Dynamical Structure and Spectral Properties of Input-Output Networks

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Abstract

We develop a dynamic production network model and study its spectral properties. The model features adjustment costs of changing inputs and thus gradual recovery from temporary TFP shocks. First, we explicitly solve for the output and welfare effects of temporary shocks. We show, because of adjustment costs along the recovery path, temporary shocks to sectors that generate significant sales through distant linkages to the consumer are most damaging. Second, we eigendecompose the input-output matrix and show, because higher-order linkages take longer to recover, fewer eigenvectors are needed to represent the welfare impact of sectoral shocks in the dynamic economy compared to the Domar weights. Third, we analyze the U.S. input-output structure and show the welfare impact of temporary shocks has a low-dimensional, 4-factor structure (out of 171 eigenvectors).

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1 Introduction

The study of production networks recently became an active research agenda in macroeconomics, reviving and developing the classic analysis of Leontief. The most common setting is that of the static general equilibrium environment subject to sectoral shocks.

We extend the static model of Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) by introducing a cost of change of inputs that depends on the speed of adjustment and showing that such model is first-order equivalent to a continuous-time version of the time-to-build friction as in Long and Plosser (1983). A dynamical system thus arises from a dynamic production network model in which there are costs for changing inputs of production. Following temporary productivity changes, input adjustment costs lead to a gradual recovery of the economy back to the steady state. While the steady state of the economy coincides with the static model—thus sectoral size captures the impact of permanent shocks Hulten (1978)—the transition path of the dynamic transmission of shocks across sectors is, in general, different from the standard setting, and even temporary shocks may have lasting impact. Our primary goal is to characterize the dynamic path of the propagation of shocks through the input-output linkages and the determinants of the output trajectory and welfare. We show, because of dynamic adjustments, sectors that are important for permanent shocks are fundamentally different from sectors that are important for temporary shocks.

Input-output adjustment cost implies slow-moving input-output quantities; the model therefore features an entire matrix of endogenous state variables, one for each input-output pair of sectors. Nevertheless, our tractable formulation enables us to derive explicit, closed-form solution for the entire time path of equilibrium allocations. We first characterize the transition path of sectoral output and aggregate consumption. To isolate the dynamics that arise from adjustment costs of the endogenous state variables, we consider an economy that have experienced temporary negative sectoral TFP shocks, which disrupted input-output linkages, and we study the recovery dynamics of these linkages after TFP recovers. Due to adjustment costs, the use of intermediate goods can only recover gradually over time. The gradual recovery of inputs then translates into the gradual recovery of the output in sectors that use those inputs. As the transition takes time, the speed of which is determined by the magnitude of adjustment costs, the path of adjustment has non-trivial welfare implications for a consumer who discounts the future.

We sharply characterize the impact of sectoral TFP shocks on output and consumer welfare along the recovery path. When shocks are permanent, sectoral size—also known as the “Domar weight”—captures the output elasticity to TFP shocks; shocks matter more when they affect large sectors, and after controlling for size, network structure no longer
matters, just as in a static model. By contrast, when shocks are temporary, welfare impact of sectoral shocks depends crucially on the network structure. Specifically, the welfare impact of temporary shocks is the affected sector’s Domar weight multiplied by its Katz centrality, which we show to be a natural notion of upstreamness: a sector is more Katz-central if a disproportionate fraction of its output goes through long supply chains before reaching the final consumer. Intuitively, shocks are more welfare consequential if they generate large and long-lasting impact on consumption; shocks to high Domar weight sectors generate large immediate impact, and shocks to upstream sectors are long-lasting due to adjustment costs.

The measure of welfare impact of temporary shocks also reveals that the magnitude of adjustment cost provides a multi-scale representation of the economy. In the static model, the Domar weight can be written as the product between the vector of consumption weights and the power series of the input-output matrix, and each subsequent term in the power series captures a higher round of network effect through which sectoral shocks affect output. The welfare impact of temporary shocks in our dynamic setting has a similar power series representation, but it disproportionately down-weights lower powers of the input-output matrix while up-weighting higher powers. The re-weighting reflects the fact that temporary shocks affect welfare only if they have lasting impact on output, and higher-order linkages—represented by higher powers of the input-output matrix—take longer to recover and therefore translate into greater welfare impact. The reweighting therefore alters the relative importance of temporary shocks’ local versus global effects, and it depends on the product between the speed of adjustment and the consumer’s discount rate. By varying this product, the input-output economy is represented at different scales or the levels of coarseness spotlighting the relative importance of the higher-order links and thus the importance of the global versus local structures.

We show that the determinants of the flow of output (that is, the solution to the dynamical system of differential equations) is closely related to the properties of the static input-output matrix. The analysis of welfare impact of productivity shocks also adds new considerations in analyzing the temporal structure of the flow. The dynamic approach that we use parallels the modern mathematical literature on the analysis of networks used in spectral graph theory (Chung (1997), Grigor’yan (2018), Spielman (2019)) that takes an inherently dynamical view. Our main methodological aim is to bring these techniques of dynamical systems to the analysis of production networks.

Our exact closed-form solution hinges on the log-linear nature of the Cobb-Douglas production network and our adjustment cost formulation. We show that under a general, constant-returns-to-scale formulation, our characterization continues to hold as a first-order approximation around the steady-state.
Summarizing our first set of theoretical results of the paper: we develop a tractable formulation of dynamic adjustments in production networks. Despite the model featuring a large number of state variables, we derive explicit, closed-form solution for the entire sequence of equilibrium allocations as well as the welfare impact effects of temporary shocks. We show shocks to sectors that generate significant sales through distant linkages to the consumer are disproportionately damaging to the economy.

Our second set of theoretical results is the characterization of the main driving forces of the welfare impact of temporary shocks. Since we have shown the importance of the higher-order production links, this naturally leads us to analyze the spectral or eigendecomposition of the input-output matrix—its eigenvectors and eigenvalues. The main reason for this is that once the matrix is diagonalized, the powers of it, which represent the higher order production links, take a particularly simple form.

The concise decomposition of the welfare effects of shocks as a combination of the eigenvectors and the power series of the eigenvalues is our second main theoretical result. The eigendecomposition provides a basis for sectoral shocks such that, along each dimension of the basis, the recovery path of the economy features a constant rate of convergence towards the steady-state. Specifically, consider a temporary shock vector that is itself an eigenvector of the input-output matrix. The impact of the shock along the entire transition path becomes a continuously decayed version of the initial shock, with rate of decay governed by the eigenvalue. An important corollary of this logic shows a marked contrast with the eigendecomposition of the Domar weights, and thus with the static economy. Because dynamic adjustment costs significantly down-weight the direct and initial rounds of network effects—as these effects recover quickly and are not long-lasting—our model effectively up-weights the higher powers in the power series of eigenvalues, thereby up-weighting the relative importance for the shock profiles with greater eigenvalues and down-weighting the shock profiles with the lower eigenvalues. This implies that significantly fewer eigencomponents are needed to represent the welfare impact of the shocks in a dynamic economy. In other words, the dynamic economy may have a factor structure where the small set of factors can capture the importance of temporary shocks. In contrast, the Domar weights may be significantly higher dimensional: because the Domar weights do not discount the direct and initial rounds of network effects, even eigenvectors with small eigenvalues may have a sizable contribution in explaining TFP shocks in the static model. In summary, sectoral shocks may not have a low-dimensional representation in the static model but may have one in our dynamic model. The concise representation of complex high dimensional systems via a few reduced coordinates is also the primary goal of the well developed literature on nonlinear dimensionality reduction using spectral methods (e.g., Coifman, Kevrekidis, Lafon, Maggioni, and Nadler
We conduct an empirical analysis the U.S. input-output network using our model. Because our adjustment cost formulation is first-order equivalent to the time-to-build friction in Long and Plosser (1983), we calibrate these costs using industry level backlog ratios, a measure of delays between order and delivery (Zarnowitz (1962), Meier (2020)), from the U.S. Census M3 survey of manufacturers. We compute the welfare impact of temporary shocks, and we show, as a notion of sectoral importance, our measure differs significantly from Domar weights. While Domar weights pick up large sectors such as hospitals, restaurants, and retail, our measure picks up large and upstream sectors such as motor vehicle parts manufacturing, basic chemical manufacturing, and sectors that provides services to commercial activities (agencies, brokerages, and insurance). We also show that, given the rich structure of the U.S. input-output network, sectoral shocks have significantly heterogeneous dynamic properties. The GDP recovers quickly from shocks to finance, oil and gas extraction, and petroleum and coal products manufacturing—three large sectors that are traditionally viewed as important because of their high Domar weights—with half-lives averaging to 4.6 months; by contrast, the average half-life is more than twice as long for GDP recovery from shocks to the manufacturing of communication equipments, motor vehicles, and motor vehicle parts—three heavy-manufacturing sectors that are relatively upstream.

Finally, we conduct a spectral analysis of the U.S. production network. We first show that 95 percent of the welfare effect of temporary sectoral shocks can be represented by the low-dimensional projection onto four eigenvectors (out of 171). That is, the U.S. input-output network has a very low-dimensional (4-factor) structure. In contrast, for the Domar weight almost all of the 171 eigenvectors are important, and, therefore, the Domar weight is a high-dimensional object. We identify the groups of sectors that form the four key eigenvectors. The first eigenvector represents shocks to the heavy manufacturing sectors. The second eigenvector additionally represents sectors relating to agencies, brokerages, and insurance and sectors covering the manufacturing of consumer goods. The third eigenvector picks up sectors representing chemicals. The fourth eigenvector represents entertainment, including radio and television broadcasting. Summarizing, we find that the welfare impact of any negative temporary shocks can be represented by only four (out of 171) eigenvectors.

**Literature**

There is a modern revival of the literature on production networks (see, e.g., reviews in macroeconomics of Carvalho (2014), Carvalho and Tahbaz-Salehi (2019), and Grassi and Sauvagnat (2019)). Carvalho (2010), Gabaix (2011) and Acemoglu, Carvalho, Ozdaglar,
and Tahbaz-Salehi (2012) show idiosyncratic sectoral productivity shocks may have aggregate impact. Jones (2011, 2013) develops a model of production networks with distortions. A number of recent papers develop various important aspects of the macroeconomic implications of the input-output and production structure of the economy: for example, Grassi (2017), Baqaei (2018), Lim (2018), Oberfield (2018), Liu (2019), Baqaei and Farhi (2019, 2020), Bigio and La’O (2020), Golub, Elliot, and Leduc (2020), vom Lehn and Winberry (Forthcoming). Our main contribution compared to these papers is to study the dynamics the economy with the adjustment costs through the lens of the spectral graph theory and dynamical system theory.¹

Methodologically, our paper is closest to Galeotti, Golub, and Goyal (2020) and Galeotti, Golub, Goyal, and Rao (2021). These papers use the spectral approach and focus on the importance of the local versus the global structure of the network games. As in our paper, the higher-order eigenvalues and eigenvectors beyond the first one are important (see also Golub and Sadler (2016) for a survey of work in which the second eigenvalue is important).²

A dynamical structure similar to our work appears in the context of the foundations of the gravity equation of Chaney (2018). In the dynamic network of the importers and exporters, the evolution of the contacts propagates from the local to the more distant neighbors governed by a differential equation. Chaney (2018) uses the Fourier theory to study this evolution while we use the spectral methods.

Kikuchi, Nishimura, Stachurski, and Zhang (2021) study a static production network using dynamic methods. They provide a general methodology of using dynamic programming by reinterpreting time as an index over decision making entities.

In a model of endogenous network formation of Taschereau-Dumouchel (2020), the planner optimally chooses to cluster firms. This creates a structure that is locally different from the models with the fixed networks. In particular, these densely built communities slow the propagation of the shocks. While Taschereau-Dumouchel (2020) studies these properties quantitatively, the spectral theory that we develop, albeit for a fixed network, allows to theoretically describe the properties of such regions. Similarly, in the asset-pricing applications such as Herskovic, Kelly, Lustig, and Van Nieuwerburgh (2020) understanding the determinants of the clusters that create comovement of the firms returns and volatilities may be important.

Our theory derives Katz centrality as a precise notion of upstreamness that relates to the upstreamness measure of Antràs, Chor, Fally, and Hillberry (2012) and the distortion

¹In a different setting, Steinerberger and Tsyvinski (2019) associate a dynamical system with the static model of optimal taxation.

centrality measure of Liu (2019), as all three measures are derived from placing an increasing sequence of weights to higher-order terms in the power series of the input-output matrix.

We now briefly discuss the relationship with the mathematical literature. The recent theoretical advances in the analysis of networks and graphs take an inherently dynamical perspective. Spectral graph theory (Chung (1997), Grigor’yan (2018), Spielman (2019)) associates a dynamical system, typically a diffusion operator, with the static network and studies the eigenvectors and eigenvalues of the Laplacian. The Leontief-inverse matrix in the input-output analysis is the inverse of the Laplacian of the network and thus admits a parallel analysis. Similarly, a classical way to analyze a non-negative matrix (Berman and Plemmons (1994)), which is the input-output network, is by associating with it a continuous time dynamical system, or a flow, represented by a system of differential equations.\(^3\) This system has an explicit solution in terms of matrix exponentials and hence the properties of solutions are intimately related to the properties of the matrix (Colonius and Kliemann (2014)). In particular, the eigenvalues and eigenvectors of the input-output matrix determine the evolution of the dynamical system and thus can be used for the analysis of shock propagations.

2 Model

In this section we generalize the static production network model of Acemoglu et al. (2012) by introducing dynamic adjustment costs in input-output linkages.

There is a representative consumer with exogenous labor supply \(\bar{\ell}\) and \(N\) production sectors that produce from labor and intermediate inputs.\(^4\) The consumer has utility

\[
V \equiv \int_0^{\infty} e^{-\rho t} \ln c(t) \, dt
\]

where \(c(t)\) is a Cobb-Douglas aggregator over sectoral goods \(j = 1, \ldots, N\):

\[
c(t) = \chi_c \prod_{j=1}^{N} (c_j(t))^{\beta_j}, \quad \sum_{j=1}^{N} \beta_j = 1,
\]

where \(\chi_c \equiv \prod_{j=1}^{N} \beta_j^{-\beta_j}\) is a normalizing constant. We refer to \(c(t)\) as aggregate consumption and GDP interchangeably.

\(^3\)The behavior of the Markov chain on the input-output network is a closely related dynamical system (Kemeny and Snell (1960)).

\(^4\)One can alternatively interpret \(\bar{\ell}\) as the total endowment of time, which the consumer endogenously allocates between leisure and labor supply, as in Long and Plosser (1983). By interpreting leisure as one of the consumption goods directly produced from time, the model can be interpreted as one with endogenous labor supply.
At each time $t$, the output of production sector $i$ satisfies
\begin{equation}
q_i(t) = \chi_i z_i(t)(\ell_i(t))^{\alpha_i} \prod_{j=1}^{N} (m_{ij}(t))^{\sigma_{ij}}, \quad \sum_{j=1}^{N} \sigma_{ij} + \alpha_i = 1,
\end{equation}
where $0 \leq \alpha_i, \sigma_{ij} \leq 1$, $\chi_i \equiv \alpha_i^{-\alpha_i} \prod_{j=1}^{N} \sigma_{ij}^{-\sigma_{ij}}$ is a normalizing constant, $z_i(t)$ is sectoral total factor productivity, $\ell_i(t)$ is the amount of labor used, and $m_{ij}(t)$ is the amount of the intermediate good of the sector $j$ used in the production of the good $i$.

From now on, wherever it does not cause confusion, we suppress dependence on time $t$ in the notation.

To use input quantity $m_{ij}$ at time $t$, sector $i$ needs to buy
\begin{equation}
s_{ij} \equiv m_{ij} \times \exp (\delta \dot{m}_{ij}/m_{ij} \times 1 (\dot{m}_{ij} > 0))
\end{equation}
units of input $j$. The term $\dot{m}_{ij} \equiv d m_{ij}(t)/dt$ is the rate of change in the quantity of intermediate input $j$ used by sector $i$. The term $\exp (\delta \dot{m}_{ij}/m_{ij} \times 1 (\dot{m}_{ij} > 0))$ captures sluggish upward adjustment of inputs and is an iceberg cost that producer $i$ has to incur when it raises the quantity of input $j$. The cost does not arise when inputs do not expand: when $\dot{m}_{ij} = 0$, equation (4) implies that quantity of inputs purchased is equal to the quantity of inputs used for production ($s_{ij} = m_{ij}$). The parameter $\delta$ captures the ease of adjustment when input expands; as $\delta \to 0$, adjustment costs vanish. That is, dynamics are introduced through the transportation of intermediate inputs across producers. Our formulation captures the notion that, following temporary negative shocks, the recovery of input-output linkages must be gradual, and temporary shocks therefore may have lasting impact on the economy.

