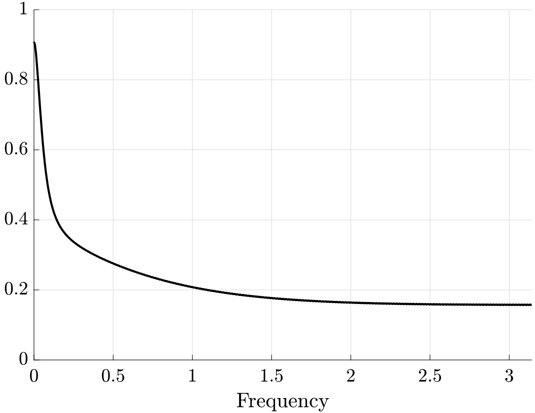
- Monetary shock anticipated two quarters ahead.
- Highly noninvertible: $R_0^2 = 0.08$. Invertibility-based identification overstates FVRs by factor $1/0.08 \approx 13$!
- Nearly recoverable: $R_2^2 = 0.87$, $R_\infty^2 = 0.88$. ▸ R_ℓ^2 ▸ SVAR



Technology shock

- Highly non-recoverable: $R_0^2 = 0.20$, $R_\infty^2 = 0.22$.
- Macro var's only informative about longest cycles of shock.
- But tight lower bound on α: $\alpha_{LB}^2 = 0.91$.

Spectral density of best two-sided predictor of shock



Outline

1. Model and parameters of interest
2. Identification
3. Illustration using structural macro model
4. Inference
5. Application: monetary shocks
6. Summary

Partial identification robust confidence intervals

- All parameters of interest are interval- or point-identified.
- Confidence interval procedure:
  1. Estimate *reduced-form* VAR for $(y_t', z_t)'$.
  2. Compute sample analogues of population bounds.
  3. Plug into Imbens & Manski (2004) and Stoye (2009) formulas.
- CIs for parameters as well as for identified sets.
- Prove non-parametric validity under "sieve VAR" asymptotics.
- Test invertibility using VAR Granger causality test.
  Giannone & Reichlin (2006); Forni & Gambetti (2014); Stock & Watson (2018)

Outline

1. Model and parameters of interest
2. Identification
3. Illustration using structural macro model
4. Inference
5. Application: monetary shocks
6. Summary

Importance of monetary shocks

- Gertler & Karadi (2015) estimate effect of monetary shocks on interest rate, IP, CPI, Excess Bond Premium. Gilchrist & Zakrajšek (2012)
- SVAR-IV approach on U.S. monthly data. IV: high-freq. changes in 3-month FFR futures prices around FOMC announcements.
- Our question: How *important* is the monetary shock in determining fluctuations of real and financial variables?
- Consider two different interest rates: FFR or 1-year Treasury.
- Sample: 1990:1–2012:6. AIC selects $p = 6$ reduced-form VAR lags.
- 1,000 iterations of i.i.d. recursive residual bootstrap.

