Robust Portfolio Optimization with Jumps*

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We study an infinite horizon consumption-portfolio allocation problem in continuous time where asset prices follow Lévy processes and the investor is concerned about potential model misspecification of his reference model. We derive optimal portfolio holdings in closed form in the presence of model uncertainty, where we analyze perturbations to the reference model in the form of both drift and jump intensity perturbations. Furthermore, we present a method for calculating error-detection probabilities by means of Fourier inversion techniques of the conditional characteristic function in the case when the measure change follows a jump-diffusion process.

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1. Introduction

The study of dynamic intertemporal portfolio choice problems in continuous time has a long history dating back to Merton (1971). In Merton's model, the investor's optimization problem consists of how to optimally choose his consumption, as well as determining optimal portfolio allocation in a riskfree and a risky asset in order to maximize his expected lifetime utility. The sources of risk in this framework are all diffusive so that sudden large changes in the underlying risky assets are unlikely to occur. More recently, researchers have extended Merton's framework allowing for discontinuities in the price process. The dynamic portfolio allocation problem has been studied when asset prices are driven by jump processes, including Poisson, stable or more general Lévy processes. For instance Kallsen (2000), Choulli and Hurd (2001), or Cvitanić et al. (2008) study the dynamic consumption-portfolio selection problem when the risky asset follows a Lévy process. However, when jumps are included, it becomes substantially more difficult to solve for the investor's optimal portfolio holdings in closed form. Aït-Sahalia et al. (2009) are the first to provide a closed-form solution to the dynamic consumption-portfolio selection problem, when asset prices are driven by Lévy processes. However, the literature mentioned above treats the parameters, such as the expected return, jump intensity and jump size distribution as if they were known. In practice, these parameters are unknown and therefore the investor faces a considerable amount of model uncertainty. One way to account for this model uncertainty is to argue that the investor has a specific model in mind but fears that it is misspecified, in other words he beliefs that the true model lies in a set of alternative models which are statically close to his reference model. Among this class of models, which are obtained by perturbing the reference model, the investor is unable to detect the true underlying model. Therefore, by considering a perturbed version of his reference model, the investor can guard himself against potential model misspecification by making consumption and portfolio choices that are robust across the set of alternative models. In other words, robust portfolio rules are designed to work well not only when the underlying model describing the asset dynamics is correctly specified, but they should also perform reasonably well in the case when the model is misspecified.

The notion of (model) robustness or ambiguity aversion has been extensively studied in the literature on continuous time consumption and asset allocation problems. One way of introducing (Knightian) ambiguity aversion is through the formulation of multiple priors preferences as presented by Gilbao and Schmeidler (1989). Given such preferences, optimal decisions are taken under the premise that state variables are governed by the worst-case probability model among a set of candidate models. Chen and Epstein (2002) formulate an inter-temporal recursive multiple-priors utility problem that incorporates Knightian ambiguity aversion.[1] In this paper however, we build upon the penalty-based framework of robust decision making which has been pioneered by Hansen and Sargent.[2] This notion of robustness has been extensively employed for solving consumption and portfolio allocation problems in the diffusive setting.[3] For instance, Uppal and Wang (2003) derive a model of inter-temporal portfolio choice of an investor who takes model misspecification into account. Trojani and Vanini (2004) solve two versions of a robust control problem and examine its impact on the resulting asset allocations.[4] Maenhout (2004) studies an inter-temporal portfolio problem of an investor who worries about model misspecification and shows that, if the investor seeks robust decision rules then the demand for equities is significantly reduced. Additionally, Maenhout (2006) extents the robust portfolio allocation analysis by a allowing for a time-varying mean-reverting risk premium and shows that while the desire for robustness lowers the total equity share, the proportion of the inter-temporal hedging demand is increased. More recently, the concept of robustness with respect to model misspecification has also been applied to models of the term structure of nominal interest rates. For instance, Ulrich (2013) employs a robust decision making framework to analyze how model uncertainty with respect to monetary policy affects the term premium on nominal bond yields. Kleshchelski and Vincent (2007) present an equilibrium model of the term structure in a robust control setting where consumption growth exhibits stochastic volatility. They show that, if the representative agent demands optimal policies that are robust to model misspecification substantially amplifies the effect of conditional heteroskedasticity in consumption growth,

[1] An extension of this formulation of ambiguity aversion in continuous time is given in Leippold et al. (2007) where the authors combine learning based on optimal Bayesian updating and ambiguity aversion.

[2] For a general treatment of Robust Control Theory see the book by Hansen and Sargent (2008) or for applications of robustness see Anderson et al. (2003), Hansen et al. (2006), Cogley et al. (2008), Hansen and Sargent (2010) and Hansen and Sargent (2011).

[3] However, even though this RMPU formulation and the constraint penalty-based approaches to model uncertainty aversion as in Anderson et al. (2003) are derived from the same axiomatic preference description, they differ in the representation of agents' perception of ambiguity in a fundamental way. RMPU is a locally constrained problem, so that the equivalence between multiplier and constraint robust-control problem as in Hansen et al. (2006) cannot be invoked.

[4] Furthermore, Trojani and Vanini (2000) derive explicit and easily understandable robust consumption and investment rules that can be compared to those of a non-robust decision-maker in Merton's model (see Merton (1969)).

As of this writing, the literature on robust asset allocation and consumption problems with risky assets that follow jump-diffusive or Lévy processes is still sparse. Most recently, Wachter (2013) introduces a framework with time-varying probability of rare event risk in which consumption follows a mixed jump diffusion process and solves an optimal consumption and portfolio choice problem in closed form. However, in her model she does not address model misspecification. The paper by Lui et al. (2005) employs a pure-exchange economy framework with a representative agent who faces model uncertainty with respect to jumps in the underlying aggregate endowment (rare events) in order to study the equilibrium equity price. They show that the corresponding equity premium consists of diffusion and jump-risk premia, which are driven by risk aversion, and the model uncertainty premium. The paper by Drechsler (2013) uses a similar robust decision making framework in an equilibrium model in order to capture salient features of equity and options markets when the risky assets follow a jump-diffusion process but has to rely on numerical methods to obtain a solution.[5]
The novelty of our approach is to introduce robustness concerns of the investor when making his consumption and portfolio choice decisions, when the underlying risky asset follows a Lévy process. We introduce model misspecification, with respect to the drift and jump intensity parameters and are still able to solve for the investors' optimal consumption and portfolio allocation in closed-form. Additionally, we derive a semi-closed form formula for detection-error probabilities, i. e. the likelihood that the investor selects the wrong model, which gives a quantitative upper bound on the set of alternative models which seem reasonably close to his reference model.

The remainder of this paper is organized as follows. Section 2 introduces the general robust portfolio allocation problem. Section 3 derives optimal robust portfolio weights under both drift and jump intensity perturbation. Section 4 derives a semi-explicit expression for error-detection probability. Section 5 concludes. The Appendix contains further derivations and technical details.

[5] An interesting extension of the jump-diffusive model studied by Drechsler (2013) is introduced in Branger et al. (2014) where they allow the jump intensity to follow a more general self-exciting jump-processes, also called Hawkes processes (see Hawkes (1971), Hawkes (1971) or Hawkes and Oakes (1974)).

2. A Robust Portfolio Allocation Problem

We consider an infinite horizon expected utility maximization problem where the investor chooses his consumption level and allocates his funds between a risky asset and a riskless asset. The investor has a particular model in mind which represents his best estimate of the risky asset dynamics under a benchmark or reference probability measure. However, the investor mistrusts his reference model and fears that it is misspecified in the sense that he believes that the true model lies in a set of alternative models that are statistically difficult to distinguish from it. In order to mitigate the effect of potential model misspecification on his utility, the investor wants to choose optimal consumption and portfolio holdings that are robust with respect to small perturbations of his reference model. This is equivalent to considering the dynamics of the risky asset under a worst - case or robust measure.

2.1. Asset Price Dynamics under the Reference and Robust Measure

We assume a complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions where we denote by $\mathbb{P}$ the reference probability measure. The investment set available to the investor at any time $t \geq 0$ consists of a riskless asset (locally deterministic) with price $S_{0,t}$ and a risky asset with price $S_{1,t}$ following compensated exponential Lévy process. More precisely, the dynamics are given by

$$\frac{dS_{0,t}}{S_{0,t}} = rdt,\ S_{0,0} > 0 \tag{1}$$

$$\frac{dS_{1,t}}{S_{1,t_-}} = (r+R)dt + \sigma dB_t + Jd\widetilde{Y}_t,\ S_{1,0} > 0,\ \mathbb{P} - a.s. \tag{2}$$

where $r \geq 0$ is the riskless return, $R \in \mathbb{R}$ denotes the excess return of the risky asset over the riskfree asset, $\sigma > 0$ is a constant volatility parameter, $B_t = (B_t)_{t\geq 0}$ is a Brownian motion under $\mathbb{P}$ and $J \in (-1, 1)$ is a jump scaling factor. $\widetilde{Y}_t = Y_t - \Lambda_t$ is a pure compensated jump process with Lévy measure $\lambda\nu(dz)$, where $\lambda \geq 0$ is a fixed jump intensity parameter and the measure ν satisfies $\int_{\mathbb{R}} \min(1, |z|)\nu(dz) < \infty$ so that jumps have finite variation. We denote by $\Lambda_t = \lambda\kappa t$, and $\kappa = \mathbb{E}[Z_t] < \infty$ the predictable compensator of the jump process Y_t. In the sequel, we assume that

Y_t is a compensated compound Poisson process, i. e. $\widetilde{Y}_t = \sum_{n=1}^{N_t} Z_n - \Lambda_t$, where N_t is a scalar Poisson process with jump intensity λ. The jump sizes Z_n are independent of N_t and are assumed to be *i.i.d* with Lévy measure $\nu(dz)$. Then the risky assets dynamics under the reference measure can be expressed as

$$\begin{aligned}\frac{dS_{1,t}}{S_{1,t_-}} &= (r+R)dt + \sigma dB_t^{\vartheta} + Jd\widetilde{Y} \\ &= (r+R)dt + \sigma dB_t^{\vartheta} + J\left(dY_t - \mathbb{E}\left[Z_t dN_t\right]\right) \\ &= (r+R)dt + \sigma dB_t^{\vartheta} + J\left(dY_t - \int z\nu(dz)\mathbb{E}\left[dN_t\right]\right) \\ &= \left(r + R - \lambda J\int z\nu(dz)\right)dt + \sigma dB_t + JdY_t,\ S_{0,1} > 0,\ \mathbb{P} - a.s. \end{aligned} \tag{3}$$