The goods and labor market clearing conditions are
\begin{equation}
q_j = c_j + \sum_{i=1}^{N} s_{ij} \text{ for all } j, \quad \bar{\ell} = \sum_{i=1}^{N} \ell_i.
\end{equation}
For simplicity, we assume goods delivered to the consumer are not subject to adjustment costs, and neither is the use of labor across production sectors.\footnote{These choices are made for expositional simplicity and are without loss of generality. We can always accommodate adjustment costs in the purchase of labor or the consumption good by creating a fictitious production sector that buys the consumption bundle and sells to the consumer or buys labor and sells to other producers.}

**Discussion of the Adjustment Cost Formulation** The specific form of adjustment costs in (4) warrants some discussion. First, in our baseline formulation, input usage is slow to expand but may shrink instantaneously. This is for expositional simplicity; the formulation enables us to focus on recovery path of the economy after temporary TFP shocks, so that we can isolate the dynamics that arise due to adjustment costs and abstract away from the direct TFP dynamics, which are already well-understood from existing production network
models. In section 3.8, we extend the analysis to the more general case with both upward and downward adjustment costs.

Second, adjustment costs in (4) take an exponential form. This is for analytic tractability. It is well-known that static log-linear (i.e., Cobb-Douglas) production network models have closed-form solutions; as we show below, the exponential adjustment cost formulation implies a log-linear law of motion for the state variables (intermediate input quantities used, $m_{ij}$), so that there exists a closed-form solution for the entire dynamic path of all endogenous variables in the economy. In Section 3.5, we show our exponential adjustment cost formulation is first-order equivalent to a time-to-build formulation that is the continuous-time analogue of Long and Plosser (1983). In Section 3.6 we extend our main result to buyer-seller-specific adjustment costs. In Section 3.7 we extend our main result to a significantly more general environment environment where both the production networks and adjustment costs are non-parametric, and we show our main result holds as a first-order approximation around the steady-state.

**Equilibrium and Steady State**

We study a competitive equilibrium in which all producers are forward-looking, and they take the entire dynamic path of prices and adjustment costs as given while choosing the path of production inputs in order to maximize present discounted value of future profits. Specifically, let $z(t) \equiv \{z_j(t)\}_{j=1}^N$ denote the set of time-$t$ productivities, let $\Xi(t) \equiv \{s_{ij}(t), m_{ij}(t), c_j(t), q_j(t), \ell_j(t), c(t)\}_{i,j=1}^N$ be the input-output quantity allocation at time $t$, and let $P(t) \equiv \{p_j(t), w(t), r(t)\}_{j=1}^N$ be the set of sectoral prices, wage rate, and interest rate at time $t$, where $p_j(t)$ is the revenue that producer $j$ gets per unit of good $j$ sold. We normalize the consumer price index to one: $1 = \prod_{j=1}^N p_j(t)^{\beta_j}$ for all $t$.

**Definition. (Equilibrium)** Given the initial allocation of intermediate inputs $\{m_{ij}(0)\}_{i,j=1}^N$ and the sequence of sectoral productivity $z(\cdot)$, an equilibrium is the dynamic path of allocation and prices $\Xi(t)$, $P(t)$ such that labor and input markets clear according to (5), and all producers choose the time path of production inputs in order to maximize present discounted value of future profits while taking as given the entire path of prices, wages, and the interest rates $P(t)$, as well as the law of motion for intermediate inputs as in (4).

**Definition. (Steady-State)** A steady-state equilibrium is one in which $z(t)$, $\Xi(t)$, $P(t)$ are all time invariant.

In what follows, we use boldface to denote vectors (lower case) and matrices (upper case). Let $\Sigma \equiv [\sigma_{ij}]$ denote the matrix of input-output elasticities, and let $\beta$ denote the $N \times 1$ vector of consumption weights. Let $\alpha$ be the vector of sectoral value-added shares. Let
\( \gamma' \equiv \beta' (I - \Sigma)^{-1} \) be the vector of Domar weights, i.e., sectoral sales relative to GDP in a decentralized equilibrium.

**Our Steady-State is the Equilibrium of Acemoglu et al. (2012)** When \( \delta = 0 \), the economy does not feature adjustment costs, and the model becomes a repeated version of the static economy in Acemoglu et al. (2012).

When \( \delta > 0 \), the economy features adjustment costs. However, allocations and prices in the steady state of this dynamic model coincide with those the static equilibrium in Acemoglu et al. (2012). The Hulten’s theorem holds across steady-states: the Domar weight \( \gamma_i \) of sector \( i \) characterizes the *steady-state importance* of each sector’s TFP. Specifically, let \( c^{ss} \) denote the steady-state consumption per period; then

\[
\ln c^{ss} = \text{const} + \gamma' \ln z.
\]

In other words, Domar weights capture the welfare impact of permanent TFP shocks. We now turn to the analysis of temporary shocks.

**3 Recovery Dynamics After Negative TFP Shocks**

Consider a production network affected by temporary negative TFP shocks to some sectors. These shocks reduce sectoral production and may propagate through input-output linkages and affect output in other sectors. After these negative shocks recede, how quickly does the economy recover? We show, when production linkages take time to recover, the topology of a production network is a key determinant of its resilience to negative shocks.

**Figure 1.** Two stylized example networks

(a) A horizontal production economy  
(b) A vertical production chain

To illustrate the intuition, consider two example networks. In Figure 1, panel (a) shows the network structure of a horizontal production economy. Here, labor is the only factor of production in each sector \( i \in \{1, \ldots, N\} \), and sectors \( i \) does not use any of the goods \( j \in \{1, \ldots, N\} \) in production. All of the goods are part of the consumption bundle \( c \). In this
setting without input-output linkages, adjustment costs are irrelevant, and traditional static analysis holds: negative TFP shocks to larger sectors—those with higher Domar weights—have greater effects on the GDP.

Now consider panel (b), which shows the network structure of a vertical production chain. Here, labor is the only factor of production of sector 1, and each subsequent sector \( i \) uses inputs only of the sector \( i - 1 \). Only good \( N \) is used in final consumption \( c \). Consider a temporary decline in sector 1’s productivity \( z_1 \). Sector 1’s output declines for the duration of the negative shock; moreover, because sector 2 requires good 1 as inputs, the output of sector 2 declines as well, and in fact output declines in all sectors \( i \in \{1, \ldots, N\} \). After the initial TFP shock dissipates and as \( z_1 \) reverts, output in sector 1 recovers immediately. However, because of adjustment costs in the recovery of input-output linkages, sectoral output for all \( i \geq 2 \) may stay extendedly depressed, and the economy as a whole—measured by the consumption aggregator \( c(t) \), i.e., the GDP—may take a long time to recover. By contrast, a temporary reduction in sector \( N \)’s TFP \( z_N \) has no lasting impact on the economy, which recovers immediately after the shock dissipates. More generally, in the vertical network of panel (b), the economy recovers more slowly from negative shocks that affect relatively upstream sectors. This is despite the fact that all sectors along the supply chain may have identical Domar weights.

We now formalize the analysis and analyze sectoral susceptibility in a production network from our dynamic perspective.

### 3.1 Negative TFP Shocks and Transitional Dynamics

We analyze an economy initially in a steady-state with sectoral log-productivities \( \{\ln z_i\}_{i=1}^N \), and we consider temporary, negative TFP shocks that reduce sectoral productivities at time zero to \( \{\ln z_i - \tilde{z}_i\}_{i=1}^N \). We use \( \tilde{z}_i > 0 \) to denote the absolute value in logs of the negative shocks for sector \( i \). For expositional simplicity, we assume sectoral TFP reverts back *instantaneously* to the pre-shock steady-state levels \( \{z_i\}_{i=1}^N \). We use \( t = 0^- \) and \( t = 0^+ \) to respectively index the time at and after the negative TFP shocks. When the negative shocks are present at \( t = 0^- \), log-GDP declines by \( \gamma' \tilde{z} \) relative to its steady-state level, where \( \gamma' \) is the Domar weights, consistent with Hulten (1978) and Acemoglu et al. (2012). We now analyze the dynamic path of sectoral output and GDP during the recovery, from \( t = 0 \) onwards.\(^6\)

\(^6\)The assumption, that inputs collapse immediately following negative shocks, is an exposition device; it enables us to focus on the recovery dynamics of the economy after TFP reverts back to steady-state level. In section 3.8 we extend the model to incorporate both upward and downward adjustment costs.
gradually over time and cannot jump instantaneously. Hence, sectoral output increases exactly in proportion to the TFP recovery at $t = 0$, and the total output in sector $j$ exceeds the total quantity of good $j$ used as production inputs. The excess output is dispensed as the adjustment costs required to expand input $j$ for the future. With passage of time, sectors continue to expand the use of inputs, sectoral output continues to expand even though TFP is constant. Eventually the economy converges back to the initial steady-state as $t \to \infty$.

Given that the quantity of intermediate inputs $\{m_{ij}\}$ cannot jump upwards, the entire matrix of input-output quantities are the state variables of the economy. Despite the large $N \times N$ state space, and the potentially complex forward-looking decisions of firms, we can in fact solve the entire equilibrium transitional path in closed-form. To do so, take note that despite the presence of adjustment frictions, the economy is efficient, and the competitive equilibrium coincides with the planner’s solution in choosing allocations to maximize consumer welfare. Hence, we directly solve for the path of sectoral allocations under the planner solution.

The planner’s problem is to choose the path of production (labor and intermediate) input allocations in order to maximize consumer welfare, taking time-0 allocations of intermediate inputs as given:

(Planner’s problem) \[ V (\{m_{ij}(0)\}) = \max_{\{\ell_i(t), s_{ij}(t)\}} \int e^{-\rho t} \sum_j \beta_j \ln c_j \, dt \] (6)

s.t. \[ c_j = z_j \ell_j (t)^{\alpha_j} \prod_k m_{jk} (t)^{\sigma_{jk}} - \sum_i s_{ij} (t), \] (7)

\[ \dot{m}_{ij} / m_{ij} = \delta^{-1} (\ln s_{ij} - \ln m_{ij}), \] (8)

and the labor market clearing condition (5). Equation (7) is derived from the market clearing condition (5) for good $j$, and equation (8) reflects the law of motion for an expanding path of intermediate inputs under adjustment costs. The time-0 allocations of intermediate inputs $\{m_{ij}(0)\}$ form the initial conditions of the planner’s problem and depend on the magnitude of the negative TFP shocks. Also note that the indicator variable $1 (m_{ij} \geq 0)$ in equation (4) is absent in the law of motion (8); this is because along the recovery path, inputs always weakly increase over time in all sectors, thus the case where inputs purchased $s_{ij}$ is less than the quantity $m_{ij}$ currently in use—resulting in a downward jump of the state variable $m_{ij}$—is irrelevant for the planning problem. This also shows that our baseline formulation with without downward adjustment costs is just an expositional device: the temporary TFP shocks at time $t = 0^-$ pin down the initial conditions $\{m_{ij}(0)\}$, and we study the dynamic path of the economy as it converges to the steady-state given these initial conditions.

**Lemma 1.** For any initial condition $\{m_{ij}(0)\}$, the solution to the planner’s problem features a constant fraction of each good $j$ being sent to the consumer and to each input user $i$ along
the entire transitional path: \( c_j(t) / q_j(t) \) and \( s_{ij}(t) / q_j(t) \) are time invariant for all \( i, j \). The solution also features a constant labor allocation along the entire transition path, as \( \ell_j(t) / \bar{\ell} \) is also time invariant for all \( j \). These fractions are independent of the initial state variables at time 0.

Proof. See Appendix A.1.

It is well known that under Cobb-Douglas preferences and production functions, the fraction of each good \( j \) sent to each input user \( i \) a static model is invariant to sectoral TFP shocks. Lemma 1 shows that under a log-linear law of motion (8) for the state variables, which is derived from our formulation of exponential adjustment costs (4), the fraction of each good sold to each buyer is invariant along the entire transition path is independent of the initial state variables. This is the key property that leads to tractability of our subsequent analysis despite the large state space of our model.

We now use Lemma 1 to provide a closed-form solution to the transition dynamics. Define

\[
x_j(t) \equiv \ln \sum_{i=1}^{N} s_{ij}(t) - \ln \sum_{i=1}^{N} m_{ij}(t).
\]

(9)

\( x_j(t) \) is the log-ratio between the quantity of good \( j \) supplied to and used by other producers. In a steady-state, the ratio \( \sum_i s_{ij} / \sum_i m_{ij} \) is equal to one, and \( x_j = 0 \) for all \( j \). Away from a steady-state, the ratio captures the proportional adjustment costs incurred for expanding input \( j \) in production. For any input \( j \), a temporary shock always generates a common initial proportional decline in \( m_{ij} \) across input users \( i \). Lemma 1 and the law of motion (8) further imply that both \( m_{ij}(t) \) and \( s_{ij}(t) \) grow at rates independent of \( i \) during the transition, giving us the following result: \( \delta^{-1} x_j(t) \) is the rate at which all sectors expand their use of input \( j \).

**Lemma 2.** Consider a temporary TFP shock that arrives at time 0 and recovers immediately. In the planner’s solution, along the entire transition path \( t \) and for every input \( j \), \( \frac{d \ln m_{ij}(t)}{dt} = \delta^{-1} x_j(t) \) for all \( i \).

Proof. See Appendix A.2.

We are now ready to derive the law of motion for sectoral output and GDP.

**Proposition 1. Laws of Motion for Sectoral Output and GDP.** Consider a TFP shock vector \( \tilde{z} \) that affects the steady-state economy at time zero and reverts back instantaneously.

1. The law of motion for sectoral output vector \( q \) is

\[
\frac{d \ln q}{dt} = \delta^{-1} \Sigma x(t), \quad \text{with initial condition } \ln q(0) = \ln q^{ss} - \Sigma (I - \Sigma)^{-1} \tilde{z},
\]

(10)

where \( I \) is the identity matrix and \( \Sigma \equiv [\sigma_{ij}] \) is the matrix of input-output coefficients.
2. The law of motion for GDP is
\[
\frac{d \ln c(t)}{dt} = \delta^{-1} \beta' \Sigma x(t), \quad \text{with initial condition } \ln c(0) = \ln c^{ss} - \beta' \Sigma (I - \Sigma)^{-1} \tilde{z}. \tag{11}
\]

3. The path of the log-ratio between quantity supplied and used of each input satisfies the ODE system
\[
\frac{d x(t)}{dt} = -\delta^{-1} (I - \Sigma) x(t), \quad \text{with the initial condition } x(0) = \tilde{z}.
\]

To understand this Proposition, first suppose the negative TFP shocks were permanent. Output declines in sectors directly affected by the shocks. Moreover, because of production linkages, output also declines in sectors that purchase inputs—directly or indirectly—from the shocked sectors. The total impact of negative shocks on sectoral output is captured by 
\[-(I - \Sigma)^{-1} \tilde{z},\]
where the Leontief inverse \((I - \Sigma)^{-1} \equiv I + \Sigma + \Sigma^2 + \cdots\) captures the infinite rounds of higher order effects through input-output linkages. This is indeed the finding of Acemoglu, Akcigit, and Kerr (2015) in the standard, static production network model.

Our Proposition 1 instead pertains to the persistent impact of temporary shocks. We now discuss these laws of motion’s intuitions, which help illustrate the model forces. As sectoral TFP recovers instantaneously, log-output directly recovers by \(\tilde{z}\); hence, at time \(t = 0\), sectoral output satisfies
\[
\ln q(0) = \ln q^{ss} - (I - \Sigma)^{-1} \tilde{z} + \tilde{z}.
\]

The input-output linkages destroyed by the negative shocks take time to recover. Because \(\delta^{-1} x_j(t) = \dot{m}_{ij}/m_{ij}\) captures the rate at which all producers expand their use of input \(j\), the output of sector \(i\) grows at rate
\[
\dot{q}_i/q_i = \delta^{-1} \sum_{j=1}^{N} \sigma_{ij} x_j(t), \tag{12}
\]
which is equation (10) in scalar form. Note that after time \(t = 0\), the expansion in output is entirely due to the recovery of input-output linkages and not because of TFP changes; the impact of input \(j\)’s recovery on \(i\)’s output is captured by \(\sigma_{ij}\), the elasticity of \(i\)’s output with respect to input \(j\). The law of motion (11) for GDP follows from the fact that a constant fraction of each good is sent to the consumer (Lemma 1); hence, log-GDP’s deviation from the steady-state is the consumption share \(\beta\)-weighted log-deviation in sectoral output.