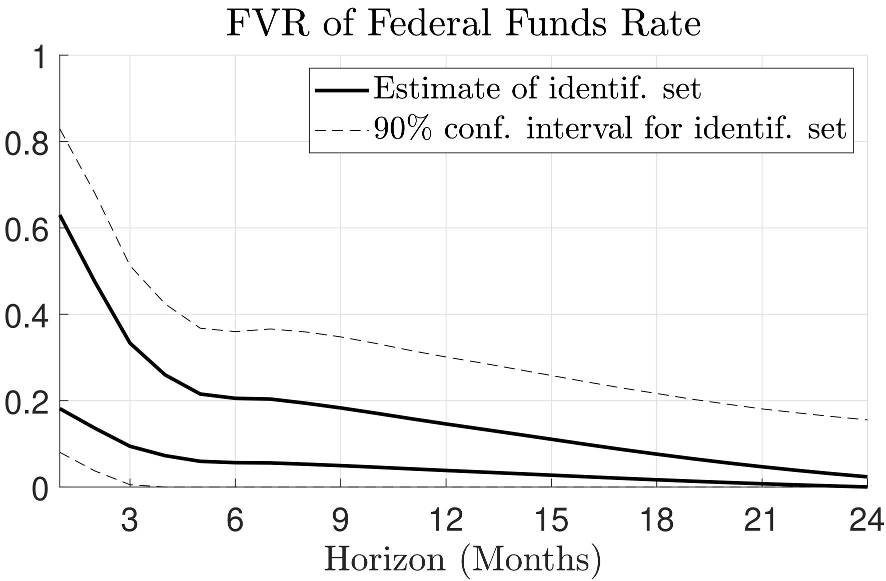
▸ SVAR-IV

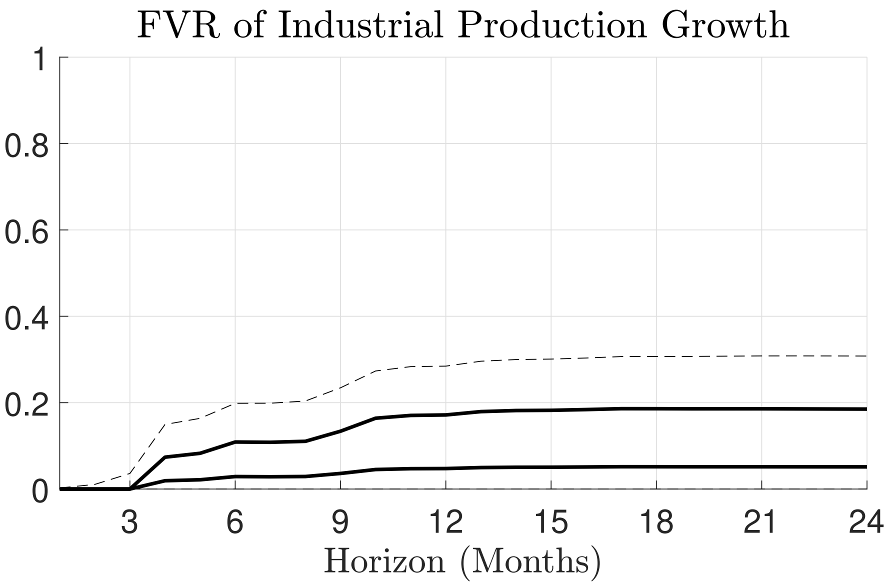
Degree of invertibility/recoverability

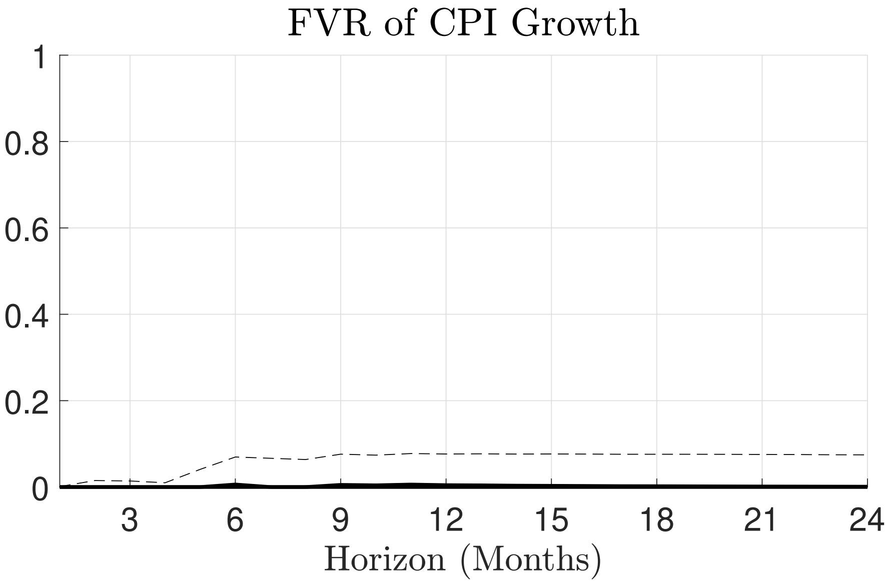
- Reject invertibility of monetary shock with FFR.

		FFR	1-year rate
R_0^2	Bound estimates	$[0.196, 0.684]$	$[0.118, 0.922]$
	90% conf. interval	$[0.097, 0.877]$	$[0.029, 1.000]$
R_∞^2	Bound estimates	$[0.282, 1.000]$	$[0.119, 1.000]$
	90% conf. interval	$[0.190, 1.000]$	$[0.028, 1.000]$
Granger causality p-value		0.0001	0.390

90% confidence interval for identified set (IS).
Upper bound of IS for R_∞^2 equals 1 by construction.