In order to introduce the notion of model misspecification we need to specify a set of alternative or worst - case robust dynamics which are statistically close to the reference dynamics in Equation (3). For this purpose we specify an equivalent probability measure which we denote by $\mathbb{P}^{\vartheta}$ and in the sequel refer to it as the robust measure. Given this setup, the investor considers alternative models under the robust measure $\mathbb{P}^{\vartheta}$ which take the general form

$$\frac{dS_{1,t}}{S_{1,t_-}} = \left(r + R + \sigma h_t - \lambda^{\vartheta} J\int z\nu(dz)\right)dt + \sigma dB_t^{\vartheta} + JdY_t^{\vartheta},\ S_{0,1} > 0,\ \mathbb{P}^{\vartheta} - a.s. \tag{4}$$

A first inspection of equation (4) shows that the drift has changed from $(r+R)$ under the measure $\mathbb{P}$ to $(r+R+\sigma h_t - \lambda^{\vartheta} J \int z\nu^{\vartheta}(dz))$ under the measure $\mathbb{P}^{\vartheta}$ where $(h_t)_{t\geq 0}$ is a continuous $\mathcal{F}_t$ -measurable function of the Markovian state $S_{1,t}$ with the same dimensionality as the Brownian motion, i.e. one dimensional. In what follows, we refer to h_t as a drift perturbation function, since it perturbs the drift dynamics of the risky asset under $\mathbb{P}$ and does not affect the jump component $\widetilde{Y}_t$. Secondly, the stochastic process $B_t^{\vartheta} = (B_t^{\vartheta})_{t\leq 0}$ is a Brownian motion but now under the perturbed or robust measure $\mathbb{P}^{\vartheta}$. Lastly, the other set of perturbations affect the jump component $\widetilde{Y}_t^{\vartheta}$ in Equation (4), namely the jump intensity and the jump size distribution under $\mathbb{P}^{\vartheta}$. The jump intensity λ is transformed into λ^{ϑ} under the robust probability measure as follows

$$\lambda^{\vartheta} = e^{a}\lambda,\ a \in \mathbb{R}$$

where a is a scalar jump intensity perturbation parameter that amplifies or diminishes the jump intensity. From Equation (4) we observe that perturbating the intensity has two effects on the risky assets dynamics. First, it alters the drift and second it changes the frequency of jumps occurring in the Poisson process N_t. For instance, when jumps are negative only, the compensation will lead to higher expected returns when assets prices are low and vice versa to lower expected returns when asset prices are high which is consistent with the empirical risk and return trade-off observed in financial markets. In other words, compensating the jump process leads to S_t carrying a risk premium for intensity misspecification. The jump size distribution under $\mathbb{P}^{\vartheta}$ has Lévy measure

$$\nu^{\vartheta}(dz) = \nu(dz; b),\ b \in \mathbb{R}^L,\ L \geq 1$$

where b is a set of possibly vector valued perturbation parameters. For instance, if the jump size distribution is normal, $Z_n \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma^2)$, a jump size perturbed model may read $Z_n \overset{i.i.d}{\sim} \mathcal{N}(\mu + \delta_\mu, \sigma^2 v_\sigma)$, where $\delta_\mu \in \mathbb{R}$ shifts the mean from μ under $\mathbb{P}$ to $\mu + \delta_\mu$ under $\mathbb{P}^{\vartheta}$ and likewise the variance is scaled by $v_\sigma > 0$. Thus for $a = \delta_\mu = 0$ and $v_\sigma = 1$ we get back the jump distributions of the reference model under the measure $\mathbb{P}$.

2.2. Measure Change for Itô Semimartingales

The dynamics of the compensated exponential Lévy process under the reference as well as under the robust measure are linked through a specific likelihood ratio or Radon-Nikodym density process ϑ_t. This density process not only allows to change the dynamics of the risky asset but also, as it will be shown in the next section, restricts the size of alternative models that are statistically difficult to distinguish from the reference model. To be more precise, let $\mathbb{P}^{\vartheta}$ be the robust or perturbed measure which is absolutely continuous with respect to the reference measure $\mathbb{P}$. Fix $T > 0$ and define $\vartheta_t = (\vartheta_t)_{t\in[0,T]} = \frac{d\mathbb{P}^{\vartheta}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \vartheta_t = \vartheta_t^D \vartheta_t^J$ where ϑ_t^D is a $(\mathcal{F}_t, \mathbb{P})$- martingale that defines the measure change of the continuous part of the stochastic process and ϑ_t^J, also a $(\mathcal{F}_t, \mathbb{P})$- martingale, that defines the measure change of the discontinuous or jump part.[6] In a jump diffusive setting, where the jumps follow a compound Poisson process, there are three ways to change the measure, i. e. from $\mathbb{P}$, the reference measure to $\mathbb{P}^{\vartheta}$, the robust measure.

1. Measure change of the diffusive part through ϑ_t^D affecting the drift and the Brownian motion.

[6]The change of measures of the diffusive and jump part factor only when the continuous and the jump part of the stochastic process are independent, which is the case in the above, since $[N, W]_t = 0$.

2. Measure change of the jump part through ϑ_t^J by changing the jump intensity of the process under $\mathbb{P}^\vartheta$.

3. Measure change of the jump part through ϑ_t^J by changing the jump size of the process under $\mathbb{P}^\vartheta$.[7]

From Girsanovs' theorem for Itô-semimartingale, for the diffusive drift measure change ϑ_t^D, with $(h_t)_{t\geq 0}$ a progressively measurable process, we have that

$$B_t^\vartheta = B_t - \int_0^t h_s ds \tag{5}$$

is a Brownian motion with respect to the measure $\mathbb{P}^\vartheta$. Then it follows that, an absolutely continuous change of measure can be represented by an exponential $(\mathbb{P}, \mathcal{F}_t)$- martingale ϑ_t satisfying

$$\vartheta_t^D = \exp\left\{\int_0^t h_s dB_s - \int_0^t \frac{h_s^2}{2} ds\right\},\ \mathbb{E}_0\left[\vartheta_t\right] = \vartheta_0 = 1,\ \mathbb{P} - a.s. \tag{6}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$ denotes the conditional expectation up to time t with respect to the measure $\mathbb{P}$ and likewise we define by $\mathbb{E}_t^\vartheta\left[\cdot\right]$ the conditional expectation up to time t with respect to the robust measure $\mathbb{P}^\vartheta$.

Concerning the jump part of the risky asset, we let $N = (N_t)_{t\in[0,T]}$ be a Poisson process with jump intensity $\lambda > 0$ on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ and $T > 0$ again fixed. We want to change the intensity λ of the Poisson process $(N_t)_{t\in[0,T]}$ on $\mathbb{P}$ to a jump intensity λ^ϑ under the robust measure $\mathbb{P}^\vartheta$. Likewise, we want to perturb the Lévy measure such that the jumps have distribution $\nu^\vartheta(dz) = \nu(dz; b)$ under the robust measure. The appropriate measure change $\vartheta^J = (\vartheta_t^J)_{t\in[0,T]}$ is

$$\left.\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \vartheta_t^J = e^{(\lambda-\lambda^\vartheta)t}\prod_{n=1}^{N_t}\frac{\lambda^\vartheta}{\lambda}\frac{\nu^\vartheta(Z_n)}{\nu(Z_n)},\ \mathbb{E}\left[\vartheta_t^J\right] = \vartheta_0^J = 1,\ \mathbb{P} - a.s. \tag{7}$$

which is a $(\mathcal{F}_t, \mathbb{P}^\vartheta)$- martingale and satisfies

$$d\vartheta_t^J = \vartheta_{t\_}^J d\left(H_t - \lambda^\vartheta\right) - \vartheta_t^J d\left(N_t - \lambda t\right),\ H_t = \sum_{n=1}^{N_t}\frac{\lambda^\vartheta}{\lambda}\frac{\nu^\vartheta(Z_n)}{\nu(Z_n)}. \tag{8}$$

where H_t is a compound Poisson process and $H_t - \lambda^\vartheta t$ is a $(\mathcal{F}_t, \mathbb{P}^\vartheta)$-martingale. Therefore the density process in Equation (7) is a right-continuous, adapted process with left limits (càdlàg). By changing the perturbation parameters $h_t,\ a$ and b we control the discrepancy between the dynamics of the

[7] One distinguishes between a finite number of jump sizes and the case where the jump sizes can take on a continuum of values.

risky asset under the reference measure with respect to its dynamics under the robust measure. Therefore, the more h_t, a and b deviate from their no-perturbation values, the more different the dynamics of the risky asset become under the reference with respect to the robust measure, i.e. the set of alternative models expands. However, the possible set of models under consideration has to be restricted to a subset of models which are statistically difficult to distinguish from the reference model. A popular statistical tool to measure 'distances' between two probability distributions is relative entropy which we discuss in the next section.

2.3. Relative Entropy Growth Bounds and Error-Detection Probabilities

The alternative set of possible models that are similar in a statistical sense are tightly linked to the measure change ϑ_t. Given two probability measures $\mathbb{P}$ and $\mathbb{P}^{\vartheta}$, growth in entropy of $\mathbb{P}^{\vartheta}$ relative to $\mathbb{P}$ over the time interval $[t, t+\Delta t]$ is defined as

$$G(t, t+\Delta t) = \mathbb{E}_t^{\vartheta}\left[\log\left(\frac{\vartheta_{t+\Delta t}}{\vartheta_t}\right)\right], \rightarrow \mathcal{R}(\vartheta_t) \overset{def}{=} \lim_{\Delta t \to 0} \frac{G(t, t+\Delta t)}{\Delta t} \qquad \forall t \geq 0 \tag{9}$$

Thus the set of admissible model misspecification can be characterized as

$$\{\vartheta_t : \mathcal{R}(\vartheta_t) \leq \eta, \ \forall t \geq, \ \eta \geq 0\} \tag{10}$$

where η is a constant that defines an upper bound on the set of alternative models.[8] As $\eta \to 0$ the investor gets fully confident about his reference model, while increasing η expands the set of alternative models that are statistically further away from the reference model, in other words overall model uncertainty increases. Due to the independence of the diffusive and the jump component, the measure change is given by $\vartheta_t = \vartheta_t^D \vartheta_t^J$, which implies that relative entropy growth is simply the sum of the two components 'drift' and 'jump', namely $\mathcal{R}(\vartheta_t) = \mathcal{R}(\vartheta_t^D) + \mathcal{R}(\vartheta_t^J)$. Therefore, by varying the perturbation function h_t and perturbation parameters a and b we regulate the space of admissible models within the set $[0, \eta]$, $\forall t \geq 0$. An immediate question that arises in this context is: What is a reasonable value for η? Anderson et al. (2003) provide a statistical tool for model detection based on the log of the measure change ϑ_t in the form of detection-error probabilities in order to quantify the

[8] While the upper bound on relative entropy growth rate is constant in this model, it is possible to make it time-varying such as for instance in Ulrich (2010), Ulrich (2012) or Drechsler (2013).

amount of model uncertainty that seems plausible to the investor. The basic intuition behind this test statistic is, given the right model is $\mathbb{P}$ and a finite time series sample of the state variable (risky asset) of length $T-t$, how likely will the investor mistakenly assume the data have been generated by model $\mathbb{P}^{\vartheta}$ instead of the true model $\mathbb{P}$. The detection-error probability is precisely quantifying the likelihood that the investor is going to select the wrong model. Thus if the true model is $\mathbb{P}$, the investor will falsely reject it for model $\mathbb{P}^{\vartheta}$ based on a time series sample of length $T-t$ whenever $\log(\vartheta_T) > 0$. Conversely, if the true model is $\mathbb{P}^{\vartheta}$ he will erroneously reject it for model $\mathbb{P}$ whenever $\log(\vartheta_T) < 0$.