Finally, to derive the law of motion for \(x_j(t)\), the log-ratio between quantity supplied and quantity used of each input \(j\), note that by Lemma 1, a constant share of \(q_j(t)\) is sent
to each sector as inputs $s_{ij}(t)$; hence, equation (9) implies
\[
\frac{dx_j(t)}{dt} = \frac{d\ln q_j(t)}{dt} - \frac{d\ln (\sum_i m_{ij}(t))}{dt}.
\]
The first term captures the rate at which sectoral output expands; the second term is the rate at which the quantity of good $j$ as intermediate inputs expands. By (8), the second term is equal to $\delta^{-1}x_j(t)$; hence, in the vector form,
\[
\frac{dx}{dt} = \delta^{-1}\Sigma x - \delta^{-1}x.
\]
The ODE system for $x(t)$ has an explicit solution in terms of the matrix exponential:
\[
x(t) = e^{-\delta^{-1}(I-\Sigma)t}\tilde{z},
\]
where matrix exponential for any generic matrix $M$ is defined as $e^M \equiv \sum_{k=0}^{\infty} \frac{M^k}{k!}$.

Intuitively, immediately after TFP recovers at time 0, since all inputs are constant, the log-ratio between quantity supplied and quantity used as intermediate inputs for each good $j$ is exactly captured by the magnitude of TFP recovery in sector $j$, i.e., $x(0) = \tilde{z}$. As production linkages recover over time and as the economy converges back to the steady-state, $x(t)$ converges to the zero vector. The term $\delta$ modulates the rate of convergence; the system converges at a faster rate if adjustment cost $\delta$ is small. The next proposition describes the time paths or the flow of the sectoral outputs and consumption.

**Proposition 2. Flow of Output and Consumption.** The flow of sectoral output satisfies
\[
\ln q(t) = \ln q^{ss} - \Sigma (I-\Sigma)^{-1} e^{-\delta^{-1}(I-\Sigma)t}\tilde{z}
\]
and the flow of aggregate consumption satisfies
\[
\ln c(t) = \ln c^{ss} - \beta'\Sigma (I-\Sigma)^{-1} e^{-\delta^{-1}(I-\Sigma)t}\tilde{z}.
\]

**Proof.** See Appendix A.3. \qed

When productivity in sector $j$ recovers, the sector’s output expands immediately, which gradually translates into the expansion of input $j$ used in other sectors $i$, thereby causing $i$’s output to expand gradually over time. The vector $(- (I - \Sigma)^{-1} \tilde{z} + \tilde{z}) = -\Sigma (I - \Sigma)^{-1} \tilde{z}$ captures the extent to which log-sectoral outputs at $t = 0$ are below their steady-state levels; it can be re-written as
\[
-\Sigma (I - \Sigma)^{-1} \tilde{z} = - \left( \sum_{s=1}^{\infty} \Sigma^s \right) \tilde{z}
\]
where each successive term in the summation captures a higher round of input-output linkages to be recovered from the initial shock. The expression $(-\Sigma (I - \Sigma)^{-1} e^{-\delta^{-1}(I-\Sigma)t})$ is the log-deviation in output relative to steady-state levels at time $t$; it is the continuous time
analogue of the discrete partial sum \(- \sum_{s=t}^{\infty} \Sigma^s\) that goes from \(s = t\) to \(s = \infty\). By varying \(t\), the expression captures the fact that input-output linkages recover gradually, and higher rounds of linkages take longer to recover. Intuitively, a discrete sum would have implied that after \(t\) periods of recovery, the loss in output is entirely attributable to the input-output linkages higher than the \(t\)-th round, as all all prior rounds of input-output linkages have recovered. As we show below, our continuous formulation implies that every round of linkages recovers continuously as time passes, but higher rounds of linkages recover more slowly.

The rate of recovery is inversely related to \(\delta\). As \(\delta \to 0\), the convergence towards the steady-state becomes instantaneous:

\[
\lim_{\delta \to 0} \left( -\Sigma (I - \Sigma)^{-1} e^{-\delta^{-1}(I-\Sigma)t} \right) = 0 \quad \text{for any } t > 0.
\]

More broadly, this proposition shows that the properties of the dynamical system described by the gradual adjustment of the economy are tightly related to the properties of the input-output matrix via the sequence of its powers \(\Sigma^s\). The parameter \(\delta\) modulates the speed of adjustment.

### 3.2 Welfare Impact of Sectoral Shocks

We now characterize the impact of sectoral TFP shocks on consumer welfare. Let \(V^{ss}\) denote consumer welfare in the initial steady state.

**Proposition 3. Welfare Impact of Temporary TFP Shocks.** Let

\[
v' \equiv \frac{1}{\rho} \left[ \beta'(I - \Sigma)^{-1} - \beta' \left( I - \frac{\Sigma}{1 + \rho \delta} \right)^{-1} \right].
\]

The impact of temporary, negative TFP shocks \(\tilde{z}\) on welfare is

\[
V(\tilde{z}) - V^{ss} = \int_{0}^{\infty} e^{-\rho s} (\ln c(s) - \ln c^{ss}) ds = - v' \tilde{z}.
\]

**Proof.** See Appendix A.4. \(\square\)

The vector \(v'\) captures the welfare impact of temporary, negative TFP shocks. When \(\delta = 0\), recovery is instantaneous, and temporary shocks have no impact on consumer welfare. The first term, \(\frac{1}{\rho} \beta'(I - \Sigma)^{-1}\), is proportional to the sectoral Domar weight and captures the impact on welfare of permanent negative TFP shocks. The second term \(\frac{1}{\rho} \beta'(I - \frac{\Sigma}{1 + \rho \delta})^{-1}\) captures the effect of input-output recovery.

It is informative to rewrite \(v'\) as

\[
v' = \frac{1}{\rho} \beta' \sum_{s=0}^{\infty} (1 - (1 + \rho \delta)^{-s}) \Sigma^s
\]

\(\text{(15)}\)
and compare the expression with sectoral Domar weights:

$$\gamma' = \beta' \sum_{s=0}^{\infty} \Sigma^s. \quad (16)$$

The Domar weight captures the impact of permanent TFP shocks on aggregate consumption, and each term $\beta' \Sigma^s$ in the power series captures the $s$-th round of network effect: $\beta'$ captures the first round, direct effect of TFP on consumption, $\beta' \Sigma$ captures the indirect effect of sectoral TFP on other producers who supply to the consumer, and so on. The Domar weight is also equal to a sector’s sale relative to GDP in a decentralized equilibrium, and each term $\beta' \Sigma^s$ in the power series captures the revenue from the $s$-th round indirect sales to the consumer.

In our dynamic model, temporary shocks may have lasting effect on output and welfare precisely because of higher-order linkages $\Sigma^s$, $s > 0$. With adjustment costs ($\delta > 0$), input-output linkages are slow to recover, and $(1 - (1 + \rho\delta)^{-s}) \Sigma^s$ captures the utility loss due to the slow recovery of the $s$-th order linkages. Effectively, the power series in (15) disproportionally removes the initial entries in (16) while keeping the tail entries unchanged:

$$v' = \frac{1}{\rho} \beta' [(1 - 1) \Sigma^0 + (1 - (1 + \rho\delta)^{-1}) \Sigma^1 + (1 - (1 + \rho\delta)^{-2}) \Sigma^2 + ...]. \quad (17)$$

Note the weight on $\Sigma^0$ is 0 and the weight on $\Sigma^s$ converges to 1 as $s \to \infty$.

We now summarize this proposition: in the presence of adjustment frictions, shocks to sectors that generate significant sales through distant linkages to the consumer are disproportionately damaging to the economy. These shocks have large and lasting impact on GDP even as sectoral TFP recovers.

**Alpha centrality and global versus local influence** The welfare impact measure $v'$ of temporary shocks is connected to the notion of alpha centrality in a network represented by the input-output matrix. The alpha centrality for $\alpha \in (0, 1]$ is defined as:

$$\iota'_\alpha \equiv \beta' (I - \alpha\Sigma)^{-1}. \quad (18)$$

Intuitively, this is a centrality measure where a parameter $\alpha$ is used to weigh the higher order input-output linkages, represented by the powers of the matrix $\Sigma$:

$$\iota'_\alpha \equiv \beta' \left[ \Sigma^0 + \alpha \Sigma^1 + \alpha^2 \Sigma^2 + ... \right]. \quad (19)$$

The $i$-th entry in $\beta' \Sigma^s$ captures the sales (relative to GDP) that sector $i$ generates through $s$ rounds of linkages before reaching the final consumer.

A related way to think about centrality is in terms of a random walk on the network, where $\Sigma_{ij}$ is the probability of reaching $j$ from $i$ in one walk. The $ij$-th entry in $\Sigma^s$ then
measures the probability of reaching \( j \) from \( i \) in the walks of length \( s \). As parameter \((\alpha \leq 1)\) decreases, shorter walks become more important, and local influences carry higher significance. When \( \alpha \) increases, longer walks become more important, and global influences carry higher significance. In the limit case as \( \alpha \to 1 \), the walks of any length carry identical weights, and the alpha centrality measure becomes the Domar weight. In this sense, alpha centrality tunes between rankings based on short walks (local influence) and those based on long walks (global influence) (Benzi and Klymko (2015)).

The welfare impact measure \( \nu' \) is thus proportional to the difference in the alpha centralities \( \iota'_a \alpha - \iota'_a \), where \( \alpha_1 = 1 \) and \( \alpha_2 = (1 + \rho \delta)^{-1} \). It corresponds to a generalized version of alpha centrality:

\[
\tilde{\iota}' \equiv \beta' \left[ a_0 \Sigma^0 + a_1 \Sigma^1 + a_2 \Sigma^2 + \ldots \right],
\]

for some sequence \( \{a_0, a_1, \ldots\} \). Assuming that such weighted power series converge, this measure weights the walks of length \( k \) with the parameter \( a_k \). In the case of alpha centrality with \( \alpha < 1 \), \( a_k = \alpha^k \) and is geometrically decreasing from \( a_0 = 1 \) and \( a_\infty = 0 \). The welfare measure \( \nu' \) is a generalized alpha centrality with \( a_k = 1 - \alpha^k_2 \) and thus increasing between \( a_0 = 0 \) and \( a_\infty = 1 \). Because the measure \( \nu' \) captures the welfare losses due to slow recovery of inputs, it relatively prioritizes the longer walks or higher order input output linkages and thus the global over local influences.

The term \((1 + \rho \delta)^{-1}\) in (14) also defines a one-parameter family of economies that can be thought of as a multi-scale representation of the static input output matrix. Specifically, the speed of adjustment and the discount factor of the agent determine the scale—the relative importance of the higher-order links and thus the importance of the global versus local structures.

### 3.3 Vertical Example Revisited

It is now instructive to revisit the examples in Figure 1. In the horizontal economy of panel (a), there are no input-output linkages; consequently, \( \nu \) is the zero vector, and temporary shocks that recover instantaneously have zero impact on this economy. By contrast, temporary shocks may have lasting impact in the vertical economy of panel (b), with the network diagram reproduced below, along with input-output table of this economy. Sector 1 is the most upstream and sector \( N \) is the most downstream.
In this vertical economy, each successive power of the input-output matrix contains a smaller identity sub-matrix in the bottom-left and zeros otherwise, and the Leontief-inverse is a lower-triangular matrix of ones. For example, when $N = 4$,

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Sigma^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Sigma^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (I - \Sigma)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By construction, the Domar weight is identically one for all sectors, $\gamma' \equiv \beta' (I - \Sigma)^{-1} = 1'$. Permanent TFP shocks in every sector has identical impact on welfare. The welfare impact $\nu$ of temporary shocks is no longer constant, however:

$$\nu' = \rho \delta \left[ 1 - (1 + \rho \delta)^{-N}, \ldots, 1 - \left(\frac{1}{1+\rho\delta}\right)^2, 1 - \frac{1}{1+\rho\delta}, 0 \right]$$

Hence, temporary shocks to an upstream sector $i$ are more damaging than to a downstream sector $j > i$, despite all sectors having the same Domar weight.

Figure 2 shows the path of GDP over time when each sector in the vertical economy (with $N = 4$ sectors) is affected by a temporary, negative TFP shock. As the figure demonstrates, because adjustment costs compound—sector $i$‘s production needs to recover before sector $i+1$‘s inputs can expand—shocks to relatively upstream sectors have long-lasting effects: the economy takes the longest time to recover from shocks to sector 1—the most upstream—and recovers instantaneously from shocks to sector 4. Consequently, $v_1 > v_2 > v_3 > v_4$ as the measure $\nu'$ integrates the entire path of output losses, discounting the future at rate $\rho$.

Our model also has rich predictions on the recovery path of sectoral output following temporary shocks. Figure 3 shows the path of sectoral output over time when sector 1 in the vertical economy (with $N = 4$ sectors) is affected by a temporary TFP shock. The figure shows that the more downstream a sector is from the original shock, the longer it takes for this sector’s output to recover. After a temporary TFP shock to sector 1, sector 1’s output recovers immediately once the TFP recovers, but the output loss lasts longer in sector 2, and even longer in sectors 3, and so on. This is because each round of input-output linkages take time to recover, and the further downstream is sector from the original shock, the more
rounds of linkages were destroyed by the initial shock and therefore need additional time to recover.

3.4 Connection to Katz Centrality and Upstreamness

With adjustment costs, temporary shocks are damaging if they affect the sectors with significant sales through distant linkages to the consumer. We now formally illustrate this point by showing that the welfare impact measure $v_i$ is the product between sector $i$’s Domar weight ($\gamma_i$) and its Katz (1953) centrality. In our context, Katz centrality is a measure of upstreamness: it captures the network-adjusted distance of sectoral supply to the final consumer. Hence, temporary shocks are more damaging to the economy if they affect large sectors that are also upstream and supply disproportionate fractions of outputs to other upstream producers.

Let $\eta_i \equiv v_i / \gamma_i$ be the welfare impact of a temporary shock to sector $i$ relative to the sectoral size. Let $\Theta$ be the input-output demand matrix whose $in$-th entry is $\theta_{in} \equiv \sigma_{ni} \gamma_n / \gamma_i$, i.e., the fraction of sector $i$’s output sold to sector $n$. Intuitively, in a decentralized equilibrium, entries of the input-output elasticity matrix $\Sigma$ coincide with input expenditure shares that are obtained by dividing the value of intermediate inputs by the sales of the buyer, whereas entries of $\Theta$ are obtained by dividing the value of inputs by the sales of the supplier.

**Proposition 4.** Welfare impact of temporary shocks is the Domar weight times the Katz centrality.
Figure 3. Time path of sectoral output losses from temporary shocks to sector 1 (upstream) in the vertical economy

\[
\eta = \delta \left[ \sum_{s=1}^{\infty} \left( \frac{1}{1 + \rho \delta} \Theta \right)^s \right] 1.
\]

Proof. See Appendix A.5.

Katz centrality can be re-written implicitly as \( \eta = \delta \frac{1}{1 + \rho \delta} \Theta 1 + \frac{1}{1 + \rho \delta} \Theta \eta \), or, in scalar form,

\[
\eta_i = \frac{\delta}{1 + \rho \delta} \sum_{n=1}^{N} \Theta_{in} + \frac{1}{1 + \rho \delta} \sum_{n=1}^{N} \Theta_{in} \eta_n.
\]

The first term on the right-hand-side is a constant \( \left( \frac{\delta}{1 + \rho \delta} \right) \) times the total fraction of sector \( i \)'s output supplied to intermediate producers (rather than the consumer). The second term is the average Katz centrality of the producers that use good \( i \) as inputs, weighted by the fraction of \( i \)'s output sold to each buyer, and scaled down by a factor \( \frac{1}{1 + \rho \delta} \). Hence, a sector is Katz-central if it supplies a disproportionate fraction of output to other Katz-central producers.