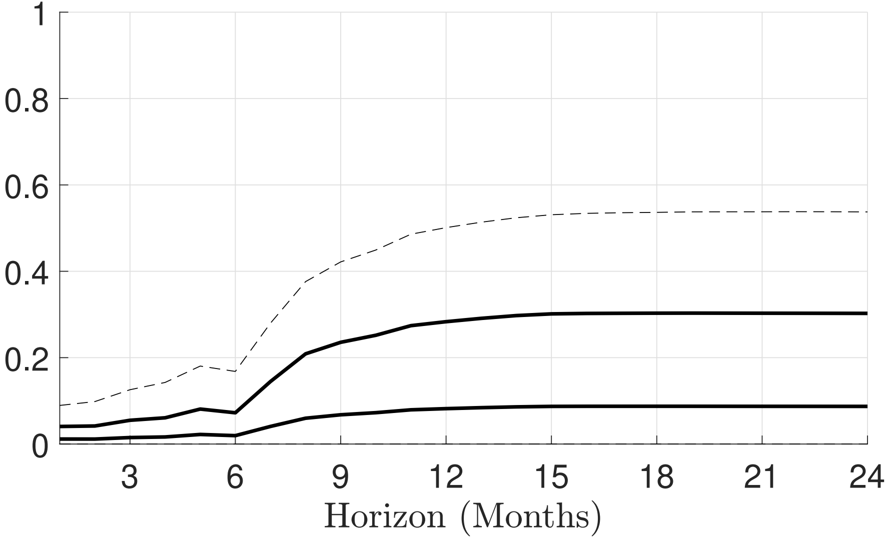

FVR of Federal Funds Rate
Estimate of identif. set
90% conf. interval for identif. set
1
0.8
0.6
0.4
0.2
0
3
6
9
12
15
18
21
24
Horizon (Months)


FVR of Industrial Production Growth
1
0.8
0.6
0.4
0.2
0
3
6
9
12
15
18
21
24
Horizon (Months)



FVR of CPI Growth
1
0.8
0.6
0.4
0.2
0
3
6
9
12
15
18
21
24
Horizon (Months)

FVR of Excess Bond Premium



Outline

1. Model and parameters of interest
2. Identification
3. Illustration using structural macro model
4. Inference
5. Application: monetary shocks
6. Summary

Summary

- SVMA-IV model: attractive semiparametric alternative to SVARs.
- Known how to identify IRFs. We provide remaining tools: variance decompositions, historical decompositions, degree of invertibility.
- Informative bounds regardless of invertibility.
- Provide sufficient conditions for point ID, weaker than invertibility.
- Partial ID robust confidence intervals. Easy to compute.
- Application: Informative upper bounds on importance of monetary shocks for fluctuations in IP, CPI, and financial spread.

Thank you!

Examples of external IVs

- Narrative monetary shocks. Romer & Romer (2004)
- Narrative fiscal shocks. Mertens & Ravn (2013); Mertens & Montiel Olea (2018); Ramey & Zubairy (2018)
- High-frequency asset price changes around FOMC announcements. Barakchian & Crowe (2013); Gertler & Karadi (2015)
- Oil supply disruptions. Hamilton (2003)
- Large oil discoveries. Arezki, Ramey & Sheng (2017)
- Utilization-adjusted TFP growth. Fernald (2014); Caldara & Kamps (2017)
- Volatility spikes. Carriero et al. (2015)

Back

Examples of variance decomposition applications

- TFP shocks. Kydland & Prescott (1982); KPSW (1991)
- Monetary shocks. Romer×2 (1989); Christiano, Eichenbaum & Evans (1999)
- Investment efficiency shocks. Justiniano, Primiceri & Tambalotti (2010)
- News shocks. Schmitt-Grohé & Uribe (2012)
- Risk shocks. Christiano, Motto & Rostagno (2014)
- Demand/sentiment shocks. Angeletos, Collard & Dellas (2017)
- Business cycle accounting. Cochrane (1994); Smets & Wouters (2007)

Back

Multiple instruments

$$\overset{n_y \times 1}{y_t} = \Theta(L) \overset{n_\varepsilon \times 1}{\varepsilon_t}, \quad \Theta(L) = \sum_{\ell=0}^{\infty} \overset{n_y \times n_\varepsilon}{\Theta_\ell} L^\ell,$$

$$\overset{n_z \times 1}{z_t} = \sum_{\ell=1}^{\infty} (\Psi_\ell z_{t-\ell} + \Lambda_\ell y_{t-\ell}) + \underbrace{\overset{1\times 1}{\alpha} \overset{n_z \times 1}{\lambda} \varepsilon_{1,t}}_{\text{exclusion}} + \overset{n_z \times n_z}{\Sigma_v^{1/2}} \overset{n_z \times 1}{v_t} .$$

- $\|\lambda\| = 1$, first nonzero element positive.
- Define vector of projection residuals

$$\tilde{z}_t \equiv z_t - E(z_t \mid \{y_\tau, z_\tau\}_{-\infty<\tau<t}).$$

- Model is testable: $s_{y\tilde{z}}(\omega)$ has rank-1 factor structure.
- If model is consistent with data, then λ is point-identified. If Σ_v is unrestricted, identification analysis is *as if* we observed the scalar IV

$$\check{z}_t \equiv \frac{1}{\lambda' \operatorname{Var}(\tilde{z}_t)^{-1} \lambda} \lambda' \operatorname{Var}(\tilde{z}_t)^{-1} \tilde{z}_t.$$

Instruments correlated with multiple shocks

- Also consider an extended model where the IVs z_t correlate with the first n_{ε_x} of the n_ε structural shocks (i.e., drop exclusion restriction).