2.4. Wealth Dynamics and Utility Specification under Robustness

We denote by $X_t = (X)_{t\geq 0}$ the investor's wealth at time t. Let $\omega_{0,t} = w_{0,t}/X_t$ be the percentage of wealth (or portfolio weight) invested in the risk free asset and by $\omega_{1,t}$ the percentage of wealth invested in the risky asset. Let $\omega_{1,t} = w_{1,t}/X_t$ be an adapted predictable càdlàg process and $w_{i,t},\ i \in \{0,1\}$ is the absolute amount of money invested into asset i. Then, the portfolio weights satisfy $\omega_{0,t}+\omega_{1,t} = 1$. The investor consumes at an instantaneous rate C_t at time t. Under the robust dynamics given in Equation (4) his wealth evolves as follows

$$
\begin{aligned}
dX_t &= \omega_{0,t}X_t\frac{dS_{0,t}}{S_{0,t}} + \omega_{1,t}X_t\frac{dS_{1,t}}{S_{1,t_-}} - C_t dt \\
dX_t &= \left[X_t\left(r + \omega_t R + \sigma h_t \omega_t - \omega_t \lambda^{\vartheta} J \int z\nu^{\vartheta}(dz)\right) - C_t\right]dt + \omega_t X_t \sigma dW_t^{\vartheta} + \omega_t X_t J d\widetilde{Y}_t^{\vartheta},
\end{aligned} \tag{11}
$$

with $X_0 > 0$, $\mathbb{P}^{\vartheta} - a.s.$ and we have set $\omega_t = \omega_{1,t}$. The investor's robust consumption and portfolio allocation problem is to choose in a first step, a set of worst-case functions $\{h_s\}_{t\leq s<\infty}$ and worst-case parameters a, b, and in a second step to select admissible consumption and portfolio holdings $\{C_s, \omega_s\}_{t\leq s<\infty}$ that maximize his expected utility of consumption under the worst-case scenario. More formally speaking, let $\beta \in (0,\infty)$ be his subjective discount or 'impatience' rate, the optimal robust consumption and portfolio problem is given by

$$
\max_{\{C_s,\omega_s\}_{t\leq s<\infty}} \min_{\{h_s\}_{t\leq s<\infty},a,b} \mathbb{E}_t^{\vartheta}\left[\int_t^{\infty} e^{-\beta s}U(C_s)ds\right] \tag{12}
$$

subject to the entropy growth constraint

$$\mathcal{R}(\vartheta_t) \leq \eta, \qquad \forall t \geq 0 \tag{13}$$

and his wealth constraint in Equation (11). We define by

$$V = V(X_t, t) = \max_{\{C_s, \omega_s\}_{t \leq s < \infty}} \min_{\{h_s\}_{t \leq s < \infty}, a, b} \mathbb{E}_t^{\vartheta} \left[\int_t^{\infty} e^{-\beta s} U(C_s) ds \right] \tag{14}$$

the value function associated to the optimal stochastic robust control problem in Equation (12). Since we work in incomplete markets and thus the martingale measure is not unique, we have to rely on (standard) stochastic dynamic programming techniques. Then, using Itô's formula for semi-martingales the perturbed Hamilton-Jacobi-Bellman (HJB) equation characterizing the optimal robust consumption and portfolio allocation problem is given by

$$\begin{aligned} 0 = \max_{\{C_t, \omega_t\}} \min_{\{h_t\}, a, b} e^{-\beta t} U(C_t) + \frac{\partial V(X_t, t)}{\partial t} + \frac{\partial V(X_t, t)}{\partial X} \left[X_t \left(r + \omega_t R + \sigma h_t \omega_t - \omega_t \lambda^{\vartheta} J \int z \nu(dz) \right) - C_t \right] \\ + \frac{1}{2} \frac{\partial^2 V}{\partial X^2} \omega_t^2 X_t^2 \sigma^2 + \lambda e^a \int \left[V(X_{t_-} + X_{t_-} \omega_t J z, t) - V(X_{t_-}, t) \right] \nu(dz, b) \end{aligned} \tag{15}$$

subject to

$$\mathcal{R}(\vartheta_t) \leq \eta \tag{16}$$

and the transversality condition $\lim_{t \to \infty} \mathbb{E}_t^{\vartheta} [V(X_t, t)] = 0$. Then by standard time-homogeneity arguments for infinite horizon problems implies that

$$\begin{aligned} e^{\beta t} V(X_t, t) &= \max_{\{C_s, \omega_s\}_{t \leq s < \infty}} \min_{\{h_s\}_{t \leq s < \infty}, a, b} \mathbb{E}_t^{\vartheta} \left[\int_t^{\infty} e^{-\beta (s-t)} U(C_s) ds \right] \\ &= \max_{\{C_{t+u}, \omega_{t+u}\}_{t \leq u < \infty}} \min_{\{h_{t+u}\}_{t \leq u < \infty}, a, b} \mathbb{E}_t^{\vartheta} \left[\int_t^{\infty} e^{-\beta u} U(C_{t+u}) ds \right] \\ &= \max_{\{C_u, \omega_u\}_{0 \leq u < \infty}} \min_{\{h_u\}_{0 \leq u < \infty}, a, b} \mathbb{E}_0^{\vartheta} \left[\int_0^{\infty} e^{-\beta u} U(C_u) du \right] \\ &\equiv L(X_t) \end{aligned} \tag{17}$$

where the third equality follows because the optimal robust control is Markov and $L(X_t)$ is independent of time. Thus $V(X_t, t) = e^{-\beta t} L(X_t)$. Then the optimal robust control problem in Equation

(15) reduces to a time-homogeneous problem in $L(X_t)$

$$0 = \max_{\{C_t,\omega_t\}} \min_{\{h_t\},a,b} U(C_t) - \beta L(X_t) + \frac{\partial L(X_t)}{\partial X} \left[X_t \left(r + \omega_t R + \sigma h_t \omega_t - \omega_t \lambda^{\vartheta} J \int z\nu(dz) \right) - C_t \right]$$
$$+ \frac{1}{2} \frac{\partial^2 L(X_t)}{\partial X^2} \omega_t^2 X_t^2 \sigma^2 + \lambda e^a \int \left[L(X_{t\_} + X_{t\_}\omega_t J z) - L(X_{t\_}) \right] \nu(dz; b) \tag{18}$$

subject to

$$\mathcal{R}(\vartheta_t) \le \eta \tag{19}$$

and the transversality condition $\lim_{t\to\infty} \mathbb{E}_t^{\vartheta}\left[L(X_t)\right] = 0$. To find a solution of the problem in (18) subject to the entropy growth constraint in (19), we formulate it as a Lagrange optimization problem with inequality constraints. In a first step, we determine the optimal robust control policies h_t^*, a^* and b^*, by solving a constraint optimization problem. We denote by $\mathcal{L} = \mathcal{L}(C_t, \omega_t, h_t, a, b, \theta)$ the Lagrangian associated to the problem given in Equation (18) and by θ the corresponding Lagrange-multiplier of the entropy growth constraint in Equation (19). Then the first order optimality conditions for the minimization part of Equation (18) are given by

$$\frac{\partial \mathcal{L}}{\partial h_t} = \sigma\omega_t - \frac{\partial}{\partial h_t} \theta \left(\mathcal{R}(\vartheta_t) - \eta \right) = 0, \tag{20}$$
$$\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial}{\partial a} \lambda e^a \int \left[L(X_{t\_} + X_{t\_}\omega_t J z) - L(X_{t\_}) \right] \nu(dz, b) - \frac{\partial}{\partial a} \theta \left(\mathcal{R}(\vartheta_t) - \eta \right) = 0, \tag{21}$$
$$\frac{\partial \mathcal{L}}{\partial b_l} = \frac{\partial}{\partial b_l} \lambda e^a \int \left[L(X_{t\_} + X_{t\_}\omega_t J z) - L(X_{t\_}) \right] \nu(dz; b_l) - \frac{\partial}{\partial b_l} \theta \left(\mathcal{R}(\vartheta_t) - \eta \right) = 0, \; l = 1, \ldots, L, \tag{22}$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = \mathcal{R}(\vartheta_t) - \eta = 0, \qquad \theta \ge 0, \qquad \theta \left(\mathcal{R}(\vartheta_t) - \eta \right) = 0 \tag{23}$$

Each Equation in (20) to (23) summarizes two opposing effects. For instance, from Equation (20), the left term $\frac{\partial \mathbb{E}^{\vartheta}[V(X_t)]}{\partial h_t} = \sigma\omega_t$ characterizes the marginal impact on the investor's utility that results from increasing perturbation. The right term in Equation (20) $\frac{\partial}{\partial h_t}\theta\left(\mathcal{R}(\vartheta_t) - \eta\right)$ captures the associated increase in detectability of the robust model. Thus under the robust measure, the perturbation of the reference model is such that its effect is most harmful to the investor's utility and simultaneously remaining difficult to detect statistically. To solve the problem, from the first order conditions in Equation (20) to (23) we obtain the optimal amount of perturbation of the drift component h_t^*, jump intensity a^* and size perturbation $b^* \in \mathbb{R}^L$, that satisfy the growth entropy constraint, the complementary slackness condition and non-negativity constraint of θ^* in (23). Having obtained a set of optimal robust control parameters $\{h_t^*, a^*, b^*\}$, we then plug them back into the Lagrangian and solve the corresponding perturbed HJB equation for the optimal consumption policy C_t^* and portfolio weights ω^*. Thus given $\{h_t^*, a^*, b^*\}$, the first order condition for the investor's optimal consumption and portfolio policies are given by,

$$\frac{\partial \mathcal{L}(h_t^*, a^*, b^*, \theta^*)}{\partial C_t} = \frac{\partial U(C_t)}{\partial C_t} = \frac{\partial L(X_t)}{\partial X} \to C_t^* = \left[\frac{\partial U}{\partial X}\right]^{-1} \frac{\partial L(X_t)}{\partial X} \tag{24}$$

$$\begin{aligned}\frac{\partial \mathcal{L}(h_t^*, a^*, b^*, \theta^*)}{\partial \omega_t} &= \frac{\partial}{\partial \omega_t}\left[\frac{\partial L(X_t)}{\partial X}\left[X_t\left(r + \omega_t R + \sigma h_t \omega_t - \omega_t \lambda^{\vartheta} J \int z\nu(dz)\right) - C_t\right]\right. \\ &\left. + \frac{1}{2}\frac{\partial^2 L(X_t)}{\partial X^2}\omega_t^2 X_t^2 \sigma^2 + \lambda e^{a^*} \int \left[L(X_{t_-} + X_{t_-}\omega_t J z) - L(X_{t_-})\right]\nu(dz; b^*)\right]\end{aligned} \tag{25}$$

The first order condition for optimal consumption is standard and says that at the optimum, marginal utility of consumption must be equal to the marginal utility of wealth. Since U is concave, the investor wants to smooth consumption. From Equation (25) we obtain the optimal portfolio allocations as a function of the perturbation parameters $\{h_t^*, a^*, b^*\}$. We are now going to discuss first the case where we the case when both drift and jump intensity are being distorted.