Katz centrality is a natural notion of upstreamness.\(^7\) Recall from (16) that the sectoral Domar weight can be written as \( \gamma' = \beta' \sum_{s=0}^{\infty} \Sigma^s \), where the \( i \)-th component of \( \beta' \Sigma^s \) captures the sales of sector \( i \) (relative to GDP) that reaches the final consumer through \( s \)-rounds of input-output linkages. Antràs et al. (2012) defines an upstreamness measure that captures

\(^7\)The Katz centrality is also isomorphic to the distortion centrality of Liu (2019) in a production network with constant market imperfection wedges.
the average number of rounds it takes for sectoral output to reach the final consumer:

\[ U_{p_i} = 1 \cdot \frac{\beta_i}{\gamma_i} + 2 \cdot \frac{[\mathbf{\beta}' \mathbf{\Sigma}]_i}{\gamma_i} + 3 \cdot \frac{[\mathbf{\beta}' \mathbf{\Sigma}^2]_i}{\gamma_i} + \ldots = \sum_{s=0}^{\infty} a_s \cdot \frac{[\mathbf{\beta}' \mathbf{\Sigma}^s]_i}{\gamma_i}, \text{ with } a_s = s + 1. \]

More generally, \( \sum_{s=0}^{\infty} \frac{a_s \cdot [\mathbf{\beta}' \mathbf{\Sigma}^s]_i}{\gamma_i} \) is a measure of sector \( i \)'s upstreamness for any increasing and convergent sequence \( \{a_s\}_{s=0}^{\infty} \) because such a sequence up-weights sectoral sales that are more distant to the consumer. Katz centrality can also be written in this form using the sequence \( a_s = \rho^{-1} (1 - (1 + \rho \delta)^{-s}) \).

### 3.5 Connection to Time-to-Build and Long and Plosser (1983)

We show our law of motion (8) for intermediate inputs, microfounded by exponential adjustment costs, is to first-order equivalent to a continuous-time formulation of the time-to-build specification in Long and Plosser (1983).

Specifically, suppose there are no adjustment costs but instead, after each intermediate input \( j \) is produced, it must go through logistical delays before it can arrive at the production lines of input-using sector \( i \). In Long and Plosser (1983)'s discrete-time formulation, goods arrive with one period delay; because our model is in continuous-time, we assume intermediate inputs arrive from sellers to buyers following a Poisson process with rate \( \delta^{-1} \), corresponding to an exponentially distributed delay with mean \( \delta \). Formally, let \( a_{ij}(t) \) denote the stock of good \( j \) sold to but have not arrived at sector \( i \) by time \( t \). The law of motion for \( a_{ij} \) is

\[ \dot{a}_{ij} = s_{ij} - m_{ij}, \quad (18) \]

which states that the rate of change in the stock of good \( j \) on its way to sector \( i \) is the difference between the quantity of new purchase (\( s_{ij} \)) and the quantity of arrival (\( m_{ij} \)).

Given that goods arrive with Poisson rate \( \delta^{-1} \), the quantity of arrival follows \( m_{ij}(t) = \delta^{-1} a_{ij}(t) \), and, combining with equation (18), we derive the law of motion for the use of intermediate inputs:

\[ \dot{m}_{ij} = \delta^{-1} (s_{ij} - m_{ij}). \quad (19) \]

Under this microfoundation, the parameter \( \delta \) can also be interpreted as the backlog ratio: it measures the average delay between when inputs are ordered and delivered.

Under the time-to-build formulation, the law of motion (19) states that the rate of change in \( m_{ij} \) is linear in the difference between new purchase orders \( s_{ij} \) and quantity delivered \( m_{ij} \). By contrast, the law of motion (8) in our baseline adjustment cost formulation states that the growth rate in \( m_{ij} \) is linear in the log-difference (\( \ln s_{ij} - \ln m_{ij} \)). The two formulations are equivalent to first-order; that is, when \( s_{ij}/m_{ij} \) is close to one, equation (19) can be re-written
as
\[ \frac{\dot{m}_{ij}}{m_{ij}} = \delta^{-1} \left( \frac{s_{ij} - \bar{m}_{ij}}{m_{ij}} \right) \approx \delta^{-1} \left( \ln s_{ij} - \ln m_{ij} \right). \]

Hence, when TFP shocks are small—so that allocations at time 0 are not too far from the steady-state—our model predictions on the path of sectoral output closely matches the predictions of a dynamic network model with time-to-build. The main advantage of our log-linear formulation is tractability; as we have shown, a log-linear law of motion (8) affords us closed-form solutions for the entire path of sectoral output, thereby enabling us to derive substantive analytic insights of how the network structure affects the economy’s susceptibility to and recovery after temporary shocks. In Section 3.7 below we further show that even with a substantially more general, non-parametric adjustment cost formulation, which nests the linear formulation à la time-to-build as a special case, our characterizations in preceding sections still hold as first-order approximations around the steady-state.

3.6 Heterogeneous Adjustment Costs and the Network Structure

In the baseline model, we assume a common adjustment cost parameter \( \delta \) for all sector-pairs. The following Proposition extends our main welfare result (Proposition 3) to a setting with buyer-seller-pair specific adjustment costs, \( \delta_{ij} \). That is, during the recovery following a temporary TFP shock, the law of motion for inputs \( j \) used in sector \( i \) follows
\[ \frac{\dot{m}_{ij}}{m_{ij}} = \delta_{ij}^{-1} \left( \ln s_{ij} - \ln m_{ij} \right), \]
replacing the law of motion (8) in the planner’s problem.

**Proposition 5. Welfare Impact of Temporary TFP Shocks Under Sector-Pair Specific Adjustment Costs.** Let \( \delta \) denote the matrix whose \( ij \)-th entry encodes the adjustment cost \( \delta_{ij} \) for input \( j \) used in sector \( i \). Let \( \Omega \) denote the matrix whose \( ij \)-th entry is \( \frac{\sigma_{ij}}{1 + \rho \delta_{ij}} \); then the impact of temporary, negative TFP shocks \( \tilde{z} \) on welfare is
\[ V(\tilde{z}) - V^{ss} = \int_{0}^{\infty} e^{-\rho s} (\ln c(s) - \ln c^{ss}) \, ds = -v' \tilde{z}, \]
where
\[ v' \equiv \frac{1}{\rho} \beta' \left[ (I - \Sigma)^{-1} - (I - \Omega)^{-1} \right]. \]

**Proof.** See Section A.6 of the Appendix.

Intuitively, the formula in Proposition 5, which encodes heterogeneous adjustment costs, is similar to the homogeneous adjustment cost formulation in Proposition 3, simply replacing
\[ \Sigma_{1+\rho\delta} \equiv \left[ \frac{\sigma_{ij}}{1+\rho\delta} \right] \] by \[ \Omega \equiv \left[ \frac{\sigma_{ij}}{1+\rho\delta_i} \right] \]. The welfare impact \( v \) can be re-written as

\[ v' = \frac{1}{\rho} \left[ \Sigma^0 - \Omega^0 + (\Sigma^1 - \Omega^1) + (\Sigma^2 - \Omega^2) + \cdots \right] \]

which has a similar interpretation to (17).

The size-adjusted welfare impact \( \eta_i \equiv v_i/\gamma_i \) can be written as

\[ \eta_i = \sum_n \frac{\delta_{ni}}{1+\rho\delta_{ni}} \theta_{in} + \sum_n \frac{\theta_{in}}{1+\rho\delta_{ni}} \eta_n \]  

(20)

where recall \( \Theta \) is the input-output demand matrix whose \( in \)-th entry is \( \theta_{in} = \sigma_{ni} \gamma_n/\gamma_i \), i.e., the fraction of sector \( i \)'s output sold to sector \( n \). It is a generalized version of the Katz centrality that accounts for heterogeneity in adjustment costs \( \delta_{ni} \). Sector \( i \) has high \( \eta \) if it supplies disproportionately to buyers \( n \) with high \( \eta \) and if the buyers of input \( i \) are subject to substantial adjustment costs. In other words, high-\( \eta \) sectors are those whose output travels through many high-adjustment-cost producers, directly and indirectly, before reaching the consumer.

In general, heterogeneity in the adjustment costs do play a role in determining the size-adjusted welfare impact of temporary shocks. However, in a vertical network as in Section 3.3, a more upstream sector always has a higher \( \eta \), regardless of the magnitude of the adjustment costs. Specifically, from equation (20), the size-adjusted welfare impact in the 4-sector vertical network follows

\[ \left[ \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{array} \right] \propto \left[ \begin{array}{c} \frac{1}{1-(1+\rho\delta_{43})(1+\rho\delta_{32})(1+\rho\delta_{21})} \\ 1 - \frac{1}{1-(1+\rho\delta_{43})(1+\rho\delta_{32})} \\ \frac{1}{1+\rho\delta_{43}} \\ 0 \end{array} \right] \]

For any adjustment cost parameters \( \delta_{43}, \delta_{32}, \delta_{21} > 0 \), we always have \( \eta_1 > \eta_2 > \eta_3 > \eta_4 \), that size-adjusted welfare impact aligns in rank order with upstreamness. This is because input flows are one directional, such that a more upstream sector’s output is always subject to more compounding of adjustment costs. In Section 5.1, we show that this insight extends to the real-world production network of the U.S. economy: input flows are mostly one directional, such that the size-adjusted welfare impact of temporary shocks align very well with sectoral upstreamness, and in fact the heterogeneity in adjustment costs is quantitatively unimportant.

### 3.7 General Production Functions and Adjustment Costs

Our baseline model is highly tractable, as we obtain closed-form solution for the entire recovery path of the economy. Such tractability is achieved through a combination of log-
linearity in the Cobb-Douglas production functions and in the law of motion for intermediate inputs. In this section, we extend our main welfare result (Proposition 3) to a non-parametric setting, where a version of our welfare formulas continues to hold locally (to first-order) around the steady-state.

Specifically, consider an economy environment in which we replace the Cobb-Douglas consumption and production functions in (2) and (3) with non-parametric aggregators that are homogeneous of degree one:

\[ c(t) \equiv c(\{c_j(t)\}_{j=1}^N), \quad q_i(t) = f_i(z_i(t), \ell_i(t), \{m_{ij}(t)\}_{j=1}^N). \]

Likewise, we consider a non-parametric adjustment cost process

\[ \dot{m}_{ij} = g_{ij}(s_{ij}, m_{ij}), \]

with the requirement that \( \dot{m}_{ij} = 0 \) when \( s_{ij} = m_{ij} \), and that \( g_{ij}() \) is locally homogeneous of degree one when \( s_{ij} = m_{ij} \).

Suppose we observe the economy in a steady-state. The following result characterizes the welfare impact of temporary shocks evaluated locally at the steady-state. Let \( \beta_j \equiv \frac{\partial \ln c}{\partial \ln c_j} \) denote the steady-state consumption elasticity with respect to good \( j \) (\( \beta \equiv [\beta_j] \) is the corresponding vector), and \( \sigma_{ij} \equiv \frac{\partial \ln q_i}{\partial \ln m_{ij}} \) is producer \( i \)'s output elasticity with respect to input \( j \) in steady-state (\( \Sigma \equiv [\sigma_{ij}] \) is the corresponding matrix).

Finally, let \( \omega_{ij}^{-1} \equiv \frac{\partial \ln g_{ij}}{\partial \ln s_{ij}} \) denote the rate at which purchased inputs \( s_{ij} \) expand the quantity of production inputs \( m_{ij} \), again evaluated at the steady-state.

**Proposition 6. Welfare Impact of Temporary TFP Shocks with Non-Parametric Network and Adjustment Costs.** Let \( \Omega \) be the matrix whose \( ij \)-th entry is \( \sigma_{ij} / (1 + \rho \omega_{ij}) \). Around the steady-state, the welfare impact of temporary, negative TFP shocks is

\[ \frac{dV(\tilde{z})}{d\tilde{z}} \bigg|_{\tilde{z}=0} = -\frac{1}{\rho} \left[ \beta' (I - \Sigma)^{-1} - \beta' (I - \Omega)^{-1} \right]. \]

**Proof.** See Section A.7 of the Appendix.

In a decentralized steady-state, the vector of consumption elasticities coincides with the vector of consumption shares \( \beta \equiv \frac{d \ln c}{d \ln c_j} \); the matrix of input-output elasticities also coincides with the matrix of input-output expenditure shares. Hence Proposition 6 shares the same empirical mapping as in our baseline model, where \( \beta \) and \( \Sigma \) can be evaluated using empirical input-output tables despite the model being fully non-parametric. The reduced-form object \( \omega_{ij} \), which parametrizes the rate at which purchased inputs \( s_{ij} \) expand production inputs \( m_{ij} \) under the adjustment cost process, has similar interpretation to the adjustment cost parameter

---

That is, \( \frac{dg_{ij}(s,m)}{ds} s + \frac{dg_{ij}(s,m)}{dm} m = g_{ij}(s, m) \) when \( s = m \).
cost parameter in our baseline model. When $\omega_{ij}$ is common across all $i,j$, the formula in Proposition 6 coincides with Proposition 3; when $\omega_{ij}$ is sector-pair specific, then the formula coincides with that in Proposition 5 if $\omega_{ij}$ is replaced by $\delta_{ij}$.

3.8 Two-Sided Adjustment Costs

Our baseline adjustment cost formulation (4) assumes inputs must expand gradually but can shrink instantaneously. As an expositional device, this assumption enables us to isolate the transitional dynamics that arise solely from endogenous input-output adjustments and not from TFP changes. In essence, negative TFP shocks that occur only at time $t = 0^-$ provide the initial conditions $m_{ij}(0)$ for the economy at $t = 0$, and all subsequent dynamics arise due to the recovery of input-output linkages.

In this section, we generalize our formulation so that inputs adjust slowly both upwards and downwards. Specifically, consider the model in Section 2, but replace equation (4) with

$$s_{ij} \equiv m_{ij} \times \exp (\delta \dot{m}_{ij}/m_{ij} \times 1(\dot{m}_{ij} > 0) + \dot{\delta} m_{ij}/m_{ij} \times 1(\dot{m}_{ij} < 0)).$$  \hspace{2cm} (21)

While $\delta > 0$ continues to parametrize the adjustment cost for expanding inputs, the additional parameter $\delta \geq 0$ captures the downward adjustment process of inputs. Our baseline formulation corresponds to the case $\dot{\delta} = 0$. When $\delta > 0$, the formulation translates into the following law of motion for production inputs:

$$\dot{m}_{ij}/m_{ij} = \begin{cases} 
\delta^{-1} (\ln s_{ij} - \ln m_{ij}) & \text{if } s_{ij} \geq m_{ij} \\
\delta^{-1} (\ln s_{ij} - \ln m_{ij}) & \text{if } s_{ij} < m_{ij}.
\end{cases}$$

That is, $m_{ij}$ can no longer jump downwards; instead, even when the purchased quantity of inputs $s_{ij}$ is below the current quantity $m_{ij}$ used in production, the inputs used for production will only decline slowly over time.

Because inputs are slow to adjust in both directions, a temporary TFP shock that lasts only for an infinitesimal period will have vanishingly small effect on consumer welfare. To meaningfully analyze the impact of negative shocks, we instead study a negative TFP shock vector $-\tilde{z}$ that hits the economy at time $0$ and recovers at time $T$.

Proposition 7. Laws of Motion under Two-Sided Adjustment Costs. Consider a TFP shock vector $\tilde{z}$ that arrives at time $0$ and recedes at time $T$ in an economy initially in steady-state.

1. The law of motion for sectoral output vector $q$ is

$$\frac{d \ln q}{dt} = \begin{cases} 
\delta^{-1} \Sigma x \ (t) & \text{if } t \in (0, T), \\
\delta^{-1} \Sigma x \ (t) & \text{if } t > T,
\end{cases}$$

25
with boundary conditions \( \ln q(0) = \ln q^s - \tilde{z} \) and \( \ln q(T) = \lim_{t \to T^-} \ln q(t) + \tilde{z} \).

2. The law of motion for GDP is

\[
\frac{d \ln c(t)}{dt} = \begin{cases} 
\delta^{-1} \beta' \Sigma x(t) & \text{if } t \in (0, T), \\
\delta^{-1} \beta' \Sigma x(t) & \text{if } t > T,
\end{cases}
\]

with boundary conditions \( \ln c(0) = \ln c^s - \beta' \tilde{z} \) and \( \ln c(T) = \lim_{t \to T^-} \ln c(t) + \beta' \tilde{z} \).

3. The law of motion for the log-ratio between quantities of inputs supplied and used is

\[
\frac{dx(t)}{dt} = \begin{cases} 
-\delta^{-1} (I - \Sigma) x(t) & \text{if } t \in (0, T), \\
-\delta^{-1} (I - \Sigma) x(t) & \text{if } t > T,
\end{cases}
\]

with boundary conditions \( x(0) = -\tilde{z} \) and \( x(T) = \lim_{t \to T^-} x(t) + \tilde{z} \).