$$\overset{n_y\times 1}{y_t} = \Theta(L) \overset{n_\varepsilon \times 1}{\varepsilon_t}, \quad \Theta(L) = \sum_{\ell=0}^{\infty} \overset{n_y \times n_\varepsilon}{\Theta_\ell} L^\ell,$$

$$\overset{n_z\times 1}{z_t} = \sum_{\ell=1}^{\infty}(\Psi_\ell z_{t-\ell} + \Lambda_\ell y_{t-\ell}) + \overset{n_z \times n_{\varepsilon_x}}{\Gamma} \overset{n_{\varepsilon_x}\times 1}{\varepsilon_{x,t}} + \overset{n_z \times n_z}{\Sigma_v^{1/2}} \overset{n_z \times 1}{v_t}.$$

- Derive sharp bounds for FVR wrt. $\Gamma\varepsilon_{x,t}$:

$$FVR_{i,\ell} \equiv 1 - \frac{\operatorname{Var}(y_{i,t+\ell} \mid \{y_\tau\}_{-\infty<\tau\le t}, \{\Gamma\varepsilon_{x,\tau}\}_{t<\tau<\infty})}{\operatorname{Var}(y_{i,t+\ell} \mid \{y_\tau\}_{-\infty<\tau\le t})}.$$

  Lower bound available in closed form. Upper bound solves (convex) semidefinite programming problem.

- Example: Assume $n_{\varepsilon_x} = n_z$ and Γ nonsingular. Then FVR wrt. $\Gamma\varepsilon_{x,t}$ equals the FVR wrt. $\varepsilon_{x,t}$. Mertens & Ravn (2013)

Back

Forecast variance decomposition

- Forecast variance decomposition (FVD):

$$FVD_{i,\ell} \equiv 1 - \frac{\text{Var}(y_{i,t+\ell} \mid \{\varepsilon_\tau\}_{-\infty<\tau\leq t}, \{\varepsilon_{1,\tau}\}_{t<\tau<\infty})}{\text{Var}(y_{i,t+\ell} \mid \{\varepsilon_\tau\}_{-\infty<\tau\leq t})}$$

$$= \frac{\sum_{m=0}^{\ell-1} \Theta_{i,1,m}^2}{\sum_{j=1}^{n_\varepsilon} \sum_{m=0}^{\ell-1} \Theta_{i,j,m}^2}.$$

- FVR $\neq$ FVD unless all shocks are invertible (SVAR).
  Forni, Gambetti & Sala (2018)

▸ Identif ◂ Back

Sufficient conditions for invertibility/recoverability

$$y_t = \Theta(L)\varepsilon_t$$

- Assuming $n_\varepsilon = n_y \dots$
- All shocks invertible if all roots of $x \mapsto \det(\Theta(x))$ outside unit circle.
- All shocks recoverable if no roots of $x \mapsto \det(\Theta(x))$ on unit circle.

◂ Back

Bias of SVAR-IV

- VAR(∞) forecast error: $u_t \equiv y_t - E(y_t \mid \{y_\tau\}_{-\infty<\tau<t})$.
- SVAR-IV: If all $n_\varepsilon = n_y$ shocks are invertible, then $\varepsilon_{1,t} = \gamma' u_t$, where

$$\gamma \equiv (\Sigma'_{u\tilde{z}} \Sigma_u^{-1} \Sigma_{u\tilde{z}})^{-1/2} \Sigma_u^{-1} \Sigma_{u\tilde{z}}, \quad \Sigma_{u\tilde{z}} \equiv \text{Cov}(u_t, \tilde{z}_t), \quad \Sigma_u \equiv \text{Var}(u_t).$$

Proposition

The SVAR-IV-(mis)identified shock is given by

$$\tilde{\varepsilon}_{1,t} \equiv \gamma' u_t = \textstyle\sum_{j=1}^{n_\varepsilon} \sum_{\ell=0}^{\infty} a_{j,\ell} \varepsilon_{j,t-\ell},$$

where $\{a_{j,\ell}\}$ satisfy $\sum_{j=1}^{n_\varepsilon} \sum_{\ell=0}^{\infty} a_{j,\ell}^2 = 1$ and $a_{1,1} = \sqrt{R_0^2}$.

The associated SVAR-IV impulse responses are given by

$$\tilde{\Theta}_{\bullet,1,\ell} \equiv \text{Cov}(y_t, \tilde{\varepsilon}_{1,t-\ell}) = \textstyle\sum_{j=1}^{n_\varepsilon} \sum_{m=0}^{\infty} a_{j,m} \Theta_{\bullet,1,\ell+m},$$

and the impact impulse responses satisfy $\tilde{\Theta}_{\bullet,1,0} = (R_0^2)^{-1/2} \Theta_{\bullet,1,0}$.