3. Explicit Robust Portfolio weights: Drift versus Jump Intensity Perturbation

In order to derive explicit results, we need to make some assumptions about the investor's utility, the Lévy measure characterizing the jump sizes and the amount of perturbation of the reference model we allow for. We consider an investor with power utility, $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and CRRA coefficient $\gamma \in (0,1) \bigcup (1,\infty)$ and $U(c) = -\infty$ whenever $c \leq 0$. To obtain fully explicit portfolio weights, we do not perturb the Lévy measure, i.e. $\nu(dz; b) = \nu(dz)$ under both measures. Furthermore, we conjecture a solution of the form

$$L(x) = \frac{K^{-\gamma} x^{1-\gamma}}{(1-\gamma)}, \tag{26}$$

for some constant K. Then, after division by $(1-\gamma)L(X_t)$ the HJB equation is given by[9]

$$0 = \max_{C_t,\omega_t} \min_{h_t,a} \frac{U(C_t)}{(1-\gamma)L(X_t)} - \frac{\beta}{(1-\gamma)} + \left[r + \omega_t R + \sigma h_t \omega_t - \omega_t \lambda^{\vartheta} J \int z\nu(dz) - \frac{C_t}{X_t}\right]$$
$$-\frac{1}{2}\gamma\omega_t^2\sigma^2 + \frac{\lambda^{\vartheta}}{(1-\gamma)} \int \left[(1+\omega_t J z)^{1-\gamma} - 1\right]\nu(dz), \text{ subject to } \mathcal{R}(\vartheta_t) \le \eta. \tag{27}$$

It is worth noting that the objective function is time independent. This follows because γ, R, σ, J and λ are constant which implies that the optimal drift perturbation parameter h_t will also be independent of time, $h_t = h,\ \forall t \ge 0$. Furthermore, since $\left[R + \sigma h_t - \lambda^{\vartheta} J \int z\nu(dz)\right]\omega_t - \frac{1}{2}\gamma\omega_t^2\sigma^2 + \frac{\lambda^{\vartheta}}{(1-\gamma)} \int \left[(1+\omega_t J z)^{1-\gamma} - 1\right]\nu(dz)$ does not depend on the investor's time t wealth X_t, the optimal portfolio share will not only be time- but also be state-independent, i. e. $\omega^*(X_t, t) = \omega^*,\ \forall t \ge 0$. However, contrarily to the setting for instance introduced in Aït-Sahalia et al. (2009), the optimal portfolio allocation will be a function of the perturbation parameters $\{h^*, a^*\}$, in other words, $\omega^* = \omega^*(h^*, a^*)$. Lastly, we now derive the measure change, to characterize the set of alternative models. When there is no jump size perturbation, the Radon-Nikodym derivative in (8) reduces to

$$d\vartheta_t^I = (e^a - 1)\vartheta_{t\_}^I d\hat{N}_t, \qquad \vartheta_0^I = 1 \text{ with } \widehat{N}_t = N_t - \lambda t \tag{28}$$

whose solution is given by

$$\eta_t^J = \exp\left\{aN_t - \lambda\left(e^a - 1\right)t\right\},\ \eta_0^J = 1. \tag{29}$$

Therefore, together with Equation (6) characterizing the measure change of the diffusive part we arrive at

$$\mathcal{R}(\vartheta_t) = \mathcal{R}(\vartheta_t^D) + \mathcal{R}(\vartheta_t^J) = \frac{1}{2}h_t^2 + e^a\lambda\left(a-1\right) + \lambda. \tag{30}$$

The investor's consumption and portfolio choice problem is summarized by Equation (27) and (30) which limits the set of alternative models. The solution to this problem is given by a two step-procedure. In a first step, which corresponds to the min-part in Equation (27), the investor has to decide how rich the alternative set of models is, he considers reasonably close to his reference model. In doing so, he specifies his preference for robustness with respect to small perturbations of his reference model by optimally choosing $\{h^*, a^*\}$.[10] In a second step, he has to decide on his

[9]Given $\{h_t^*, a^*\}$ and θ^*, $\left[R + \sigma h_t\right]\omega_t - \frac{1}{2}\gamma\omega_t^2\sigma^2$ and $\frac{\lambda^{\vartheta}}{(1-\gamma)} \int \left[(1+\omega_t J z)^{1-\gamma} - 1\right]\nu(dz)$ are both concave in ω_t, thus any solution to (27) will always have a unique maximizer.

[10]If we were to allow for jump size perturbation, the investor would of course, additionally need to decide upon the optimal jump size perturbation b^*.

optimal consumption and portfolio policies. With CRRA preferences, from the first order condition of consumption in Equation (24) we obtain

$$C_t^* = C_t^*(h^*, a^*) = K(h^*, a^*)X_t \tag{31}$$

where we require that $X_t > 0$ such that consumption remains non-negative. Then given $\{h^*, a^*\}$ evaluating Equation (27) at optimal consumption C_t^* and portfolio holdings ω_t^* the constant $K = K(h^*, a^*)$ is given by

$$K = \frac{(1-\gamma)}{\gamma}(r + R\omega^*) - \frac{\beta}{\gamma} - \frac{1}{2}(1-\gamma)\omega^{*2}\sigma^2 - \frac{\lambda(1-\gamma)}{\gamma}\int\left[(1+\omega^* Jz)^{1-\gamma} - 1\right]\nu(dz) \tag{32}$$

which will be fully determined once we have first solved for the optimal perturbation parameters $\{h^*, a^*\}$ and secondly obtained the optimal portfolio holdings ω^*. Furthermore, from Equation (31) and (32) it follows that optimal consumption is affected by robustness concerns. This result is in contrast to the case of robust control problems when the sources of randomness are only diffusive, as for instance studied in Maenhout (2004), Trojani and Vanini (2004) or Sbuelz and Trojani (2008). Thus Equation (24) shows that the same conclusion in the case the risky asset has continuous dynamics do not carry over to the case where the risky asset follows a jump-diffusion process. We now discuss the joint perturbation of the drift and jump intensity.

3.1. Explicit Portfolio Weights: Joint Drift and Jump Intensity Perturbation

We now discuss the case where the investor is concerned about potential drift and jump intensity misspecification simultaneously.[11] In order to obtain fully explicit portfolio weights, we need to be more precise about the level of risk aversion, the Lévy measure and the treatment of the entropy growth constraints. In the sequel, we focus on a Lévy measure $\nu(dz)$ defined on $[0, 1]$ and set the deterministic jump scaling factor $J \in [-1, 0]$. This implies that we only consider negative jumps in the asset price dynamics since those are the ones the investor is more concerned about as they are more harmful to his utility. To be more precise we choose $\nu(dz)$ to follow a power law under both

[11] We do not consider jump size misspecification, i. e. $\nu^\vartheta(dz) = \nu(dz)$, since this type of perturbation leads to highly-nonlinear first order conditions of both the jump size perturbation parameter b as well as for the optimal portfolio holdings which can only be resolved numerically.

measures, i. e.

$$\nu^{\vartheta}(dz) = \nu(dz) = dz/z, \text{ if } z \in (0, 1] \tag{33}$$

Concerning the treatment of the entropy growth constraints, we use a separation argument. Since total relative entropy separates into a diffusive and a jump part, i. e. $\mathcal{R}(\vartheta_t) = \mathcal{R}(\vartheta_t^D) + \mathcal{R}(\vartheta_t^J)$, we can treat entropy growth with respect to the drift and jump part independently.[12] However, this implies that, the total maximal amount of robustness η is the sum of the maximal amount of robustness with respect to drift misspecification, denoted by η^D and the maximal amount of robustness with respect to jump intensity misspecification, denoted by η^J. Therefore $\eta = \eta^D + \eta^J \leftrightarrow 1 = \widetilde{\eta}^D + \widetilde{\eta}^I$ with $\widetilde{\eta}^i = \eta^J/\eta$, $i \in \{D, I\}$ where $\widetilde{\eta}^D$, $\widetilde{\eta}^I$ denote the share of total amount of robustness in drift and jump intensity perturbation, respectively. Then the entropy growth constraints are given by

$$\mathcal{R}(\vartheta_t^D) \leq \eta^D, \qquad \mathcal{R}(\vartheta_t^J) \leq \eta^J, \ \forall t \geq 0. \tag{34}$$

Define as $\mathcal{L}(\omega_t, h_t, \theta^D, \theta^I)$ the Lagrangian associated to the constraint HJB problem in (27) with Lagrange multipliers θ^D and θ^I. Then taking $\gamma = 2$ we obtain

$$\begin{aligned} \mathcal{L}(\omega, h, a, \theta^D, \theta^I) &= \omega R + \sigma h \omega - \omega \lambda J e^a - \omega^2 \sigma^2 + \lambda e^a \log\left(1 + \omega_t J\right) \\ &\quad + \theta^D \left(\frac{1}{2} h^2 - \eta^D\right) + \theta^I \left(\lambda e^a (a - 1) + \lambda - \eta^J\right). \end{aligned} \tag{35}$$

The necessary first order optimality conditions are

$$\frac{\partial \mathcal{L}(\omega, h, a, \theta^D, \theta^J)}{\partial h} = \sigma \omega_t + \theta^D h = 0, \ \rightarrow h^* = -\frac{\sigma \omega_t}{\theta^D}, \qquad \theta^D \geq 0 \tag{36}$$

$$\frac{\partial \mathcal{L}(\omega, h, a, \theta^D, \theta^J)}{\partial a} = \lambda e^a \left(a \theta^I + \log\left(1 + J\omega\right) - wJ\right)\right) = 0, \tag{37}$$

$$\rightarrow a = \frac{\omega J - \log\left(1 + J\omega\right)}{\theta^J}, \ \theta^J \geq 0 \tag{38}$$

$$\frac{\partial \mathcal{L}(\omega, h, a, \theta^D, \theta^J)}{\partial \theta^D} = \eta^D - \frac{1}{2} h^2 = 0, \ \rightarrow \theta^{D*} = \pm \sqrt{\frac{\sigma^2 \omega^2}{2 \eta^D}} \tag{39}$$

$$\frac{\partial \mathcal{L}(\omega, h, a, \theta^D, \theta^J)}{\partial \theta^J} = \eta^J - \lambda e^a (a - 1) - \lambda = 0. \ \rightarrow \theta^{J*} = \frac{\omega J - \log\left(1 + J\omega\right)}{1 + W\left(\frac{\eta^J - \lambda}{e \lambda}\right)} \geq 0 \tag{40}$$

[12] A similar idea has also been used by Ulrich (2010) and Ulrich (2012) to solve a model ambiguity problem where there risks are only diffusive. In our particular case, a joint entropy growth constraint gives rise to first order conditions of the drift and the jump intensity perturbation parameter which cannot be solved analytically and one needs to rely on numerical techniques to solve the system.