Proof. See Appendix A.8.

Proposition 7 extends our earlier Proposition 1 to the more general setting with two-sided adjustment costs. Intuitively, as TFP experiences discrete jumps at times \( t = 0 \) and \( T \), sectoral output and GDP also experiences discrete jumps at these times. Hence, the dynamics in this economy are jointly determined by TFP changes and the adjustment process. At all other times \( t \notin \{0, T\} \), however, output changes continuously and follows a differential equation system that is analogous to our earlier characterization, with the additional nuance that the speed of evolution depends on the direction of inputs changes as parametrized by \( \delta \) and \( \tilde{\delta} \).

Our next result characterizes the welfare impact of temporary negative shocks under two-sided adjustment costs.

**Proposition 8. Welfare Impact of Temporary TFP Shocks.** The impact of a temporary, negative TFP shock \( \tilde{z} \) that affects the economy during \( t \in [0, T] \) on welfare is

\[
V(\tilde{z}; T, \delta, \delta) - V^{ss} = -\frac{1}{\rho} \beta' \left( I - \frac{1}{1 + \rho \tilde{\delta}} \Sigma \right)^{-1} \left( I - e^{-\tilde{\delta}^{-1}((1 + \rho \tilde{\delta})I - \Sigma)T} \right)
\]

\[
- (I - \Sigma)^{-1} \left( e^{-\rho T} - e^{-\tilde{\delta}^{-1}((1 + \rho \tilde{\delta})I - \Sigma)T} \right)
\]

\[
- \left[ \left( I - \frac{1}{1 + \rho \delta} \Sigma \right)^{-1} - (I - \Sigma)^{-1} \right] \left( e^{-\rho T} - e^{-\tilde{\delta}^{-1}((1 + \rho \tilde{\delta})I - \Sigma)T} \right) \tilde{z}
\]

(22)

Proof. See Appendix A.9.

Under two-sided adjustment costs (\( \tilde{\delta} > 0 \)), the initial impact of the negative shock is no longer instantaneous and instead depends on the duration \( T \) of the shock; the recovery takes
longer if the negative shock lasts longer (larger $T$) or if the initial impact realizes sooner (lower $\delta$). The first two lines on the right-hand side of (22) captures the welfare impact due to GDP losses when the negative shock is in place $t \in [0, T)$, and the third line captures the welfare impact due to GDP losses along the recovery path $t \geq T$. Proposition 8 nests our earlier Proposition 1 as a special case under the limit $\delta \to 0$ and $T \to 0$.

**Figure 4.** Time path of GDP losses from sectoral shocks in the vertical economy

![Figure 4](image.png)

Figure 4 demonstrates the time paths of GDP losses in the vertical economy of Section 3.3 for different sectoral shocks in the reasonable scenario where input contractions are faster than expansions ($\hat{\delta} < \delta$). First consider a negative shock to the downstream sector 4. The path of GDP is depicted by the dotted line. Because the shock does not destroy input-output linkages—no production sector uses good 4 as inputs—the GDP moves in tandem with sector 4’s TFP; the GDP collapses when the negative shock arrives at $t = 0$ and recovers completely at time $T$ once the negative shock recedes.

By contrast, consider a negative shock to the upstream sector 1. The path of GDP is depicted by the solid line. When the shock arrives at $t = 0$, the GDP does not collapse immediately—because production inputs are sticky even as purchased inputs collapse—but instead declines gradually over time. Analogously, the GDP recovers gradually after the shock recedes.

Our earlier analysis shows that, in the presence of upward adjustment costs, the economy recovers more slowly from negative shocks to the upstream sector. Figure 4 demonstrates that, conversely, in the presence of downward adjustment costs, shocks to the upstream sector takes longer to negatively affect GDP. The closer is $\hat{\delta}$ to zero, the faster are the downward adjustments, and the less important is this new effect. Because it is reasonable to assume
that input contractions are faster than expansions ($\hat{\delta} < \delta$), and because our primary interest is in analyzing the recovery dynamics after negative shocks, from now on for expositional clarity we focus on our baseline specification with the limit $\hat{\delta} \to 0$ and $T \to 0$.

4 Spectral Properties of the Input-Output Matrix and the Dynamical System

Our preceding analysis shows that temporary shocks to large and upstream sectors are disproportionately damaging because input-output linkages are slow to recover from these shocks. We now examine determinants of the welfare impact from the spectral point of view.

Spectral analysis is a powerful tool for the analysis of dynamics over networks. As we demonstrate, the tool enables us to identify which properties of the input-output network as well as how the nature of adjustment costs matter quantitatively. In our context, spectral analysis provides a basis in the vector space of sectoral shocks, and each component of the basis represent a vector of shocks whose impact decays over time at a constant rate. By examining the associated rate of decay, captured by the eigenvalue, spectral analysis enables us to distinguish sectoral shocks that have long-lasting impact versus those shocks that the economy can recover from quickly. We uncover a low-dimensional factor structure for the temporary shocks that are welfare-relevant.

Specifically, we undertake an eigendecomposition of the input-output table, $\Sigma = U\Lambda W$, where $\Lambda$ is a diagonal matrix of eigenvalues $\{\lambda_k\}_{k=1}^{N}$ arranged in decreasing order by absolute values, and $W = U^{-1}$. For each eigenvalue $\lambda_h$, the $h$-th column of $U$ ($u_h$) and the $h$-th row of $W$ ($w_h'$) are the right- and left-eigenvectors of $\Sigma$, respectively, such that

$$\Sigma u_h = \lambda_h u_h, \quad w_h' \Sigma = \lambda_h w_h'.$$

That is, $u_h$ ($w_h'$) is the vector that, when left-multiplied (right-multiplied) by $\Sigma$, is proportional to itself but scaled by the corresponding eigenvalue $\lambda_h$.

Now consider a negative TFP shock vector that equals to $-u_h$. The first round effect lowers sectoral output by $-u_h$; the second round effect lowers sectoral output by $-\Sigma u_h = -\lambda_h u_h$; the third round effect lowers sectoral output by $-\lambda_h^2 u_h$, and so on. That is, at each round of propagation, the productivity shock vector $-u_h$ always reduces sectoral output in proportion to $-u_h$, with effects scaled-down by a factor equal to the eigenvalue $\lambda_h$ relative to the previous round. In other words, $u_h$ is the profile of TFP shocks with every round of general equilibrium effect always in proportion to the first round but decays at rate $\lambda_h$ after

\[9\text{We construct the right-eigenvectors such that the 2-norm of } u_h \text{ is equal to 1 for all } h.\]
each round of propagation.\textsuperscript{10} Generically, any TFP shock vector $\tilde{z}$ can be written as a linear combination $\{a_k\}_{k=1}^{N}$ of the right-eigenvectors

$$\tilde{z} = \sum_{k=1}^{N} a_k u_k,$$

or, in matrix notation, $\tilde{z} = Ua$. That is, the matrix of right-eigenvalues $U$ provides a basis for TFP shocks that feature constant decay along each dimension in the basis, and the linear weights $a$ are the coordinates in that basis. The weights $a$ can be recovered using the left-eigenbasis:

$$W\tilde{z} = WUa = U^{-1}Ua = a.$$

That is, $a_k = w'_k \tilde{z}$, or in vector form, $a = W\tilde{z}$.

To summarize, the right-eigenvectors capture the shock profiles whose general equilibrium impact decays at rate governed by the corresponding eigenvalues; the left-eigenvectors convert shock profiles into coordinates in the right-eigenspace. We sometimes refer to the right-eigenvectors simply as eigenvectors.

**Eigen-Decomposition of Permanent and Temporary Shocks**  We now use the eigenbases $U$ and $W$ to further decompose the aggregate impact of sectoral shocks.

**Proposition 9. Eigen-Decomposition of Permanent and Temporary Shocks.** The Domar weight can be written as

$$\gamma' = \beta' \sum_{k=1}^{N} \frac{1}{1 - \lambda_k} u_k w'_k.$$  \hspace{1cm} (23)

The welfare sensitivity $v'$ to temporary shocks can be written as

$$v' = \beta' \sum_{k=1}^{N} \frac{\delta \lambda_k}{(1 - \lambda_k)(1 + \rho \delta - \lambda_k)} u_k w'_k.$$  \hspace{1cm} (24)

**Proof.** See Appendix A.10.

The proposition turns the infinite-sum-of-power-series representation of $\gamma$ and $v$ in (16) and (15) into finite sums over eigencomponents.

\textsuperscript{10}In general, these eigenvectors and eigenvalues can be complex-valued. If the TFP shock vector is the real part of a complex eigenvector $u_h$ ($\tilde{z} = \text{Re}(u_h)$), then $\Sigma' u_h = \text{Re}(\lambda'_h u_h) \neq \text{Re}(\lambda_h) \cdot \text{Re}(\lambda'^{-1}_h u_h)$. That is, the impact no longer decays at a constant rate $\text{Re}(\lambda_h)$. Instead, the complex eigenvalues introduce oscillatory motion as the dynamical system converges to the new steady-state. Empirically, we find that the imaginary components of the input-output table’s eigenvalues are very small, implying that oscillatory effects are negligible relative to the effects that decay exponentially, so we abstract away from these complex components for expositional brevity.
To understand the implication of Proposition 9, first consider a TFP shock profile captured by \( \tilde{z} = u_k \). Note that \( w'_\ell u_k = 1 \) if \( \ell = k \) and is zero otherwise; hence, the welfare impact of a permanent shock profile \( u_k \) is captured by

\[
\gamma' u_k = \beta' \sum_{\ell=1}^{N} \frac{1}{1 - \lambda_\ell} u_\ell w'_\ell u_k = \frac{1}{1 - \lambda_k} \beta' u_k. \tag{25}
\]

That is, the shock \( u_k \) affects consumption only through the \( k \)-th eigen component, with the direct effect being \( \beta' u_k \), the \( s \)-th round indirect network effect being \( \lambda_k^s \beta' u_k \), and a cumulative effect of \( \sum_{s=0}^{\infty} \lambda_k^s \beta' u_k = \frac{1}{1 - \lambda_k} \beta' u_k \).

We now analyze \( v' u_k \), which is the welfare impact of the temporary shock \( u_k \). Since the shock affects consumption at all times only through the \( k \)-th eigen component, the impact can be re-written as

\[
v' u_k = \frac{1}{\rho} \beta' u_k \left( \sum_{s=0}^{\infty} (1 - (1 + \rho \delta)^{-s}) \lambda_k^s \right) = \frac{\delta \lambda_k}{(1 - \lambda_k)(1 + \rho \delta - \lambda_k)} \beta' u_k. \tag{26}
\]

The term \( (1 - (1 + \rho \delta)^{-s}) \) assigns zero weight to the direct effect of the shock \( (s = 0) \) and an increasing sequence of weights to higher-order network effects. The cumulative effect is scaled inversely by the discount rate \( \rho \) in the dynamic economy. The summation further simplifies to \( \frac{\delta \lambda_k}{(1 - \lambda_k)(1 + \rho \delta - \lambda_k)} \) as the proposition shows.

Any generic TFP shock vector \( \tilde{z} \) can be projected onto the right-eigenspace with \( \tilde{z} = \sum_{k=1}^{N} u_k a_k \), and \( a_k \equiv w'_k \tilde{z} \) is its \( k \)-th coordinate after the projection. The overall effect on welfare is \( -v' \tilde{z} = -\sum_{k=1}^{N} v' u_k a_k \) if the shock is temporary and is \( -\sum_{k=1}^{N} \gamma' u_k a_k \) if the shock is permanent.

As we show below, \( v' \) has a low-dimensional factor representation for the U.S. economy, where \( v' \tilde{z} \) can be approximated closely by its projection onto the first \( K \) \( (K = 4) \) eigenvectors:

\[
v' \tilde{z} \approx v' \sum_{k=1}^{K} u_k a_k.
\]

The low-dimensional representation holds across any values of \( \rho, \delta \in (0, \infty) \) and is entirely due to the structure of the U.S. input-output network. Proposition 9 hints at why this is the case and also why the Domar weight does not have a good low-dimensional approximation in the eigenspace. Intuitively, eigenvectors with large eigenvalues represent shock profiles whose higher-order effects decay slowly; these shock profiles generate disproportionately large impact via higher rounds of network effects. The fact that Domar weights do not discount the direct and initial rounds of network effects imply that even eigenvectors with small eigenvalues may have a sizable contribution in explaining TFP shocks in the static model. Permanent shocks therefore do not have a low-dimensional representation. By contrast,
because dynamic adjustment costs significantly down-weight the direct and initial rounds of
network effects, \( \mathbf{v}' \) may have a factor representation as long as \(|\lambda_k|\) declines relatively fast in \( k \).

Specifically, consider two distinct shock profiles captured by real right-eigenvectors \( \mathbf{u}_k \) and
\( \mathbf{u}_\ell \) with \(|\lambda_k| < |\lambda_\ell|\). Let \( |\gamma' u_k| / |\gamma' u_\ell| \) denote the relative importance of these components
in the static model. In our dynamic model, their relative importance is (as implied by 25 and 26)

\[
\frac{|v' u_k|}{|v' u_\ell|} = \frac{|\lambda_k| |1 + \rho \delta - \lambda_\ell|}{|\lambda_\ell| |1 + \rho \delta - \lambda_k|} \times \frac{|\gamma' u_k|}{|\gamma' u_\ell|}.
\]

That is, relative to the static model, our dynamic model up-weights the relative importance of slow-decay right-eigenvectors shock profiles (those with greater eigenvalues) and, conversely, down-weights fast-decay right-eigenvector shock profiles (those with lower eigenvalues). These difference could be very significant: as we show below, for the U.S. economy, the dominant eigenvalue is \( \lambda_1 \approx 0.54 \), and the 100th is \( \lambda_{100} \approx 0.03 \). Hence, relative to the dominant eigencomponent, the 100-th component is at least 18 times more important in the static model than in our dynamic model. This qualitative fact, that fast-decay eigencomponents are significantly less important in the dynamic model than in the static model, holds regardless of the parametrization of \( \rho \) and \( \delta \) and is the reason behind the low-dimensional representation of input-output tables in our dynamic economy.

5 Analysis of the U.S. Input-Output Table

We now turn to the 2012 U.S. input-output table published by the U.S. Bureau of Labor Statistics. We first provide an analysis of \( \mathbf{v} \), the welfare impact of temporary shocks. We show \( \mathbf{v} \) differs significantly from Domar weights \( \gamma \): while the latter is an indication of sectoral size, the former instead picks up industries that are not only large but also upstream. We show that the high-dimensional input-output table—171 by 171 sectors under broad categories of agriculture, mining, manufacturing, and services\(^{11}\)—has a low-dimensional, 4-factor structure in terms of its susceptibility to temporary shocks: the \( \mathbf{v}' \) vector essentially loads on only four eigenvectors of the \( \Sigma \) matrix. These four eigenvectors explain almost the entire variations in \( \mathbf{v}' \) and they jointly capture five clusters of sectors in the economy: 1) heavy manufacturing sectors including iron, steel, and machineries; 2) light manufacturing sectors of consumer

\(^{11}\)13 sectors from the original 184-by-184 BLS input-output table do not use or supply any intermediate inputs and therefore do not interact with the rest of the network. These sectors are all in services, including offices of dentists, individual family services, home health care services, etc. We drop these sectors when performing the eigendecomposition.
products including food and textiles; 3) chemicals; 4) agency, brokerage, and insurance; 5) entertainment. We show such a factor structure emerges only when assessing the impact of temporary shocks. In contrast, the economy does not have a low-dimensional, factor representation for permanent shocks, as the Domar weights have significant loadings on over 150 eigenvectors.

5.1 Computing and Interpreting the Welfare Impact $v$ of Temporary Shocks

Calibrating Adjustment Costs To compute the welfare impact measure $v$, we set $\rho = 4\%$ as the annual discount rate. To calibrate the adjustment cost, we rely on the time-to-build microfoundation in Section 3.5, under which $\delta$ corresponds to the average delay between when inputs are ordered and when they are delivered and can be measured using the backlog ratio, i.e., the ratio between the stock value of unfilled orders and the flow value of goods delivered (Zarnowitz (1962), Meier (2020)). We further rely on our analysis in Section 3.6 and allow for input (i.e., seller) specific adjustment costs. We measure the each sector’s backlog ratio using the seasonally-adjusted value from the U.S. Census M3 survey of manufacturers’ shipments, inventories, and orders, which provides broad-based, monthly statistical data on economic conditions in the manufacturing sector. For each sector, we compute the average backlog ratio between the years 2010 and 2019, excluding the spike in backlogs due to the COVID-19 pandemic. We impute the backlog ratio using the sample average for input-output sectors that are not in the M3 survey. The cross-sector average backlog ratio is about 3.2 months, corresponding to $\delta_j = 0.27$ at the annual frequency. The standard deviation is 2.55 months. Sectors with the highest backlog ratios are ship and boat building (11.3 months), manufacturing of transportation equipment (10.3 months), and communications equipment manufacturing (8.2 months).