Back

Static model for intuition

- For intuition, start with static version of model:

$$\underset{n_y\times 1}{y_t} = \underset{n_y\times n_\varepsilon}{\Theta_0} \underset{n_\varepsilon\times 1}{\varepsilon_t},$$

$$\underset{1\times 1}{z_t} = \underset{1\times 1}{\alpha} \varepsilon_{1,t} + \sigma_v \underset{1\times 1}{v_t},$$

$$(\varepsilon_t', v_t)' \overset{i.i.d.}{\sim} N(0, I_{n_\varepsilon+1}).$$

- Bonus: Directly applies to SVAR-IV identification with $n_\varepsilon \geq n_y$.
- Interesting objects:

$$R_0^2 = 1 - \mathrm{Var}(\varepsilon_{1,t} \mid y_t), \quad FVD_{i,1} = \frac{\Theta_{i,1,0}^2}{\mathrm{Var}(y_{i,t})}.$$

Static model: identification up to scale

- Impulse responses identified up to scale:

$$\text{Cov}(y_{i,t}, z_t) = \alpha \Theta_{i,1,0}, \quad i = 1, \ldots, n_y.$$

Relative IRs $\Theta_{i,1,0}/\Theta_{1,1,0}$ point-identified. Stock & Watson (2018)

- Degree of invertibility identified up to scale:

$$R_0^2 = \frac{1}{\alpha^2} \text{Var}(E(z_t \mid y_t)).$$

- FVDs identified up to scale:

$$FVD_{i,1} = \frac{\frac{1}{\alpha^2} \text{Cov}(y_{i,t}, z_t)^2}{\text{Var}(y_{i,t})}, \quad i = 1, \ldots, n_y.$$

- What is identified set for α?

Static model: identified set for α

- Upper bound:
$$\alpha^2 \leq \text{Var}(z_t) \equiv \alpha_{UB}^2.$$
Binds when IV is perfectly informative.

- Lower bound:
$$\alpha_{LB}^2 \equiv \text{Var}(E(z_t \mid y_t)) = \alpha^2 \,\text{Var}(E(\varepsilon_{1,t} \mid y_t)) \leq \alpha^2 \,\text{Var}(\varepsilon_{1,t}) = \alpha^2.$$
Binds when macro var's y_t are perfectly informative about shock $\varepsilon_{1,t}$.

- Bounds are sharp: Given any var-cov matrix for $(y_t', z_t)'$, can find a consistent model with any $\alpha \in [\alpha_{LB}, \alpha_{UB}]$.

- Identified set $[\alpha_{LB}, \alpha_{UB}]$ for α is an interval. Implies identified sets for IRs, FVD, and degree of inv.

Static model: interpretation of identified sets

$$R_0^2 = \frac{1}{\alpha^2}\,\text{Var}(E(z_t \mid y_t)), \quad FVD_{i,1} = \frac{1}{\alpha^2} \times \frac{\text{Cov}(y_{i,t}, z_t)^2}{\text{Var}(y_{i,t})}$$

- Express identified set for $\frac{1}{\alpha^2}$ in terms of model parameters:

$$\Bigg[\underbrace{\frac{\alpha^2}{\alpha^2+\sigma_v^2}}_{\text{IV strength}} \times \frac{1}{\alpha^2}, \quad \underbrace{\frac{1}{R_0^2}}_{\text{recoverability}} \times \frac{1}{\alpha^2}\Bigg].$$

- Identified set for FVD or R_0^2 narrower when the IV is stronger or the shock is more recoverable/invertible. Only collapses to a point when IV is perfect *and* shock is invertible.

Static model: point identification

- Point identification under any of the following auxiliary assumptions:

  1. Shock $\varepsilon_{1,t}$ is invertible/recoverable (untestable). Then $\alpha = \alpha_{LB}$ and $\varepsilon_{1,t} = \frac{1}{\alpha} E(z_t \mid y_t)$.
  2. $n_\varepsilon = n_y$ (untestable). Implies invertibility of all shocks.
  3. IV z_t is perfect, i.e., $\sigma_v = 0$ (untestable). Then $\alpha = \alpha_{UB}$ and $\varepsilon_{1,t} = \frac{1}{\alpha} z_t$.

- But auxiliary assumptions are not necessary for *partial* ID with nontrivial, informative bounds.

◂ Back

Forecast variance decomposition

- Identif. set for FVR scales with identif. set for $\frac{1}{\alpha^2}$. What about FVD?

Proposition

Let there be given a spectral density for $(y_t', \tilde{z}_t)'$ (same as'ns as before).