From the system of Equations (36) to (40) we find that the optimal drift h^* and jump intensity (a^*) perturbation parameters are given by

$$h_t^* = h^* = -\sqrt{2\eta^D},\ \forall \eta^D \geq 0, \qquad a^* = 1 + W\left(\frac{\eta^J - \lambda}{e\lambda}\right) \geq 0,\ \forall \eta^J \geq 0, \lambda \in (0, \infty) \tag{41}$$

where $W\left(\frac{\eta^J-\lambda}{e\lambda}\right) = W(\cdot, \lambda)$ denotes Lambert's W function and e is Euler's constant. $W(\cdot, \lambda)$ is plotted in Figure 1 below.[13] Note that $\lim_{\eta \to 0} W(\eta; \lambda) = W\left(\frac{-1}{e}; \lambda\right) = -1$ so that $\lambda^\vartheta = \lambda$, in other

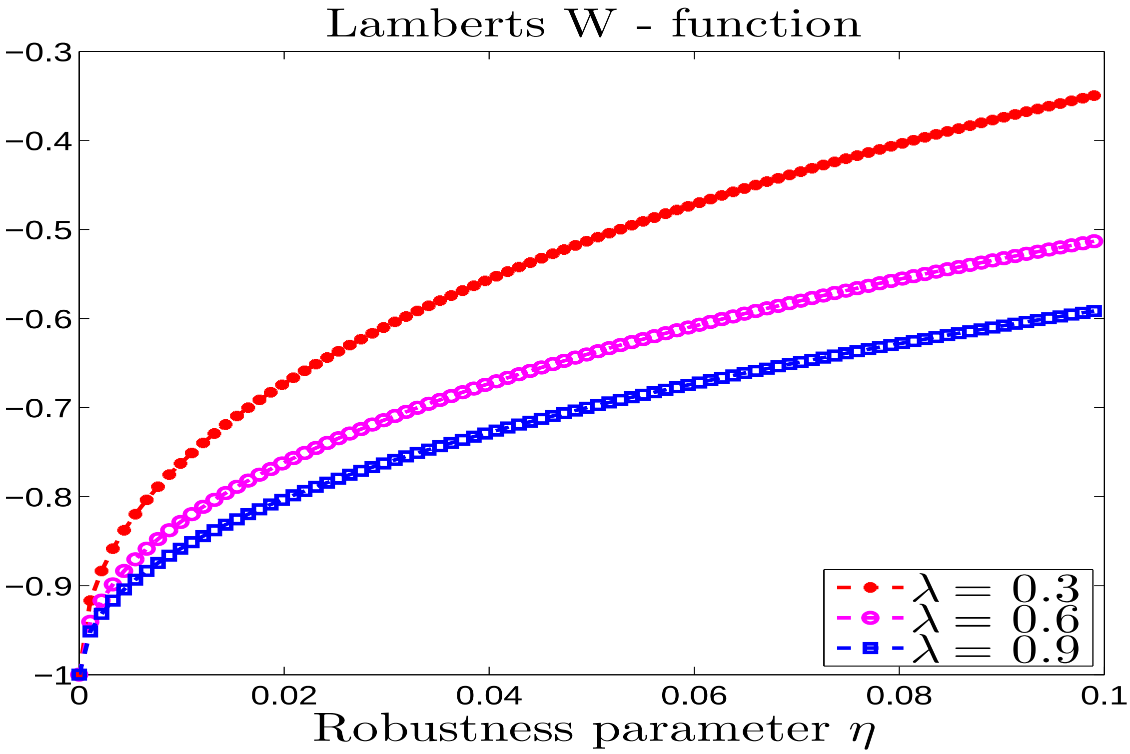

Figure 1: Lambert's W-Function for various levels of jump intensities λ.

words we are back to the case where there are no robustness concerns of the investor. Furthermore, $\lambda^\vartheta > \lambda,\ \forall \eta, \lambda > 0$. Thus the robust jump intensity under $\mathbb{P}^\vartheta$ is always higher than the jump intensity under the reference measure $\mathbb{P}$. In the case where there are no robustness concerns, i. e. $\eta^J = 0$ we have of course $\lambda^\vartheta = \lambda$. Since $\lambda > 0$, λ^ϑ is increasing in η^J as $\frac{\partial W}{\partial \eta} = \frac{W\left(\frac{\eta^J - \lambda}{e\lambda}\right)}{(\eta^J - \lambda)(1 + W\left(\frac{\eta^J - \lambda}{e\lambda}\right)} > 0\ \forall \eta^J, \lambda > 0$. This implies that the larger the set of potential models $(\eta^J \uparrow)$ is we allow for, the higher the jump frequency under the robust measure becomes.

In order to make sure that the optimal robust control variables h^* and a^* are indeed (global) mini-

[13] Since η is real, the function W is injective on the interval $-1/e \leq \eta < 0$. Furthermore, on the domain $[-1/e, \infty]$ the function $W(\cdot, \lambda)$ is real for any $\lambda > 0$ and $\eta \in [0, \infty)$.

mizers we need to check the second order optimality conditions. For the optimal drift perturbation h^* we have $\frac{\partial^2 \mathcal{L}(\omega,h,a,\theta^D,\theta^I)}{\partial h^2} = \theta^D$ and thus we need $\theta^D \geq 0$ such that $\mathcal{L}(\omega, h, a, \theta^D, \theta^I)$ is convex in h and therefore the solution in Equation (41) is indeed a (global) minimum. This implies $\theta^D = \sqrt{\frac{\sigma^2\omega^2}{2\eta}} \geq 0$ and therefore the second order Lagrange-conditions are satisfied for h^* and θ^{D*}. Along the same line of argumentation, we have $\theta^{J*} = \frac{\omega J - \log\left(1-J^2 w^2\right)}{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} \geq 0$ for any ω satisfying the solvency constraint $|\omega J| < 1$ and $1 + W\left(\frac{\eta^J-\lambda}{e\lambda}\right) \in (0, \infty)$, $\forall \lambda, \eta^J > 0$, and $\frac{\partial \mathcal{L}^2(\omega,h^*,a^*,\theta^{D*},\theta^{J*})}{\partial a^2} = e^a \lambda \theta^{J*} \geq 0$ which shows that a^* is the global minimizer. Next, using the optimal perturbation parameters h^* and a^* we can now solve for the optimal robust portfolio weights in closed form. Given $\mathcal{L}(\omega, h^*, a^*, \theta^{D*}, \theta^{J*})$ in Equation (35) the first order condition for ω is given by

$$\frac{\partial \mathcal{L}(\omega, h^*, a^*, \theta^{D*}, \theta^{J*})}{\partial \omega} = R - \sqrt{2\eta^D}\sigma - \lambda J e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} - 2\omega\sigma^2 + \lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} J/(1+\omega J) = 0 \tag{42}$$

which is a quadratic polynomial in the portfolio weights ω whose solution is given by[14]

$$\omega^*(h^*, a^*) = \frac{J(R - \sqrt{2\eta^D}\sigma) - J^2\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} - 2\sigma^2}{4J\sigma^2} + \frac{\sqrt{8J\sigma^2(R - \sqrt{2\eta^D}\sigma) + \left(J^2\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} + 2\sigma^2 - J(R - \sqrt{2\eta^D}\sigma)\right)^2}}{4J\sigma^2} \tag{43}$$

and we require that solvency constraint $|\omega^* J| < 1$ holds at the optimal portfolio allocation ω^*. In Figure 2 we plot the optimal robust portfolio weights as a function of the share of total robustness assigned to drift perturbation $\widetilde{\eta}^D$. As expected, the robust portfolio allocation is such that the amount invested into the risky asset is lower compared to the case when the investor has full confidence in his reference model ($\eta = 0$). Further, regardless of the share of η allocated to drift or jump intensity robustness, the optimal portfolio weights are more sensitive when η is low compared to the case when η is large. This implies that the marginal effect of increasing the total amount of robustness on $\omega^*(h^*, a^*)$ is declining. Next, comparing the case of only drift $\widetilde{\eta}^D = 1$ to only jump intensity $\widetilde{\eta}^D = 0$ perturbation shows that ω^* is reduced more in the case when there are solely concerns about potential drift as opposed to only jump intensity misspecification. This is a direct consequence of compensating

[14]Another case which leads to fully explicit portfolio weights is when $\gamma = 3$. The first order conditions for optimal portfolio holdings lead to a cubic equation, which given that $|wJ| < 1$, is solvable in closed form using standard methods.