Interpreting the Welfare Impact Measure Table 1 lists the top-10 most important and least important sectors for the U.S. in terms of the welfare impact of temporary sectoral shocks. As our intuitions suggest, the most important sectors are large and supply to many other producers. The top-10 list includes very large sectors such as real estate and wholesale trade, whose sales-to-GDP ratios add to 24%. The list also includes much smaller but very upstream manufacturing sectors such as chemical and metal sectors, with sales-to-GDP ratio of only 1.9% and 1.2%, respectively. In static models, their small Domar weights would imply these sectors are unimportant; in our dynamic environment, by contrast, shocks to these sectors could create long-lasting impact because of their network positions. On the right
sectoral shocks. As intuitions suggest, the most important ones are large sectors that supply
inputs and therefore do not interact with the rest of the network. These sectors are all in services, including
many service sectors.

To compute the welfare sensitivity measure

\frac{\delta}{\gamma} \equiv 4\% \quad \frac{\delta}{\gamma} \equiv 4\%

The left panel of Table 2 lists the top-10 sectors in terms of Domar weights
\gamma_i, which capture the welfare impact of permanent shocks. Compared with \nu, the Domar weight \gamma
captures sectoral size but disregards upstreamness. There are three sectors (wholesale trade,

<table>
<thead>
<tr>
<th>10 sectors with the highest \nu_i</th>
<th>10 sectors with the smallest \nu_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real estate</td>
<td>Community and vocational rehabilitation services</td>
</tr>
<tr>
<td>Motor vehicle parts manufacturing</td>
<td>Other furniture related product manufacturing</td>
</tr>
<tr>
<td>Wholesale trade</td>
<td>Gambling industries (except casino hotels)</td>
</tr>
<tr>
<td>Agencies, brokerages, etc.</td>
<td>Personal care services</td>
</tr>
<tr>
<td>Oil and gas extraction</td>
<td>Amusement parks and arcades</td>
</tr>
<tr>
<td>Management of companies enterprises</td>
<td>Grantmaking, giving services, social advocacy organizations</td>
</tr>
<tr>
<td>Advertising, public relations, etc</td>
<td>Food and beverage stores</td>
</tr>
<tr>
<td>Basic chemical manufacturing</td>
<td>Tobacco manufacturing</td>
</tr>
<tr>
<td>Government services</td>
<td>Household appliance manufacturing</td>
</tr>
<tr>
<td>Petroleum and coal products</td>
<td>Furniture and kitchen cabinet manufacturing</td>
</tr>
</tbody>
</table>

side of the table, sectors with low welfare impact are those that are small and downstream, including many service sectors.

<table>
<thead>
<tr>
<th>10 sectors with the highest \gamma_i</th>
<th>10 sectors with the smallest \gamma_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wholesale trade</td>
<td>Nonferrous metal (excl. aluminum) production</td>
</tr>
<tr>
<td>Real estate</td>
<td>Forestry</td>
</tr>
<tr>
<td>Construction</td>
<td>Basic chemical manufacturing</td>
</tr>
<tr>
<td>Hospitals</td>
<td>Agencies, brokerages, insurance related activities</td>
</tr>
<tr>
<td>Retail</td>
<td>Metal ore mining</td>
</tr>
<tr>
<td>Food services and drinking places</td>
<td>Processed steel products</td>
</tr>
<tr>
<td>Petroleum and coal products</td>
<td>Support activities for agriculture and forestry</td>
</tr>
<tr>
<td>Insurance carriers</td>
<td>Logging</td>
</tr>
<tr>
<td>Scientific R&amp;D</td>
<td>Iron and steel mills and ferroalloy manufacturing</td>
</tr>
<tr>
<td>Finance (securities, commodity contracts, funds)</td>
<td>Alumina and aluminum production and processing</td>
</tr>
</tbody>
</table>

The left panel of Table 2 lists the top-10 sectors in terms of Domar weights \gamma, which capture the welfare impact of permanent shocks. Compared with \nu, the Domar weight \gamma
captures sectoral size but disregards upstreamness. There are three sectors (wholesale trade,
real estate, and petroleum and coal products manufacturing) that are on both top-10 lists, whereas seven out of ten sectors do not overlap.

The right panel of Table 2 lists the top-10 sectors in terms of size-adjusted welfare impact of temporary shocks, $\eta_i \equiv v_i / \gamma_i$. Recall $\eta_i$ is a generalized Katz centrality; high-$\eta_i$ sectors are those whose products travel through many high-adjustment-cost producers, directly and indirectly, before reaching the consumer. The table shows that many basic manufacturing sectors have high $\eta$'s: shocks to these sectors are especially damaging relative to sectoral size because these sectors are very upstream.

Under these calibration, we find the average half-life of sectoral shocks—calculated as the minimum time needed for the GDP to recover 50% of the initial loss—to be 4.8 months, with a standard deviation of 1.8 months.

**Figure 5.** Time path of GDP losses after sectoral shocks

![GDP losses over time](image)

Figure 5 shows the time path of GDP losses in the calibrated model from temporarily shocking six sectors, one at a time. The six sectors are separated into two groups: the first group (grey lines) consists of finance, oil and gas extraction, petroleum and coal products manufacturing; the second group (black lines) consists of the manufacturing of communication equipments, motor vehicles, and motor vehicle parts. Sectors in the first group have high Domar weights and are traditionally viewed as “important”—their (3 sectors out of 171) Domar weights add to about 12% of GDP—whereas the latter three heavy-manufacturing sectors (black lines in the figure) are relatively upstream. To control for sectoral size differ-
ences and isolate the dynamic effects, we normalize the GDP losses at time-zero to be -100% for the shock to each sector.\footnote{Note that the magnitude of the time-zero GDP loss from a shock to sector $i$ is equal to the size of the shock times the sector’s Domar weight ($\ln c(0) - \ln c^T = \gamma_i \times \tilde{z}_i$). The two sectors, “motor vehicle manufacturing” and “oil and gas extraction”, have near-identical Domar weights; hence, TFP shocks of the same magnitude to these two sectors would produce near identical instantaneous effects but qualitatively different dynamic effects, as shown in Figure 5.}

The figure shows that, because adjustment costs compound, upstream sectors do experience marked real-world differences in the recovery dynamics from sectoral shocks. The half-lives of sectoral shocks to the first group average to 4.6 months, whereas the half-lives of the second group is more than twice as long and average to 9.5 months. For TFP shocks to the first group, the GDP loss one year after TFP recovery averages to 17% of the initial loss, whereas the GDP loss remains at 41% of the initial loss one year after TFP recovery of the second group. That is, conditioning on the same initial impact, the one-year GDP loss after TFP recovery from shocks to the second group is 2.4 times as large as the corresponding loss from shocks to the first group. The relative impact widens as time progresses: the three-year GDP loss after TFP recovery from shocks to the second group is 7.6 times as large as the corresponding loss from shocks to the first group.

Quantitative Evaluation of Adjustment Cost Heterogeneity

Our baseline model features a homogeneous adjustment cost parameter $\delta$, whereas our calibration so far is based on the heterogeneous adjustment cost extension in Section 3.6 and exploits the sectoral heterogeneity in 3M survey. We now show that the heterogeneity in adjustment costs is not quantitatively important; most variations in the welfare impact of temporary shocks arise from the network structure.

To demonstrate this, we construct a welfare impact measure $v^{\text{base}}$ with homogeneous adjustment cost $\delta = 0.27$ for all input-pairs (calibrated to match the mean backlog ratio of 3.2 months in the 3M survey), and define $\eta_i^{\text{base}} = v_i^{\text{base}} / \gamma_i$ as the corresponding Katz centrality. All cross-sector variations in the $\eta_i^{\text{base}}$ arise from the network structure.

Table 3 shows the pair-wise correlations (Pearson correlations below the diagonal and Spearman’s rank correlations above the diagonal) among $\eta$ (heterogeneous adjustment cost), $\eta^{\text{base}}$ (constant adjustment cost), and the upstreamness measure of Antràs et al. (2012). All three measures are near-perfectly correlated, showing that, even with heterogeneous adjustment costs, most variations in the size-adjusted welfare impact arises from the network structure.

High-$\eta$ sectors are those whose output travels through many high-adjustment-cost producers, directly and indirectly, before reaching the consumer. Why is the heterogeneity in
Table 3. Size-adjusted welfare impact correlates strongly with upstreamness

<table>
<thead>
<tr>
<th></th>
<th>η</th>
<th>η_base</th>
<th>Up</th>
</tr>
</thead>
<tbody>
<tr>
<td>η</td>
<td>-</td>
<td>0.96</td>
<td>0.96</td>
</tr>
<tr>
<td>η_base</td>
<td>0.92</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Up</td>
<td>0.92</td>
<td>1.0</td>
<td>-</td>
</tr>
</tbody>
</table>

Notes. This table shows the pair-wise correlations among η (heterogeneous adjustment cost), η_base (constant adjustment cost), and the upstreamness measure of Antrás et al. (2012). Pearson correlations are shown below the diagonal, and Spearman’s rank correlation are shown above the diagonal.

Figure 6. The input-output demand matrix of the U.S. economy

adjustment costs δ quantitatively unimportant? The answer lies in the structure of the U.S. input-output network. As discussed in Section 3.6, because adjustment costs compound through input-output linkages, a relatively upstream sector in a vertical network always has higher η, regardless of the heterogeneity in adjustment costs. Even though the U.S. economy is not a vertical network, the “compounding” intuition still applies. Figure 6 visualizes the U.S. input-output demand matrix Θ, with sectors sorted by descending upstreamness. For ease of visualization, entries are drawn in proportion to θ_{ij} and are truncated below at 4%, so that only important linkages are shown.

Figure 6 shows a striking feature of the U.S. input-output network. Once sectors are
sorted by upstreamness, the network appears hierarchical: sectors exhibit a clear pecking order and have highly asymmetric input-output relationships. The downstream sectors purchase heavily from the upstream ones—but not the reverse—as the matrix is dense below the diagonal and sparse above. A hierarchical structure is also evident below the diagonal, as upstream inputs are used more heavily by relatively upstream producers than by downstream producers.

In such a hierarchical network, the bulk of input flows are one directional, such that the most upstream sector’s output is also subject to the most compounding of adjustment costs. This is why the size-adjusted welfare sensitivity aligns very well with sectoral upstreamness, and the heterogeneity in adjustment costs matter little. Perhaps interestingly, the hierarchical feature is not special of the U.S. production network: Liu (2019) notes the similar hierarchical feature in the input-output tables of China and South Korea, and Dhyne, Kikkawa, Kong, Mogstad, and Tintelnot (2022) show the Belgium production network can be well-approximated by an acyclic network, which is a network with a lower-triangular input-output demand matrix $\Theta$.

In summary, most variations in the welfare impact—size-adjusted or not—arise from the network structure and not from the heterogeneity in adjustment costs. For expositional simplicity and to further isolate the role of the network structure, in subsequent analysis we adopt $v^{\text{base}}$ as our baseline measure of the welfare impact of temporary shocks, with a single, economy-wide adjustment cost parameter $\delta = 0.27$.

### 5.2 Factor Structure of the U.S. Input-Output Table

We now describe the next empirical results of the paper: the welfare impact $v$ of temporary shocks in the U.S. can be well-approximated in a low-dimensional sub-eigenspace.

Recall Proposition 3 shows that the welfare impact $v$ summarizes the infinite rounds of general equilibrium, network effects along the recovery path of the economy. Proposition 9 converts $v$ from an infinite summation across general equilibrium effects into a finite summation across eigencomponents (right- and left-eigenvectors, $u_k$ and $w'_k$), each of which captures a particular shock profile that features constant decay through each round of general equilibrium effects:

$$v' = \beta' \sum_{k=1}^{N} \frac{\delta \lambda_k}{(1 - \lambda_k) (1 + \rho \delta - \lambda_k)} u_k w'_k.$$  \hspace{1cm} (28)

We now show that four eigencomponents (out of $N = 171$) capture most of the variation in $v$, suggesting that these four shock profiles are all that is needed to summarize the welfare impact of temporary shocks in our dynamic model. This is not the case for permanent shocks.
in our model (or shocks in static models): the Domar weight does not have a low-dimensional representation.

Specifically, let $v'_h \equiv \delta' \sum_{k=1}^{h} \frac{\lambda_k}{(1-\rho(1-\lambda_k))} u_k w'_k$ denote the partial sum of the first $h$ eigencomponents in (28), with $v = v_{(17)}$. $v_{(h)}$ captures the welfare impact of sectoral shocks through the first $h$ eigencomponents. We show that $v_{(4)}$ approximates $v$ very well.

Table 4 shows the regression of $v_{(h)}$ on $v$ for $h \in \{1, \ldots, 6\}$ and reports the slope coefficients and adjusted $R^2$. The first 3 eigenvectors capture 76% of the variation in $v$; the first 4 eigenvectors capture 95% of the variation. That is, most of the welfare impact of any sectoral shock can be explained by the loading of the shock on the first four eigenvectors.

**Table 4. Regression of $v_{(h)}$ on $v$**

<table>
<thead>
<tr>
<th>$h$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>slope</td>
<td>0.53</td>
<td>0.82</td>
<td>1.01</td>
<td>0.97</td>
<td>0.97</td>
<td>0.96</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.39</td>
<td>0.58</td>
<td>0.76</td>
<td>0.95</td>
<td>0.96</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Figure 7 scatter plots $v_{(h)}$ against $v$ for $h \leq 6$; the red line in each panel is the 45-degree line. As the figure shows, $v'_{(4)}$ approximates $v'$ very well, and additional eigencomponents do not seem to significantly improve the fit.

**Figure 7.** Welfare impact from the first $h$ eigencomponents ($v_{(h)}$) plotted against $v$

Table 5 and Figure 8 conduct the analogous exercise for the Domar weight. We compute the partial sum $\gamma'_{(h)} = \beta' \sum_{k=1}^{h} \frac{1}{1-\lambda_k} u_k w'_k$ as the first $h$ eigencomponents of Domar weights.
(with the actual Domar weight $\gamma = \gamma_{(171)}$ c.f. equation 23). Table 5 shows that the first four eigencomponents explain almost no variation in Domar weights, with an $R^2$ close to zero. In fact, the $R^2$ remains close to zero even with 160 eigencomponents of the largest eigenvalues, and almost all eigencomponents are needed to explain variations in Domar weights. Figure 8 shows that Domar weights are almost orthogonal to approximations of up to $h \leq 6$ dimensions.

<table>
<thead>
<tr>
<th>$h$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$\cdots$</th>
<th>160</th>
<th>170</th>
</tr>
</thead>
<tbody>
<tr>
<td>slope</td>
<td>0.02</td>
<td>0.03</td>
<td>0.04</td>
<td>0.04</td>
<td>$\cdots$</td>
<td>0.47</td>
<td>1.14</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>$\cdots$</td>
<td>0.03</td>
<td>0.62</td>
</tr>
</tbody>
</table>

**Figure 8.** Domar weights from the first $h$ eigencomponents ($\gamma_{(h)}$) plotted against $\gamma$

These results imply that, to understand the dynamic impact of temporary shocks, only four shock profiles are needed to approximate the impact of all (171 dimensional) shocks. By contrast, such a low-dimensional representation does not exists for the Domar weights; hence, to understand the impact of permanent shocks, one needs information of the entire input-output table.

Figure 9 contrasts the contribution of each eigencomponent to the Domar weight $\gamma$—as determinants of the welfare response to permanent shocks—and to $v$—as determinants of the
Figure 9. The contribution of each eigencomponent to $\gamma$ (Domar weight) and $v$ (welfare impact of temporary shocks) relative to the first component.

welfare response to temporary shocks. Each point in the figure represents an eigencomponent $k \in \{1, \ldots, 171\}$. The X-coordinate is $|v'u_k|$, the $k$-th eigencomponent’s contribution to $v$. The Y-coordinate is $|\gamma' u_k|$, the $k$-th eigencomponent’s contribution to $\gamma$. For the two axes to be comparable, we normalize the contribution of the most important eigencomponent to 100% on each axis.