Given knowledge of $\alpha \in (\alpha_{LB}, \alpha_{UB}]$, the largest possible value of $FVD_{i,\ell}$ is 1 (the trivial bound); the smallest possible value is

$$\frac{\sum_{m=0}^{\ell-1} \mathrm{Cov}(y_{i,t}, \tilde{z}_{t-m})^2}{\sum_{m=0}^{\ell-1} \mathrm{Cov}(y_{i,t}, \tilde{z}_{t-m})^2 + \alpha^2 \,\mathrm{Var}\left(\tilde{y}_{i,t+\ell}^{(\alpha)} \mid \{\tilde{y}_\tau^{(\alpha)}\}_{-\infty<\tau\le t}\right)}. \qquad (\Delta)$$

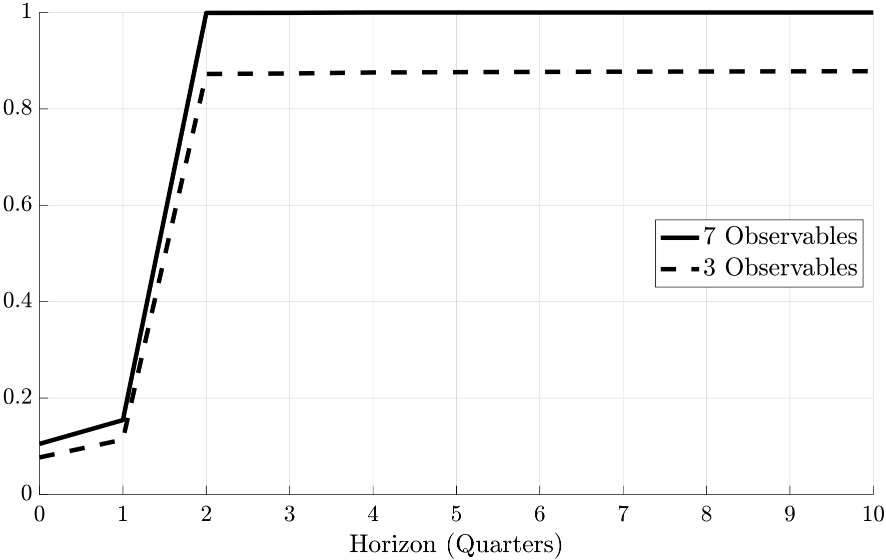
$\tilde{y}_t^{(\alpha)}$ denotes a stationary Gaussian time series with spectrum

$$s_{\tilde{y}^{(\alpha)}}(\omega) = s_y(\omega) - \tfrac{2\pi}{\alpha^2} s_{y\tilde{z}}(\omega) s_{y\tilde{z}}(\omega)^*, \quad \omega \in [0, 2\pi].$$

Expression (Δ) is monotonically decreasing in α, so the overall lower bound is attained at $\alpha = \alpha_{UB}$.

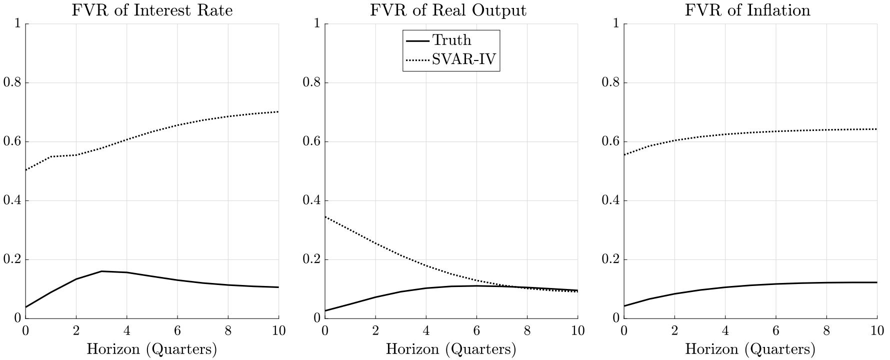
Back

Structural model: R^2_ℓ for forward guidance shock



Back

Structural model: SVAR-IV-estimated FVR of fwd. guid.



Back

Structural model: Degree of invertibility/recoverability

Observables	Monetary shock		Technology shock		Forw. guid. shock	
	R_0^2	R_∞^2	R_0^2	R_∞^2	R_0^2	R_∞^2
Baseline	0.8702	0.8763	0.1977	0.2166	0.0768	0.8807
$+ I + C$	0.9415	0.9507	0.2128	0.2384	0.0980	0.9492
$+ L$	0.9272	0.9286	0.9799	0.9816	0.0774	0.9331
All	1	1	1	1	0.1049	1

Baseline: Output, inflation, nom. interest rate.

◂ Back

Imbens-Manski-Stoye confidence intervals

- Let ϑ denote reduced-form VAR parameters. Estimator $\hat{\vartheta}$.
- Consider any identified set $[\underline{h}(\vartheta), \overline{h}(\vartheta)]$, with $\underline{h}(\cdot), \overline{h}(\cdot)$ ctsly diff.

$$\begin{pmatrix} \underline{h}(\hat{\vartheta}) \\ \overline{h}(\hat{\vartheta}) \end{pmatrix} \overset{approx}{\sim} N\left(\begin{pmatrix} \underline{h}(\vartheta) \\ \overline{h}(\vartheta) \end{pmatrix}, \begin{pmatrix} \hat{\underline{\sigma}}^2 & \hat{\rho}\hat{\underline{\sigma}}\hat{\overline{\sigma}} \\ \hat{\rho}\hat{\underline{\sigma}}\hat{\overline{\sigma}} & \hat{\overline{\sigma}}^2 \end{pmatrix} \right).$$