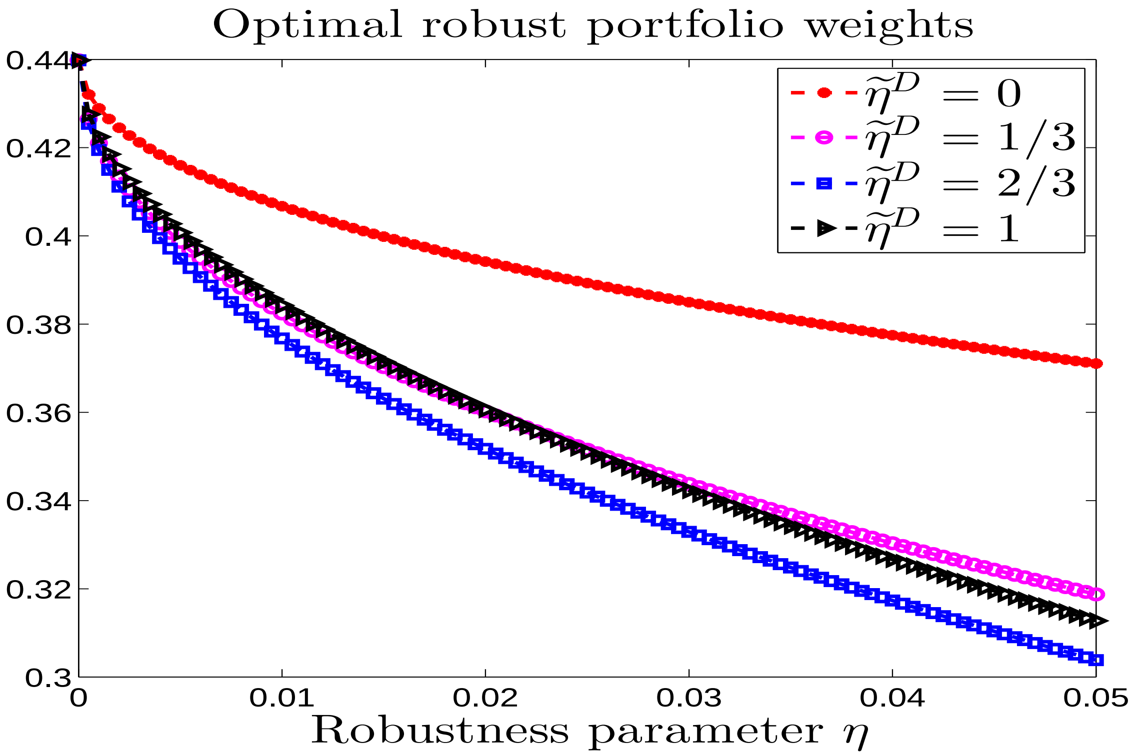

Figure 2: Comparison optimal robust portfolio weights as given in Equation (43) for $\widetilde{\eta}^D \in \{0, 1/3, 2/3, 1\}$. The selected parameters values are: $R = 0.1$, $\sigma = 0.1$, $J = -0.3$, $\lambda = 2$ and $\eta \in [0, 0.05]$.

the Lévy process. Furthermore, the optimal portfolio weights are lowest when $\widetilde{\eta}^D = 2/3$, suggesting that a very conservative investor, meaning very averse to model misspecification, would want to distribute about two thirds of the maximal amount of robustness to drift and one third to jump intensity perturbation.

There are also other settings where one can derive fully explicit portfolio weights. An interesting example is when the Poisson process is not compensated, meaning $Y_t = \sum_{n=1}^{N_t} Z_n$ and the *i.i.d.* jumps Z_n have symmetric Lévy measure about zero, i. e.

$$\lambda\nu(dz) = \begin{cases} \lambda dz/z & \text{if } z \in (0, 1], \\ -\lambda dz/z & \text{if } z \in [-1, 0) \end{cases} \tag{44}$$

so that the underlying asset exhibits both positive and negative jumps. The quantitative behavior of the optimal portfolio weights is very similar to the compensated Lévy case. An other interesting example is given when the jumps are negative only and we do not compensate the Lévy - process S_t. In this case, the portfolio weights are more sensitive to jump intensity perturbation, meaning the fraction of wealth invested into the risky asset is not only lower, but also decreases in η^J substantially faster compared to the compensated Lévy case.

3.2. Sensitivity Analysis of optimal Portfolio weights

Given the explicit portfolio weights we can now analyze their sensitivity with respect to both the asset price parameters R, σ, J, λ and the entropy growth parameters η^D and η^J. We start our analysis by comparing ω^* in Equation (43) to the classical Merton solution with and without robustness concerns, which can be obtained by letting $\lambda \to 0$. In this case, the risks are only diffusive and ω^* reduces to

$$\omega^*_{M,R} = \frac{R - \sqrt{2\eta^D}\sigma}{2\sigma^2} \tag{45}$$

From Equation (45) one can immediately see that an investor who is concerned about potential model misspecification will always invest less into the risky asset. Furthermore, for any $\lambda > 0$, $\omega^* = 0$ whenever $R = \sqrt{2\eta^D}\sigma$. This is intuitive as when the expected return under the robust measure is zero, since $\mathbb{E}_t^{\vartheta}[\widetilde{Y}_t] = 0$, the investor has no benefit in allocating funds into the risky asset. Note that

$$\frac{\partial \omega^*}{\partial R} = \frac{1}{4\sigma^2}\left(1 + \frac{\left(R - \sqrt{2\eta^D}\sigma\right) - e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} J^2\lambda + 2\sigma^2}{H}\right) > 0 \leftrightarrow 0 > J\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)}, \tag{46}$$

$$H := \sqrt{8J\sigma^2(R - \sqrt{2\eta^D}\sigma) + \left(J^2\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} + 2\sigma^2 - J(R - \sqrt{2\eta^D}\sigma)\right)^2} > 0 \tag{47}$$

from which we see that the inequality in (46) will always be satisfied under both measures. Thus increasing the expected return will lead to an increase of investment into the risky asset. Next, the partial derivative of ω^* with respect to λ is

$$\frac{\partial \omega^*}{\partial \lambda} = \frac{J\left(W\left(\frac{\eta^J-\lambda}{e\lambda}\right) - 1\right)\left(2\sigma^2 - J\left(R - \sqrt{2\eta^D}\sigma\right) + e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} J^2\lambda - H\right)}{4\sigma^2 H\left(1 + W\left(\frac{\eta^J-\lambda}{e\lambda}\right)\right)}$$

When there is no concern about potential jump intensity misspecification, i. e. $\eta^J = 0$ the optimal portfolio weights are insensitive with respect to changes in the jump intensity as $\frac{\partial \omega^*}{\partial \lambda} = 0$. This somewhat counterintuitive result is due to compensating the jump process. Increasing (decreasing) the jump intensity leads to negative jumps occurring more (less) frequently but simultaneously leads to an increase (decrease) in the compensator and thus in the expected return. Note that when $R = \sqrt{2\eta^D}\sigma$ we have that $\frac{\partial \omega^*}{\partial \lambda} = 0$ for any $\eta^J > 0$. For $\lambda \to \infty$ the optimal portfolio weight approach zero at the following rate

$$\omega^*|_{\lambda\to\infty} \sim \frac{1}{\lambda}\frac{R-\sqrt{2\eta^D}\sigma}{J^2 e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)}} + O\left(\frac{1}{\lambda^2}\right) \tag{48}$$

Thus, whenever $R - \sqrt{2\eta^D}\sigma > 0$, $\omega^* > 0$ and thus the optimal portfolio weight approaches zero from positive territory and vice versa when $R - \sqrt{2\eta^D}\sigma < 0$. Further, under the robust measure the rate is lower due to both drift (downward level shift) and jump intensity perturbation (downward scaling). Figure 3 shows that, under robustness concerns, that the amount invested into the risky

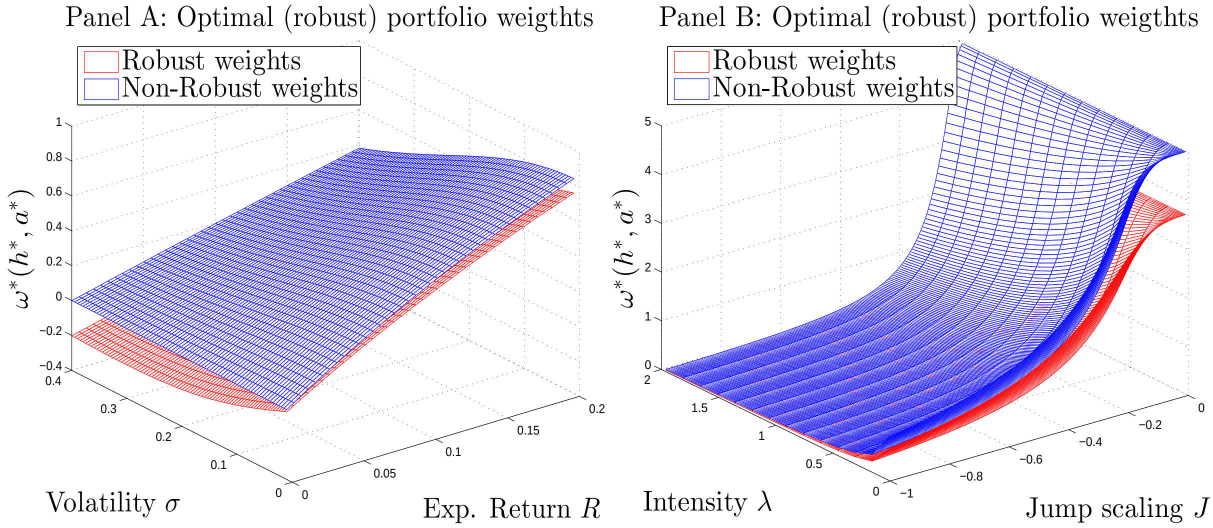

Figure 3: Comparison of robust and non-robust portfolio allocations when both drift and jump intensity are perturbed: Panel A displays $\omega^*(h^*, a^*)$ when both the expected return R and the volatility σ vary. Panel B displays $\omega^*(h^*, a^*)$ when both the jump scaling parameter J and the intensity λ vary. If not otherwise stated, the selected parameters values are: $R = 0.1$, $\sigma = 0.1$, $J = -0.3$, $\lambda = 2$ and $\eta = 0.05$.

asset is always lower than when there the investor has full confidence in his reference model. From Panel A we see that increasing volatility of the risky asset leads the investor to optimally decrease $\omega^*(h^*, a^*)$ and even short the asset whenever R is sufficiently low. Panel B shows that both J and λ are high, then the investor simply allocates all his funds into the risk free asset. Thus, the model can capture the well-documented empirical flight-to-quality behavior when jump risk λ is high and J approaches -1. However, contrarily to the case when jumps are only negative and the process is not compensated which leads the investor to optimally short the risky asset, i. e. $\omega^* \to -\infty$ as $\lambda \to \infty$ (see Aït-Sahalia et al. (2009)), in this setting our robust investor does not short the risky asset whenever J and λ are sufficiently high, but instead optimally chooses $\omega^* = 0$. Further, if either the jump scaling parameter or the intensity approach zero, i. e. the risky assets dynamics converges to a purely diffusive process, $\omega^*(h^*, a^*)$ increases non-linearly and the gap between the robust and non-robust portfolio weights widens. This suggests that for low jump risks $\lambda \downarrow$ and simultaneously low jump size scaling $J \to 0$, perturbations of the reference model have a higher impact on $\omega^*(h^*, a^*)$, meaning that there is a significant drop in the amount invested into the risky asset, compared to the case when both J and λ are high.