The figure shows that there are only four eigencomponents (black circles) with X-coordinates above 10%, meaning the fifth most important eigencomponent contributes to than 10% as the most important component for the welfare impact $v$ of temporary shocks. By contrast, for permanent shocks, a large number (101 out of 171) of eigencomponents have their contributions exceeding 10% of the most important component.

Section 4 has alluded to the reason behind the low-dimensional representation for $v$, and the lack thereof for the Domar weight. Intuitively, the Domar weight represents an non-discounted summation through the infinite rounds of network effects ($I + \Sigma + \Sigma^2 + \cdots$); the initial rounds are equally important as the later rounds. By contrast, the welfare impact of temporary shocks represents a weighted summation ($((1 - 1) I + \left(1 - \frac{1}{1+\rho}\right) \Sigma + \left(1 - \frac{1}{(1+\rho)^2}\right) \Sigma^2 + \cdots$) with increasing weights on later rounds of network effects, as they represent long-lasting damages due to adjustment costs, and initial rounds are heavily underweighted. An eigencomponent with a small eigenvalue—meaning it represents a dimension of general equilibrium effect that decays quickly over iterative powers of the input-output
matrix—has little impact on welfare if the shocks are temporary (for the component’s lack of impact at higher rounds) and but is potentially very important if the shocks are permanent.

**Which sectors do the first four eigenvectors represent?** Recall that each eigenvector $u_k$ represents a TFP shock profile, under which the network effects decay at exponentially at rate $\lambda_k$ and that the cumulative welfare impact is $-v'u_k = -\frac{\lambda_k}{(1-\lambda_k)(1+\rho\delta-\lambda_k)}\beta'u_k$.

**Figure 10.** The first four eigenvectors of $\Sigma$

Figure 10 visualizes the first four eigenvectors, which can be used to capture the welfare impact of most temporary shocks. The X-axis represent the sectoral ordering according to the BLS input-output table, which roughly arranges broad sector groups by agriculture, food manufacturing, chemical products, metals, heavy manufacturing, and services. In the figure, we indicate the broad groups of sectors that these eigenvectors represent; Tables 6 and 7 in the Appendix provide more detailed lists of sector names. The first eigenvector $u_1$ represents shocks to the heavy manufacturing sectors, including metal products, foundries, forging and stamping, and as well as the production of boiler tanks, machinery, electrical and transportation equipment. This eigenvector captures the vector of TFP shocks under
which the economic damage to GDP lasts the longest time after TFP recovers.

The second eigenvector $u_2$ negatively correlates with the first (Pearson correlation coefficient of $-0.59$). $u_2$ represents shocks to three groups of industries. First and most notably, $u_2$ has large positive entries for the two sectors relating to agencies, brokerages, and insurance. Second, $u_2$ has positive entries for the manufacturing of consumer goods including food, textile, paper products, and furniture. Third, $u_2$ has negative entries on the heavy manufacturing industries, partly neutralizing the shock profile from the first eigenvector.

The third eigenvector $u_3$ correlates positively with $u_2$—correlation coefficient 0.36—by having positive entries on the manufacturing of consumer goods. In addition, $u_3$ also includes sectors that manufacture chemicals, plastic, and rubber products.

The fourth eigenvector has close-to-zero correlations with the previous three eigenvectors. The new sector picked up by $u_4$ is radio and television broadcasting; in addition, $u_4$ also has negative entries on the manufacturing of chemicals, plastic, and rubber products, partly neutralizing the shock profiles represented by $u_3$.

Altogether, the eigenvectors $u_1$ through $u_4$ form a 4-dimensional subspace of the 171-dimensional vector space in which the U.S. input-output table lies. It may appear puzzling at first that the sectors represented by these four eigenvectors do not seem to coincide with the sectors with high welfare impacts as listed in Table 1. This is not an inconsistency: the welfare impact of any temporary TFP shock vector $\tilde{z}$ can be well-approximated by projecting $\tilde{z}$ onto this subspace, $v'\tilde{z} \approx \sum_{k=1}^{4} v'u_k a_k$, i.e., approximating $\tilde{z}$ with a linear combinations of $u_1$ through $u_4$, with coordinates $a_k = w_k'\tilde{z}$ obtained using the corresponding left-eigenvectors. Appendix Table 8 shows the 4-dimensional coordinates for shocking each of the 10 sectors with the highest welfare impact individually and no other sectors. As an example, a shock to the sector “Agencies, brokerages, and other insurance related activities” has a positive coordinate on $u_2$, which picks up shocks to this sector very strongly but also shocks to heavy manufacturing products (negatively) and consumer goods (positively); see Figure 10. To isolate the shock to agencies, brokerages and insurance, the shock loads positively on $u_1$ and negatively on $u_3$ to neutralize the other sectors picked up by $u_2$.

6 Conclusion

The dynamical view of the input-output matrix that is inherent in spectral graph theory and dynamical system theory can reveal important determinants of the structure of production networks. We have built one such dynamical system—a microfounded general equilibrium model with adjustment frictions in which economy gradually transition back to the steady-state after temporary TFP shocks. This analysis is useful to reveal the structure of the input-output matrix—varying the degree of the frictions and thus the speed of the adjustment
allows us to have a multi-scale representation of the economy. These scales represent the varying importance of the higher order links and the associated eigenvalues and eigenvectors.

References


Appendix

A  Proofs

A.1 Proof to Lemma 1

Consider the planner’s problem in (6). We use the following change of variables: let $v_{ij}(t) \equiv s_{ij}(t)/q_j(t)$ denote the fraction of good $j$ sent to sector $i$ at time $t$, and let $n_{ij}(t) \equiv \ln n_{ij}(t)$. Then consumption of good $j$ is $c_j(t) = (1 - \sum_i v_{ij}(t)) q_j(t)$. Taking logs of the production function in (3) and recognizing that TFP is constant during the recovery path, we can equivalently write the planner’s problem as

$$V(\{m_{ij}(0)\}) = \max_{\{\ell_j, v_{ij}\}} \int e^{-\rho t} \sum_j \beta_j \left( \alpha_j \ln \ell_j(t) + \sum_k \sigma_{jk} x_{jk}(t) + \ln \left( 1 - \sum_i v_{ij}(t) \right) \right) dt$$

s.t. $\dot{x}_{ij}(t) = \delta^{-1} \left( \ln v_{ij}(t) + \alpha_j \ln \ell_j + \sum_k \sigma_{jk} x_{jk} - x_{ij}(t) \right)$

$$\sum_j \ell_j = \bar{\ell}$$

In what follows, we omit the time argument whenever the context is clear. Form the current-value Hamiltonian, where for notational simplicity we supress the dependence on time for the control, state, and co-state variables:

$$H(\{\ell_j\}, \{x_{jk}\}, \{v_{ij}\}, t) = \sum_j \beta_j \left( \alpha_j \ln \ell_j + \sum_k \sigma_{jk} x_{jk} + \ln \left( 1 - \sum_i v_{ij} \right) \right)$$

$$+ \delta^{-1} \sum_{ij} \mu_{ij} \left( \ln v_{ij} + \alpha_j \ln \ell_j + \sum_k \sigma_{jk} x_{jk} - x_{ij} \right) + \lambda \left[ \bar{\ell} - \sum_j \ell_j \right].$$

By the maximum principle,

$$H_{\ell_j} = 0 \iff \frac{\alpha_j (\beta_j + \delta^{-1} \sum_i \mu_{ij})}{\ell_j} = \lambda \text{ for all } j. \quad (29)$$

$$H_{v_{ij}} = 0 \iff \frac{\beta_j}{1 - \sum_i v_{ij}} = \frac{\mu_{ij} \delta^{-1}}{v_{ij}} \quad (30)$$

$$H_{x_{jk}} = \rho \mu_{jk} - \dot{\mu}_{jk} \iff \beta_j \sigma_{jk} - \mu_{jk} \delta^{-1} + \delta^{-1} \sum_i \mu_{ij} \sigma_{jk} = \rho \mu_{jk} - \dot{\mu}_{jk} \quad (31)$$

We now show that the transversality condition $\lim_{t \to \infty} e^{-\rho t} H(\{\ell_j\}, \{x_{jk}\}, \{v_{ij}\}, t) = 0$ implies $\dot{\mu}_{jk}(t) = 0$ for all $j, k, t$; the Lemma is then immediate, i.e., $v_{ij}$ and $\ell_j$ are time-invariant for all $i, j$. 

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To show $\dot{\mu}_{jk}(t) = 0$ for all $j, k, t$, we proceed in two steps. In the first step, we define $\xi_k \equiv \sum_j \mu_{jk}$ and show $\dot{\xi}_k(t) = 0$ for all $k, t$. We then show $\dot{\mu}_{jk}(t) = 0$.

Using the definition of $\xi_k$, we sum both sides of equation (31) across $j$ and get

$$\sum_j \beta_j \sigma_{jk} - \xi_k \delta^{-1} + \delta^{-1} \sum_j \xi_j \sigma_{jk} = \rho \xi_k - \dot{\xi}_k$$

In vector notation,

$$\dot{\xi} = \delta^{-1} (I - \Sigma' + \rho \delta) \xi - \Sigma' \beta$$

Thus

$$\xi(t) = e^{\delta^{-1} (I - \Sigma' + \rho \delta)t} \xi_0 - \int_0^t e^{\delta^{-1} (I - \Sigma' + \rho \delta)(t-s)} ds \Sigma' \beta$$

Transversality implies

$$0 = \lim_{t \to \infty} e^{-\rho t} \xi(t) = \lim_{t \to \infty} e^{\delta^{-1} (I - \Sigma')t} \left[ \xi_0 - \delta (I - \Sigma' + \rho \delta)^{-1} \Sigma' \beta \right]$$

which is true only if

$$\xi_0 = \delta (I - \Sigma' + \rho \delta)^{-1} \Sigma' \beta$$

thereby implying $\dot{\xi}(t) = 0$ for all $t$.

We now show $\dot{\mu}_{ij}(t) = 0$ for all $t$. Note equation (31) can be written as

$$\beta_j \sigma_{jk} - \mu_{jk} \delta^{-1} + \delta^{-1} \xi_j \sigma_{jk} = \rho \mu_{jk} - \dot{\mu}_{jk}$$

or in matrix form

$$\dot{\mu} = (\delta^{-1} + \rho) \mu - (\beta' + \delta^{-1} \xi') \Sigma$$

Following the same strategy above, we can integrate and write

$$\mu(t) = e^{(\delta^{-1} + \rho) t} \mu_0 - (\delta^{-1} + \rho)^{-1} \left( e^{(\delta^{-1} + \rho) t} - I \right) (\beta' + \delta^{-1} \xi') \Sigma$$

The transversality condition $0 = \lim_{t \to \infty} e^{-\rho t} \mu(t)$ again implies

$$(\delta^{-1} + \rho) \mu(0) = (\beta' + \delta^{-1} \xi') \Sigma$$

and thus $\dot{\mu}(t) = 0$ for all $t$.

We now characterize the model further using the planner’s solution and provide an alternative proof to our main result, Proposition 3. This characterization will be useful when we prove Proposition 6 below, which considers non-parametric production function and adjustment cost process. First, note that $\mu_{ij}$ is the marginal value of an additional unit of $m_{ij}$ in place at time 0. Under the log-linear baseline model, $\mu_{ij}$ is time-invariant. We can actually
solve for $\mu_{ij}$ in closed-form. Using (31), note
\[
\delta \beta_j + \sum_i \mu_{ij} = (1 + \rho \delta) \mu_{jk}/\sigma_{jk}
\]
Which shows $\mu_{jk}/\sigma_{jk}$ is independent of $k$. Define $\gamma_j \equiv \mu_{jk}/\sigma_{jk}$; then
\[
\delta \beta_j + \sum_i \gamma_i \sigma_{ij} = (1 + \rho \delta) \gamma_j
\]
In matrix notation,
\[
(1 + \rho \delta) \gamma' = \delta \beta' + \gamma' \Sigma \iff \gamma' = \delta \beta' (I - \Sigma)^{-1}.
\]
To figure out the welfare impact of a temporary TFP shock, note that the impact of shock $\tilde{z}$ on state variables $\ln m_{ij}$ at time 0 is the $j$-th entry of vector $(I - \Sigma)^{-1} \tilde{z}$; that is, the TFP shock affects $m_{ij}$ to the same proportion as it affects the output of sector $j$. Hence, the impact of $\tilde{z}$ on welfare is
\[
V(\tilde{z}) - V^{ss} = - \sum_{ij} \mu_{ij} [(I - \Sigma)^{-1} \tilde{z}]_j
\]
\[
= - \sum_{ij} \gamma_i \sigma_{ij} [(I - \Sigma)^{-1} \tilde{z}]_j
\]
\[
= - \gamma' \Sigma (I - \Sigma)^{-1} \tilde{z}
\]
\[
= - \delta \beta' ((1 + \rho \delta) I - \Sigma)^{-1} \Sigma (I - \Sigma)^{-1} \tilde{z}
\]
\[
= - \frac{1}{\rho} \beta' [(I - \Sigma)^{-1} - ((1 + \rho \delta) I - \Sigma)^{-1}] \Sigma \tilde{z}
\]
\[
= - \frac{1}{\rho} \beta' \left[ (I - \Sigma)^{-1} - \left( I - \frac{\Sigma}{1 + \rho \delta} \right)^{-1} \right] \tilde{z},
\]
as Proposition 3 claims.

A.2 Proof to Lemma 2

We need to show $x_j(t) = \ln s_{ij}(t) - \ln m_{ij}(t)$ for all $i$. First note that the time-0 impact of a negative TFP shock on the state variables—the quantity of intermediate inputs $\{m_{ij}\}$—is captured by the impact on the output of intermediate suppliers. That is, $\ln m_{ij}(0) - \ln m_{ij}^{ss}$ is the same across all input-buyers $i$. Second, because $s_{ij}(t)/q_{ij}(t)$ is time-invariant (Lemma 1), we know $\ln s_{ij}(t) - \ln s_{ij}^{ss}$ is the same for all $i$. Integrating the law of motion (8), these two facts imply $\ln s_{ij}(t) - \ln m_{ij}(t)$ is independent of $i$ for all $t$, and that $\sum_{i'} m_{i'j}(t) = \sum_{i'} s_{i'j}(t)$ and both sides are time-invariant, implying $\ln s_{ij}(t) - \ln m_{ij}(t) = \ln \sum_{i'} s_{i'j}(t) - \ln \sum_{i'} m_{i'j}(t)$ for all $i$ and $t$, establishing the Lemma.
A.3 Proof to Proposition 2

We have

\[ \ln q(t) = \ln q(0) + \delta^{-1} \sum_t \int_0^t \mathbf{x}(s) \, ds \]

\[ = \ln q(0) + \delta^{-1} \sum_t \int_0^t e^{-\delta^{-1} (I-\Sigma)t} \, ds \]

\[ = \ln q^{ss} - (I-\Sigma)^{-1} \tilde{z} + \tilde{z} + \Sigma (I-\Sigma)^{-1} \left( I - e^{-\delta^{-1} (I-\Sigma)t} \right) \tilde{z} \]

\[ = \ln q^{ss} - \Sigma (I-\Sigma)^{-1} e^{-\delta^{-1} (I-\Sigma)t} \tilde{z}. \]

The expression for \( c(t) \) is derived analogously.