- $1-\beta$ conf. interval for the identified set: Imbens & Manski (2004)

$$\left[\underline{h}(\hat{\vartheta}) - \Phi^{-1}(1-\beta/2)\hat{\underline{\sigma}}\,,\ \overline{h}(\hat{\vartheta}) + \Phi^{-1}(1-\beta/2)\hat{\overline{\sigma}}\right].$$

- Could also construct CI for parameter. Stoye (2009)
- Caveat: Lower bound for α is given by supremum; not generally ctsly diff. We use conservative lower bound $\alpha^2 \geq \int_0^{2\pi} s_{\tilde{z}^\dagger}(\omega)\, d\omega$.

Back

Sieve VAR inference

- Non-parametric VAR(∞) model for $W_t \equiv (y_t', z_t)'$:

$$W_t = \sum_{\ell=1}^{\infty} A_\ell W_{t-\ell} + e_t.$$

  Stationary, non-singular, abs. summable coefficients.

- e_t i.i.d. with $E\|e_t\|^8 < \infty$.

- Parameter of interest:

$$\psi \equiv \int_0^{2\pi} h(\omega)' g(A_{\cos}(\omega), A_{\sin}(\omega), \Sigma)\, d\omega,$$

  where

$$A_{\cos}(\omega) \equiv \sum_{\ell=1}^{\infty} A_\ell \cos(\omega\ell), \quad A_{\sin}(\omega) \equiv \sum_{\ell=1}^{\infty} A_\ell \sin(\omega\ell).$$

- $h(\cdot)$ bounded. $g(\cdot)$ twice cts'ly diff. with first partial derivatives in $L_2(0, 2\pi)$ as fct of ω.

Sieve VAR inference (cont.)

- Least-squares VAR plug-in estimator:

$$\hat{\psi} \equiv \int_0^{2\pi} h(\omega)' g(\hat{A}_{\text{cos}}(\omega), \hat{A}_{\text{sin}}(\omega), \hat{\Sigma})\, d\omega.$$

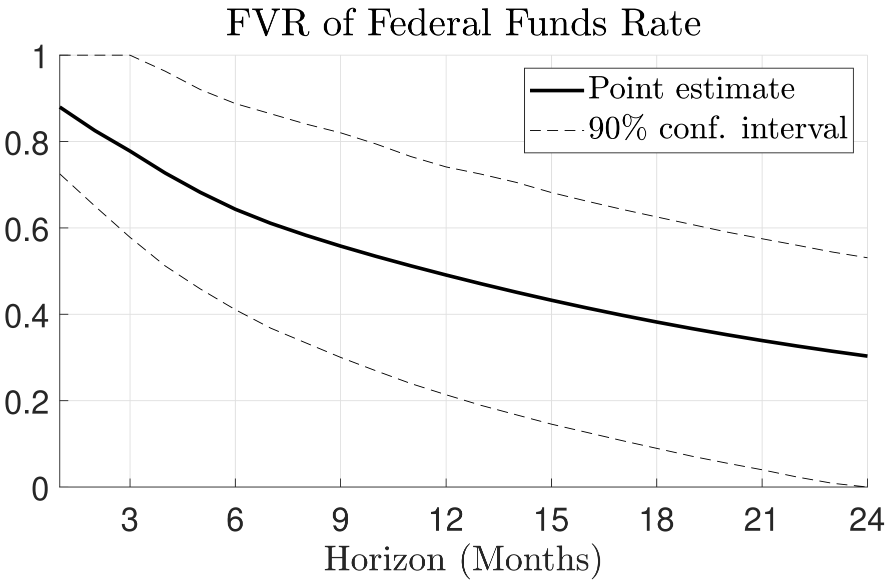
- VAR lag length $p_T \in \mathbb{N}$ satisfies

$$p_T^3/T \to 0, \quad T^{1/2} \textstyle\sum_{\ell=p_T+1}^{\infty} \|A_\ell\| \to 0.$$