4. Error-Detection Probability

The robust portfolio weights derived in the previous sections crucially depend on the amount of model uncertainty we allow for. In order to quantify how much uncertainty seems reasonable to the investor, we make use of detection error probabilities as suggested by Anderson et al. (2003). More formally, let $\xi_t = \log(\vartheta_t)$ and $\mathcal{F}_t$ be the filtration with respect to which the probabilities and expectations are conditioned, the error-detection probability $\pi(t, T; \eta)$ is defined as the conditional probability at time t of making a detection-error given a sample of length $T > 0$,

$$\pi(t, T; \eta) = \frac{1}{2} \Big(\mathbb{P}\left[\xi_T > 0 | \mathcal{F}_t\right] + \mathbb{P}^{\vartheta}\left[\xi_T < 0 | \mathcal{F}_t\right] \Big), \qquad 0 \le t \le T. \tag{49}$$

Therefore, as η increases, so does the set of admissible models for the risky asset under $\mathbb{P}$ and $\mathbb{P}^{\vartheta}$, thereby causing the detection-error probability to decrease towards zero. Thus the larger η, the easier statistical discrimination between the model dynamics under $\mathbb{P}$ and $\mathbb{P}^{\vartheta}$ becomes. Anderson et al. (2003) suggest to choose η such that the error-detection probability is at least 10%. Note that when $\pi(t, T; \eta) = 0.5$ the models are statistically indistinguishable. Following Maenhout (2006), we now derive an expression for the conditional probabilities in (49) by means of Fourier transformation of the conditional probability measures. The conditional characteristic functions of ξ_T, under the reference measure $\mathbb{P}$, denoted by $\phi_{\mathbb{P}}(k, t, T)$ and $\phi_{\mathbb{P}^{\vartheta}}(k, t, T)$ under the robust measure $\mathbb{P}^{\vartheta}$ are given by

$$\phi_{\mathbb{P}}(k,t,T) = \mathbb{E}^{\mathbb{P}}\left[e^{ik\xi_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\vartheta_T^{ik}|\mathcal{F}_t\right] \tag{50}$$

$$\phi_{\mathbb{P}^\vartheta}(k,t,T) = \mathbb{E}^{\mathbb{P}^\vartheta}\left[e^{ik\xi_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}^\vartheta}\left[\vartheta_T^{ik}|\mathcal{F}_t\right] \tag{51}$$

where $i = \sqrt{-1}$ is the imaginary unit and $k \in \mathbb{R}$ is the transform variable. Using a simple measure change of the form

$$\phi_{\mathbb{P}^\vartheta}(k,t,T) = \int_{\omega\in\Omega} \vartheta_T^{ik}\vartheta_T d\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}\left[\vartheta_T^{1+ik}|\mathcal{F}_t\right] \tag{52}$$

the Fourier transform under the robust measure can be obtained by integrating with respect to the reference measure. By an application of Feynman-Kac's theorem, we can compute the conditional expectations Equation (50) and (52) by solving a partial differential difference equation (PDDE) with appropriate boundary conditions. In order to derive this PDDE for both conditional Fourier transforms we need the dynamics of the measure change under $\mathbb{P}$ given the optimal perturbation policies h_t^* and a^*. An application of Itô's product formula for Semimartingales to the optimally perturbed ϑ_t^* shows that

$$\begin{aligned}
\vartheta_t^* = \vartheta_t^{*,D}\vartheta_t^{*,J} &= \vartheta_t^{*,D}\vartheta_t^{*,J} + \int_0^t \vartheta_{s_-}^{*,J} d\vartheta_s^{*,D} + \int_0^t \vartheta_s^{*,D} d\vartheta_s^{*,J} + \left[\vartheta^{*,D},\vartheta^{*,J}\right]_t \\
&= \vartheta_t^{*,D}\vartheta_t^{*,J} + \int_0^t \vartheta_{s_-}^{*,J} d\vartheta_s^{*,D} + \int_0^t \vartheta_s^{*,D} d\vartheta_s^{*,J}
\end{aligned}$$

Then applying Itô's formula to Equation (6) and using Equation (28) we obtain

$$d\vartheta_t^* = \vartheta_t h_t^* dB_t + \vartheta_{t_-}\left(e^{a^*}-1\right) d\hat{N}_t \tag{53}$$

From Equation (53) the dynamics of the log measure change are

$$\begin{aligned}
\xi_t &= \xi_0 + \int_0^t h_s^* dB_s - \int_0^t \left(\frac{1}{2}h_s^{*2} + \lambda\left(e^{a^*}-1\right)\right) ds + \sum_{0<s\le t}\left(\xi_s - \xi_{s_-}\right) \\
&= \xi_0 + \int_0^t h_s^* dB_s - \int_0^t \left(\frac{1}{2}h_s^{*2} + \lambda\left(e^{a^*}-1\right)\right) ds + a^* N_t
\end{aligned} \tag{54}$$

and therefore the PDDE for $\phi_{\mathbb{P}}(t) = \phi_{\mathbb{P}}(k,t,T)$ is given by

$$\begin{aligned}
0 &= \mathcal{A}\phi_{\mathbb{P}}(k,t,T) \\
&= \frac{\partial \phi_{\mathbb{P}}}{\partial t} - \frac{\partial \phi_{\mathbb{P}}}{\partial \xi}\left(\frac{1}{2}h_t^{*2} + \lambda\left(e^{a^*}-1\right)\right) + \frac{1}{2}\frac{\partial^2 \phi_{\mathbb{P}}}{\partial \xi^2} h_t^{*2} + \lambda\left(\phi_{\mathbb{P}}(k,t,T) - \phi_{\mathbb{P}}(k,t_-,T)\right)
\end{aligned} \tag{55}$$

subject to the following boundary condition

$$\phi_{\mathbb{P}}(k,T,T) = \vartheta_T^{ik} \tag{56}$$

and likewise the PDDE for $\phi_{\mathbb{P}^\vartheta}(t) = \phi_{\mathbb{P}^\vartheta}(k,t,T)$ is equivalent and given by

$$\begin{aligned} 0 &= \mathcal{A}\phi_{\mathbb{P}^\vartheta}(k,t,T) \\ &= \frac{\partial \phi_{\mathbb{P}^\vartheta}}{\partial t} - \frac{\partial \phi_{\mathbb{P}^\vartheta}}{\partial \xi}\left(\frac{1}{2}h_t^{*2} + \lambda\left(e^{a^*}-1\right)\right) + \frac{1}{2}\frac{\partial^2 \phi_{\mathbb{P}^\vartheta}}{\partial \xi^2}h_t^{*2} + \lambda\left(\phi_{\mathbb{P}^\vartheta}(k,t,T) - \phi_{\mathbb{P}^\vartheta}(k,t_-,T)\right) \end{aligned} \tag{57}$$

subject to a different boundary condition given by

$$\phi_{\mathbb{P}^\vartheta}(k,T,T) = \vartheta_T^{1+ik}.$$

The PDDE in (57) admits a unique affine solution of the form

$$\phi_{\mathbb{P}}(k,t,T) = e^{\alpha(T-t)+\beta(T-t)\xi_t} \tag{58}$$

Inserting the conjecture of Equation (58) into Equation (57) gives

$$0 = -\alpha^{'} - \beta^{'}\xi_t - \beta\left(\frac{1}{2}h_t^{*2} + \lambda\left(e^{a^*}-1\right)\right) + \frac{1}{2}h_t^{*2}\beta^2 + \lambda\left(e^{\beta a^*}-1\right).$$

Using the fact that this equation has to hold for all ξ_t and the constant terms we get two equations, namely

$$\beta(T-t) = \beta = \int_t^T \beta^{'} = K \to K = ik = \beta \tag{59}$$

$$\alpha(T-t) = \int_t^T \alpha^{'} = \left[\beta\left(\frac{1}{2}h_t^{*2} + \lambda\left(e^{a^*}-1\right)\right) + \frac{1}{2}h_t^{*2}\beta^2 + \lambda\left(e^{\beta a^*}-1\right)\right](T-t) \tag{60}$$

where the expression for β in Equation (59) follows from the boundary condition in Equation (56). The solution for Equation (57) is identical except that $\beta = 1 + ik$. Given these conditional characteristic functions $\phi_{\mathbb{P}^\vartheta}(k,t,T)$ and $\phi_{\mathbb{P}}(k,t,T)$, the error-detection probability in Equation (49) based on a sample of length $T-t$ is given by

$$\pi(t,T;\eta) = \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \left(\mathrm{Re}\left[\frac{\phi_{\mathbb{P}^\vartheta}(k,0,T)}{ik}\right] - \mathrm{Re}\left[\frac{\phi_{\mathbb{P}}(k,0,T)}{ik}\right]\right)dk$$

where $\mathrm{Re}(\cdot)$ denotes the real part of a complex number. Let $\tau = T-t$, in Figure 4 we plot the error-detection probability as a function of the robustness parameter η. As expected, $\pi(t,T;\eta)$ is monotonously decreasing in η. However, the rate of decrease is higher for low values of the robustness

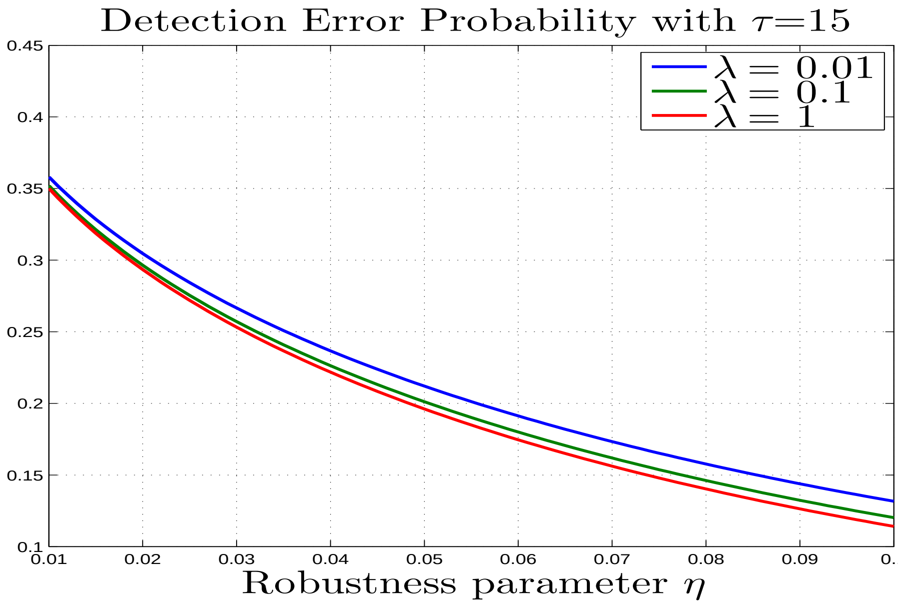

Figure 4: Error-detection probability $\pi(t, T; \eta)$ computed from a time series length of 15 years.

parameter indicating that the error-detection probability is more sensitive to changes in η in this area. Furthermore, a larger intensity under the reference measure $\mathbb{P}$ amplifies this sensitivity of $\pi(t, T; \eta)$ as η grows. This shows that it becomes statistically easier to discriminate whether a given time series realization was generated by a process under measure $\mathbb{P}$ or by a process under measure $\mathbb{P}^{\vartheta}$.