A.4 Proof to Proposition 3

We have

\[ \begin{align*}
V(\tilde{z}) - V^{ss}_0 &= \int_0^\infty e^{-\rho s} (\ln c(s) - \ln c^{ss}_0) \, ds \\
&= -\beta' \Sigma (I-\Sigma)^{-1} \int_0^\infty e^{-\delta^{-1} ((1+\rho\delta)I-\Sigma)t} \, dt \tilde{z} \\
&= -\delta\beta' \Sigma (I-\Sigma)^{-1} \left( (1+\rho\delta)I-\Sigma \right)^{-1} \tilde{z} \\
&= \frac{1}{\rho} \beta' [ (I-\Sigma)^{-1} - (1+\rho\delta)I^{-1} ] \tilde{z} \\
&= -\frac{1}{\rho} \left[ \beta' (I-\Sigma)^{-1} - \beta' \left( I - \frac{\Sigma}{1+\rho\delta} \right)^{-1} \right] \tilde{z}.
\end{align*} \]

A.5 Proof to Proposition 4

Let \( r \equiv \frac{1}{1+\rho\delta} \) and \( a' \equiv \beta' (I-r\Sigma)^{-1} \). We have

\[ \rho \eta_j = 1 - a_j/\gamma_j = 1 - \beta_j/\gamma_j - r \sum_i \sigma_{ij}a_i/\gamma_j = \sum_i \theta_{ji} - r \sum_i \theta_{ji} (1 - \rho \eta_k) \]

\[ \Rightarrow \eta = \delta r \Theta 1 + r \Theta \eta = \delta (I-r\Theta)^{-1} r \Theta 1 = \delta \left[ \sum_{s=1}^\infty \left( \frac{1}{1+\rho\delta} \Theta \right)^s \right] 1. \]

A.6 Proof to Proposition 5

Suppose adjustment cost is sector-pair-specific \( (\delta_{ij}) \). Following the proof to Lemma 1 in Appendix Section A.1, one can setup the Hamiltonian and find

\[ \beta_j + \sum_i \delta^{-1}_{ij} \mu_{ij} = (\delta^{-1}_{jk} + \rho) \mu_{jk} / \sigma_{jk} \quad (32) \]
Let \( \tilde{\gamma}_j \equiv (\delta_{jk}^{-1} + \rho) \mu_{jk}/\sigma_{jk} \), the previous equation becomes
\[
\beta_j + \sum_i \frac{1}{1 + \rho \delta_{ij}} \tilde{\gamma}_i \sigma_{ij} = \tilde{\gamma}_j \iff \tilde{\gamma}' = \beta' (I - \Omega)^{-1}
\]
where \( \Omega_{ij} \equiv \frac{\sigma_{ij}}{1 + \rho \delta_{ij}} \). Let \( \delta \circ \Omega \) be the matrix whose \( ij \)-th entry is \( \delta_{ij} \sigma_{ij} \). The welfare impact is
\[
V (\tilde{z}) - V^{ss} = -\sum_{ij} \mu_{ij} \left[ (I - \Sigma)^{-1} \tilde{z} \right]_j
\]
\[
= -\sum_{ij} \frac{\delta_{ij}}{1 + \rho \delta_{ij}} \tilde{\gamma}_i \sigma_{ij} \left[(I - \Sigma)^{-1} \tilde{z} \right]_j
\]
\[
= \tilde{\gamma}' (\delta \circ \Omega) (I - \Sigma)^{-1} \tilde{z}
\]
\[
= \beta' (I - \Omega)^{-1} (\delta \circ \Omega) (I - \Sigma)^{-1} \tilde{z}
\]
\[
= -\frac{1}{\rho} \beta' \left[(I - \Sigma)^{-1} - (I - \Omega)^{-1} \right] \tilde{z}, \quad \text{QED.}
\]

A.7 Proof to Proposition 6

We follow the proof strategy in Appendix Section A.1. Setup the planner's problem as
\[
\max_{\{s_{ij}(\cdot)\}} \int e^{-\rho t} \ln c \left\{ \ln f_j (\ell, \{x_{jk}\}) + \ln \left(1 - \sum_i v_{ij}\right) \right\}_{j=1}^N \ dt
\]
\[
\text{s.t. } \dot{x}_{ij} = g \left(\ln v_{ij} + \ln f (\ell, \{x_{jk}\}, x_{ij})\right)
\]
\[
\sum_j \ell_j = \bar{\ell}
\]
where recall \( x_{jk} \equiv \ln m_{jk} \), and \( v_{ij} \equiv s_{ij}/q_j \) is the share of good \( j \) sent to producer \( i \). Note that writing the law of motion in logs is without loss of generality; the only requirement on \( g \) is that it is locally homogeneous of degree one when the two arguments are equal; the property holds in logs if and only if it also holds in levels. Let us form the current-value Hamiltonian:
\[
H (\{\ell_j\}, \{x_{jk}\}, \{v_{ij}\}, t) = \ln c \left\{ \ln f_j (\ell, \{x_{jk}\}) + \ln \left(1 - \sum_i v_{ij}\right) \right\}_{j=1}^N
\]
\[
+ \sum_{ij} \mu_{ij} g \left(\ln v_{ij} + \ln f (\ell, \{x_{jk}\}, x_{ij})\right) + \lambda \left[\bar{\ell} - \sum_j \ell_j \right].
\]
Let \( \omega_{ij}^{-1} \equiv g_1 (\ln v_{ij} + \ln f (\ell, \{x_{jk}\}, x_{ij})) \) denote the first partial derivative of \( g \) evaluated at the steady-state for the state variable \( ij \) and \( \xi_{ij}^{-1} \equiv g_2 (\ln v_{ij} + \ln f (\ell, \{x_{jk}\}, x_{ij})) \) denote the second partial derivative of \( g \). Let \( \alpha_j \equiv \frac{\partial \ln f_j}{\partial \ln \ell_j} \) and \( \sigma_{ij} \equiv \frac{\partial \ln f_j}{\partial \ln m_{ij}} \left(= \frac{\partial \ln f_i}{\partial x_{ij}}\right) \) respectively denote
labor and intermediate elasticities evaluated at the steady-state. Let \( \beta_j \equiv \frac{\partial \ln c}{\partial \ln c_j} \) denote the elasticity of the consumption aggregator with respect to consumption good \( j \) evaluated at the steady-state. In a steady-state, \( \beta_j \) coincides with the consumer expenditure share on good \( j \), \( \sigma_{ij} \) coincides with the intermediate expenditure share of producer \( i \) on good \( j \), and \( \alpha_j \) coincides on the labor share of producer \( j \); hence these reduced-form elasticities can all be directly observed in the data.

By the maximum principle,

\[
H_{\ell_j} = 0 \iff \frac{\alpha_j (\beta_j + \sum_i \mu_{ij} \omega_{ij}^{-1})}{\ell_j} = \lambda \quad \text{for all } j. \tag{33}
\]

\[
H_{v_{jk}} = 0 \iff \frac{\beta_j}{1 - \sum_i v_{ij}} = \frac{\mu_{jk} \omega_{jk}^{-1}}{v_{ij}} \tag{34}
\]

\[
H_{x_{jk}} = \rho \mu_{jk} - \dot{\mu}_{jk} \iff \beta_j \sigma_{jk} + \mu_{jk} \xi_{jk}^{-1} + \sum_i \mu_{ij} \omega_{ij}^{-1} \sigma_{jk} = \rho \mu_{jk} - \dot{\mu}_{jk} \tag{35}
\]

Evaluating the planner’s problem at the steady-state \( \{m_{ij}^*\} \), \( \dot{\mu}_{ij} = 0 \) for all \( i, j \), and \( g (\ln v_{ij} + \ln f (\ell_j, \{x_{jk}\})) \) 0. Given that \( s_{ij}^* = m_{ij}^* \) in a steady-state—hence \( \ln s_{ij}^* = x_{ij}^* \)—and by local-homogeneity and Euler’s theorem,

\[
\omega_{ij}^{-1} \ln s_{ij}^* + x_{ij}^* \xi_{ij}^{-1} = 0 \implies \omega_{ij} = -\xi_{ij} \quad \text{for all } i, j.
\]

Equation (35) implies

\[
\beta_j + \sum_i \mu_{ij} \omega_{ij}^{-1} = \frac{\mu_{jk}}{\sigma_{jk}} (\rho + \omega_{jk}^{-1}),
\]

which coincides with (32) and we can simply interpret \( \omega_{ij} \) as the inverse of the adjustment costs. Specifically, let \( \Omega \) denote the matrix whose \( ij \)-th entry is \( \frac{\sigma_{ij}}{1 + \rho \omega_{ij}} \), then around the steady-state, the non-parametric welfare elasticity to temporary shocks is

\[
\left. \frac{dV (\tilde{z}) - V^{ss}}{d\tilde{z}} \right|_{\tilde{z} = 0} = -\frac{1}{\rho} \beta' \left[ (I - \Sigma)^{-1} - (I - \Omega)^{-1} \right], \tag{36}
\]

which follows the Proof to Proposition 5 in Appendix Section A.6.

### A.8 Proof to Proposition 7

The result follows from applying Proposition 1 for time \( t \in [0, T) \) as sectoral output adjusts downwards with adjustment cost \( \delta \).
A.9 Proof to Proposition 8

\[ V(\tilde{z}; T, \tilde{\delta}, \delta) - V^{ss} = \int_0^T e^{-\rho s} (\ln c(s) - \ln c_0^{ss}) \, ds + e^{-\rho T} \int_0^\infty e^{-\rho s} (\ln c(T + s) - \ln c_0^{ss}) \, ds \]

\[ = -\beta'(I - \Sigma)^{-1} \int_0^T e^{-\rho s} \left( I - \Sigma e^{-\delta^{-1}((I - \Sigma)s)} \right) \, ds \tilde{z} \]

\[ - e^{-\rho T} \beta' \Sigma (I - \Sigma)^{-1} \int_0^\infty e^{-\rho s} \left( e^{-\delta^{-1}(I - \Sigma)s} \right) \, ds \left( I - e^{-\delta^{-1}(I - \Sigma)T} \right) \tilde{z} \]

\[ = -\beta'(I - \Sigma)^{-1} \left( \frac{1}{\rho} (1 - e^{-\rho T}) I - \Sigma \left[ \tilde{\delta} ((1 + \rho \tilde{\delta}) I - \Sigma)^{-1} \left( I - e^{-\delta^{-1}((1 + \rho \tilde{\delta})I - \Sigma)T} \right) \right] \right) \tilde{z} \]

\[ - e^{-\rho T} \beta' \Sigma (I - \Sigma)^{-1} \delta ((I + \rho \delta) I - \Sigma)^{-1} \left( I - e^{-\delta^{-1}(I - \Sigma)T} \right) \tilde{z} \]

\[ = -\frac{1}{\rho} \beta' \left( I - \frac{1}{1 + \rho \delta} \Sigma \right)^{-1} \left( I - e^{-\delta^{-1}((1 + \rho \delta)I - \Sigma)T} \right) \tilde{z} \]

\[ + \frac{1}{\rho} \beta' (I - \Sigma)^{-1} \left( e^{-\rho T} - e^{-\delta^{-1}((1 + \rho \delta)I - \Sigma)T} \right) \tilde{z} \]

\[ + \frac{1}{\rho} e^{-\rho T} \beta' \left[ \left( I - \frac{1}{1 + \rho \delta} \Sigma \right)^{-1} - (I - \Sigma)^{-1} \right] \left( I - e^{-\delta^{-1}(I - \Sigma)T} \right) \tilde{z} \]

\[ = -\frac{1}{\rho} \beta' \left( I - \frac{1}{1 + \rho \delta} \Sigma \right)^{-1} \left( I - e^{-\delta^{-1}((1 + \rho \delta)I - \Sigma)T} \right) \tilde{z} \]

\[ + \frac{1}{\rho} \beta' (I - \Sigma)^{-1} \left( e^{-\rho T} - e^{-\delta^{-1}((1 + \rho \delta)I - \Sigma)T} \right) \tilde{z} \]

\[ + \frac{1}{\rho} \beta' \left[ \left( I - \frac{1}{1 + \rho \delta} \Sigma \right)^{-1} - (I - \Sigma)^{-1} \right] \left( e^{-\rho T} - e^{-\delta^{-1}((1 + \rho \delta)I - \Sigma)T} \right) \tilde{z} \]

A.10 Proof to Proposition 9

Consider the Domar weight

\[ \gamma' = \beta' \left( \sum_{s=0}^{\infty} \Sigma^s \right) = \beta' U \left( \sum_{s=0}^{\infty} \Lambda^s \right) W \]

\[ = \beta' \sum_{k=1}^{N} \left( \sum_{s=0}^{\infty} \lambda_k^s \right) u_k w'_k = \beta' \sum_{k=1}^{N} \frac{1}{1 - \lambda_k} u_k w'_k. \]
The welfare impact

\[ v' = \frac{1}{\rho} \beta' \sum_{s=0}^{\infty} \left( 1 - (1 + \rho \delta)^{-s} \right) \Sigma^s \]

\[ = \frac{1}{\rho} \beta' \sum_{k=1}^{N} \left( \frac{1}{1 - \lambda_k} - \frac{1}{1 + \rho \delta - \lambda_k} \right) u_k w'_k \]

\[ = \delta \beta' \sum_{k=1}^{N} \lambda_k \frac{1 - \lambda_k}{(1 - \lambda_k) (1 + \rho \delta - \lambda_k)} u_k w'_k. \]

\[ \frac{1}{\rho} \beta' \sum_{k=1}^{N} \left( \frac{1}{1 - \lambda_k} - \frac{1}{1 + \rho \delta - \lambda_k} \right) u_k w'_k. \]

B  Factor Structure of the U.S. Input-Output Table: Additional Empirical Results

Table 6. The 1st & 2nd eigenvector shock profiles: 10 largest entries by absolute value

<table>
<thead>
<tr>
<th>([u_1]_i)</th>
<th>([u_2]_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonferrous metal (except aluminum) production and processing</td>
<td>0.439</td>
</tr>
<tr>
<td>Alumina and aluminum production and processing</td>
<td>0.221</td>
</tr>
<tr>
<td>Other electrical equipment and component manufacturing</td>
<td>0.213</td>
</tr>
<tr>
<td>Railroad rolling stock manufacturing</td>
<td>0.187</td>
</tr>
<tr>
<td>Motor vehicle manufacturing</td>
<td>0.178</td>
</tr>
<tr>
<td>Steel product manufacturing from purchased steel</td>
<td>0.178</td>
</tr>
<tr>
<td>Forging and stamping</td>
<td>0.175</td>
</tr>
<tr>
<td>Boiler, tank, and shipping container manufacturing</td>
<td>0.163</td>
</tr>
<tr>
<td>Iron and steel mills and ferroalloy manufacturing</td>
<td>0.159</td>
</tr>
<tr>
<td>Motor vehicle parts manufacturing</td>
<td>0.153</td>
</tr>
</tbody>
</table>
Table 7. The 3rd & 4th eigenvector shock profiles: 10 largest entries by absolute value

<table>
<thead>
<tr>
<th>$[u_3]_i$</th>
<th>$[u_4]_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Animal slaughtering and processing</td>
<td>Radio and television broadcasting</td>
</tr>
<tr>
<td>Dairy product manufacturing</td>
<td>Animal slaughtering and processing</td>
</tr>
<tr>
<td>Animal food manufacturing</td>
<td>Dairy product manufacturing</td>
</tr>
<tr>
<td>Resin, synthetic rubber, and artificial synthetic fibers and filaments manufacturing</td>
<td>Alumina and aluminum production and processing</td>
</tr>
<tr>
<td>Plastics product manufacturing</td>
<td>Railroad rolling stock manufacturing</td>
</tr>
<tr>
<td>Textile mills and textile product mills</td>
<td>Paint, coating, and adhesive manufacturing</td>
</tr>
<tr>
<td>Grain and oilseed milling</td>
<td>Rubber product manufacturing</td>
</tr>
<tr>
<td>Sugar and confectionery product manufacturing</td>
<td>Textile mills and textile product mills</td>
</tr>
<tr>
<td>Fruit and vegetable preserving and specialty food manufacturing</td>
<td>Resin, synthetic rubber, and artificial synthetic fibers and filaments manufacturing</td>
</tr>
<tr>
<td>Animal production and aquaculture</td>
<td>Plastics product manufacturing</td>
</tr>
</tbody>
</table>

Table 8. Low dimensional representation of TFP shocks to vulnerable sectors in the U.S.

<table>
<thead>
<tr>
<th>10 sectors with the highest $v_i$</th>
<th>Loadings on the first 4 eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>Real estate</td>
<td>0.29</td>
</tr>
<tr>
<td>Wholesale trade</td>
<td>0.44</td>
</tr>
<tr>
<td>Agencies, brokerages, and other insurance related activities</td>
<td>0.89</td>
</tr>
<tr>
<td>Oil and gas extraction</td>
<td>0.29</td>
</tr>
<tr>
<td>Basic chemical manufacturing</td>
<td>0.46</td>
</tr>
<tr>
<td>Management of companies and enterprises</td>
<td>0.17</td>
</tr>
<tr>
<td>Petroleum and coal products manufacturing</td>
<td>0.23</td>
</tr>
<tr>
<td>Advertising, public relations, and related services</td>
<td>0.12</td>
</tr>
<tr>
<td>Nonferrous metal (except aluminum) production &amp; processing</td>
<td>1.60</td>
</tr>
<tr>
<td>Motor vehicle parts manufacturing</td>
<td>0.08</td>
</tr>
</tbody>
</table>