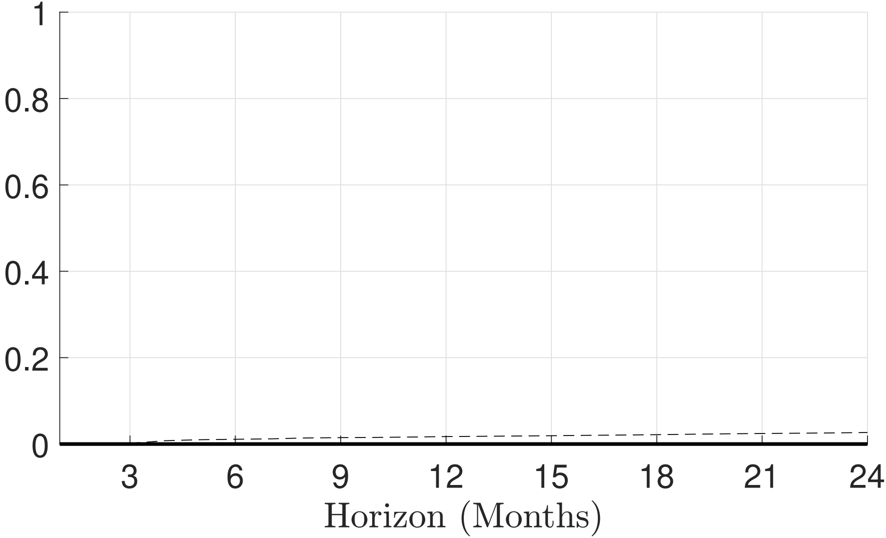
- Prove $\sqrt{T}$ asymptotic normality of $\hat{\psi}$. Berk (1974); Lewis & Reinsel (1985); Saikkonen & Lütkepohl (2000); Gonçalves & Kilian (2007)

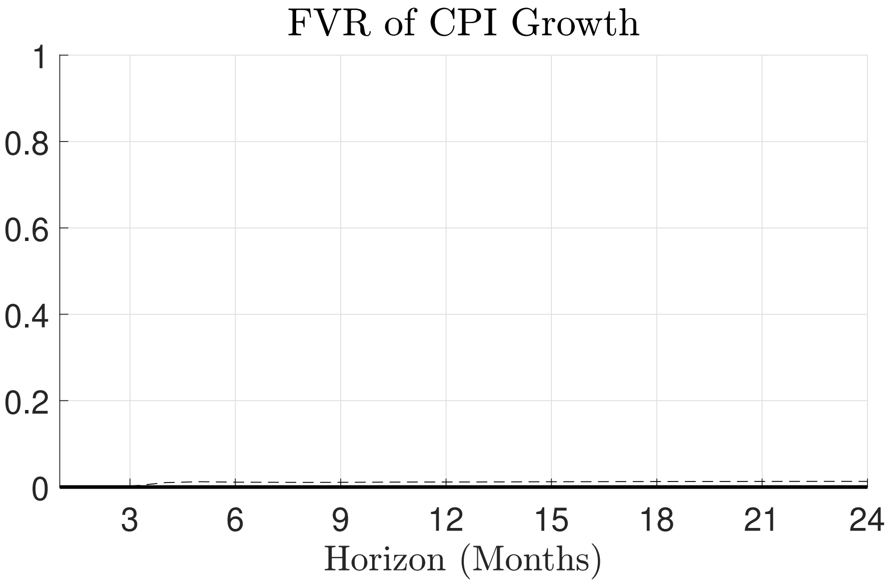
- Require asy. var. to be strictly positive, which rules out parameters on the boundary (as in SVAR-IV).

Back

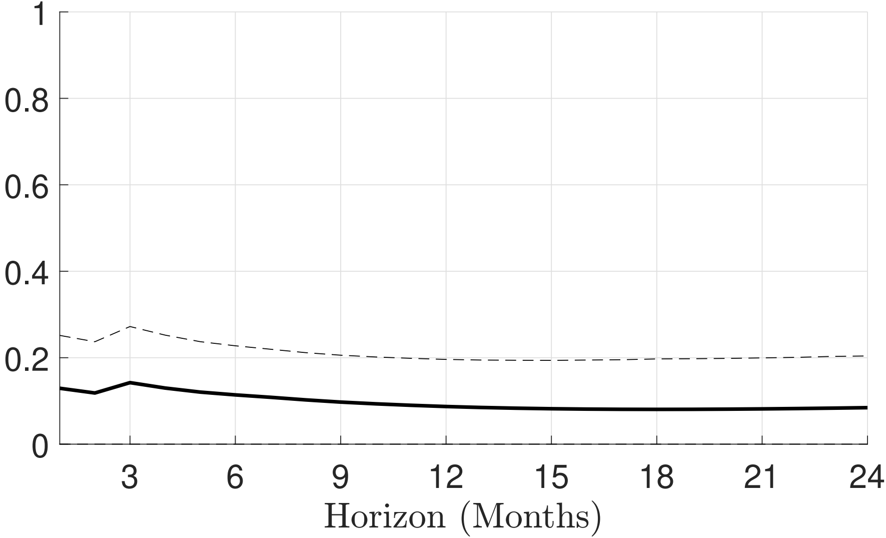

FVR of Federal Funds Rate
Point estimate
90% conf. interval
1
0.8
0.6
0.4
0.2
0
3
6
9
12
15
18
21
24
Horizon (Months)

FVR of Industrial Production Growth




FVR of CPI Growth
1
0.8
0.6
0.4
0.2
0
3
6
9
12
15
18
21
24
Horizon (Months)

FVR of Excess Bond Premium



Back