5. Conclusion

In this paper we study a robust optimal consumption and portfolio choice problem where the underlying risky asset follows a Lévy process. We introduce model misspecification with respect to drift and jump intensity parameters and derive explicit expressions for optimal consumption and portfolio rules. Our main findings are that perturbations of the drift are relatively more important than perturbation of the jump intensity in the case when jumps are symmetric about zero, i. e. the occurrence of positive and negative jumps is equally likely. However, assuming only negative jumps in the underlying asset shows that optimal portfolio weights become not only more sensitive to perturbations of the jump intensity but also, that misspecification with respect to the jump intensity becomes more important than misspecification of the drift. If we incorporate a compensated Poisson process for the jump part of the risk asset, shows that optimal allocations are now less sensitive with respect to perturbations of the jump intensity as opposed to perturbations of the drift. This is to be expected as the investor receives a compensation in the form of higher expected return when jumps are compensated. Additionally, we derive a semi-closed form solution for detection-error probabilities. Our sensitivity analysis shows that the detection-error probability is relatively insensitive to changes in the jump intensity which implies that for two processes one generated under the reference and another one generated under the robust measure, they need to have very distinct jump intensities in order to be correctly classified by the detection-error probability.

6. Appendix

A. Optimal Portfolio weights

A.1. Joint Drift and Jump Intensity Perturbation with CRRA Utility and symmetric jumps

In this section, we derive explicit portfolio weights when the risky asset follows an exponential Lévy process under the robust measure $\mathbb{P}^{\vartheta}$ of the form

$$\frac{dS_{1,t}}{S_{1,t_-}} = (r + R + \sigma h_t)dt + \sigma dB_t^{\vartheta} + JdY_t^{\vartheta},\ S_{0,1} > 0,\ \mathbb{P}^{\vartheta} - a.s.$$

and jumps follow a symmetric power law distribution as in Equation (44). Then the corresponding Lagrangian reads

$$\begin{aligned}\mathcal{L}(\omega, h, a, \theta^D, \theta^I) &= \omega R + \sigma h\omega - \omega\lambda J e^a - \omega^2\sigma^2 + \lambda e^a \log\left(1 + \omega_t J\right) \\ &+ \theta^D\left(\frac{1}{2}h^2 - \eta^D\right) + \theta^I\left(\lambda e^a(a-1) + \lambda - \eta^J\right). \end{aligned} \tag{61}$$

from which we obtain the following first order conditions

$$\theta^{D*} = \pm\sqrt{\frac{\sigma^2\omega^2}{2\eta^D}} \rightarrow h_t^* = -\sqrt{2\eta^D} \tag{62}$$

$$\theta^{J*} = \frac{-\log\left(1 - J^2\omega^2\right)}{1 + W\left(\frac{\eta^J - \lambda}{e\lambda}\right)} \rightarrow a^* = 1 + W\left(\frac{\eta^J - \lambda}{e\lambda}\right) \tag{63}$$

Note that $\theta^{J*}(\omega) \geq 0$ for any ω satisfying the solvency constraint $|\omega^* J| < 1$. Given the optimal perturbation parameters in Equation (62) and (63), the objective function is

$$\begin{aligned}\mathcal{L}(\omega, h^*, a^*, \theta^{D*}, \theta^{J*}) &= \omega R - \sigma\omega\sqrt{2\eta^D} - \omega^2\sigma^2 + \lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} \log\left(1 + \omega J\right) \\ &+ \lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} \log\left(1 - \omega J\right) \end{aligned} \tag{64}$$

and the first order condition for the optimal portfolio weight is

$$\begin{aligned}\frac{\mathcal{L}(\omega, h^*, a^*, \theta^{D*}, \theta^{J*})}{\partial\omega} &= R - \sigma\sqrt{2\eta^D} - 2\omega\sigma^2 + \lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} J/(1 + \omega J) \\ &- \lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} J/(1 - \omega J) = 0 \end{aligned} \tag{65}$$

Equation (65) is a cubic polynomial in the portfolio weights ω. Let

$$A = 2J^2\sigma^2, \qquad B^{DI} = J^2(\sqrt{2\eta^D}\sigma - R)$$
$$C^{DI} = -2\left(\sigma^2 + J^2\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)}\right), \qquad D^{DI} = R - \sqrt{2\eta^D}\sigma$$

Then Equation (65) can be written as $Ax^3 + Bx^2 + Cx + D$. Defining $a^{DI} = B^{DI}/A$, $b^{DI} = C^{DI}/A$, $c^{DI} = D^{DI}/A$ and $G^{DI} = (a^{DI^2} - 3b^{DI})/9$, $H^{DI} = (2a^{DI^2} - 9a^{DI}b^{DI} + 27c^{DI})/54$ the solution to Equation (65) subject to the solvency condition $|J\omega| < 1$ is given by

$$\omega^* = -2\sqrt{G^{DI}}\cos\left(\operatorname{acos}\left(\frac{1}{3}H^{DI}/\sqrt{G^{DI}} - \frac{2\pi}{3}\right)\right) - a^{DI}/3 \tag{66}$$

A.2. Exponential Utility: Closed - form portfolio weights with compensated exponential Lévy dynamics

In this section we consider an investor with exponential utility of the form

$$U(C) = -\frac{1}{q}e^{-qC}, \quad \text{with CARA coefficient } q > 0$$

where his wealth dynamics under the robust measure evolves as in Equation (11), that is no jump size perturbation. We analyze robust optimal portfolio holdings where the jumps sizes follow an exponential distribution, i. e. $Z_n \overset{i.i.d}{\sim} \text{Exp}(\xi)$, with $\nu(dz) = f_Z(z;\xi) = \xi e^{-z\xi}$, $z \geq 0$. Then conjecturing a solution to Equation (18) of the form $L(x) = -K/qe^{-rqx}$ where are $r > 0$ is the risk free rate in Equation (1) and K some constant to be determined, we have

$$\frac{\partial L(x)}{\partial x} = -rqL(x), \ \frac{\partial^2 L(x)}{\partial x^2} = r^2q^2L(x) \tag{67}$$

The robust control problem in Equation (18) is then given by

$$\begin{aligned} 0 = \max_{\{C_t,\omega_t\}}\min_{\{h_t\},a} & U(C_t) - \beta L(X_t) - rqL(X_t)\left[X_t\left(r + \omega_t\left(R + \sigma h_t - \lambda e^a J\int z\nu(dz)\right)\right) - C_t\right] \\ & + \frac{1}{2}r^2q^2L(X_t)\omega_t^2X_t^2\sigma^2 + \lambda e^a\int\left[e^{-rqX_{t\_}\omega_{t\_}Jz}L(X_{t\_}) - L(X_{t\_})\right]\nu(dz) \end{aligned} \tag{68}$$

subject to

$$\frac{1}{2}h_t^2 \leq \eta^D, \ \lambda e^a(a-1) + \lambda \leq \eta^D, \ \eta^D \geq 0, \ \eta^J \geq 0 \tag{69}$$

We fix $J = -1$, so that jumps are negative. Dividing Equation (68) above by $-rqL(X_t) > 0$ we obtain

$$
\begin{aligned}
0 = \max_{\{C_t,\omega_t\}} \min_{\{h_t\},a} & -\frac{U(C_t)}{rqL(X_t)} + \frac{\beta}{qr} + X_t\left(r + \omega_t\left(R + \sigma h_t + \frac{\lambda e^a}{\xi}\right)\right) - C_t \\
& - \frac{1}{2} rq\omega_t^2 X_t^2 \sigma^2 - \frac{\lambda e^a \omega_t X_t}{\xi - qr\omega_t X_t}
\end{aligned} \tag{70}
$$

subject to

$$
\frac{1}{2}h_t^2 \leq \eta^D,\ \lambda e^a(a-1) + \lambda \leq \eta^J,\ \eta^D \geq 0,\ \eta^J \geq 0 \tag{71}
$$

We define as $\mathcal{L} = \mathcal{L}(C_t, h_t, a, \theta^D, \theta^J)$ the Lagrangian corresponding to the perturbed HJB problem in Equation (68) with Lagrange multiplier θ^D and θ^J for the diffusive and jump intensity part of the entropy constraint respectively. Then we have[15]

$$
\begin{aligned}
\mathcal{L}(C_t, w, h_t, a, \theta^D, \theta^J) = & -\frac{U(C_t)}{rqL(X_t)} + w\left(R + \sigma h_t + \lambda e^a \int z\nu(dz)\right) - C_t - \frac{1}{2} rqw^2\sigma^2 \\
& - \frac{\lambda e^a}{rq} \int \left[e^{rqwz} - 1\right] \nu(dz) + \theta^D\left(\frac{1}{2}h_t^2 - \eta^D\right) + \theta^J\left(\lambda e^a(a-1) + \lambda - \eta^J\right)
\end{aligned} \tag{72}
$$

where $w = \omega_t X_t$ is the (absolute) amount of money invested into the risky asset. As before, it follows that $h_t^* = \sqrt{2\eta^D}$, $\theta^D = \sqrt{\frac{\sigma^2\omega^2}{2\eta^D}}$ and $a^* = 1 + W\left(\frac{\eta^J - \lambda}{e\lambda}\right)$ and $\theta^I = \frac{qrw^2}{\xi\left(1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)\right)(\xi - qrw)} \geq 0, \forall w \in \mathbb{R}$. Given the optimal perturbation parameters, the objective function is

$$
\begin{aligned}
\mathcal{L}(C_t, w, h^*, a^*, \theta^{D*}, \theta^{I*}) = & -\frac{U(C_t)}{rqL(X_t)} + w\left(R - \sqrt{2\eta}\sigma + \frac{\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)}}{\xi}\right) \\
& - \frac{\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} w}{\xi - qrw} - \frac{1}{2}qr\sigma^2 w^2
\end{aligned} \tag{73}
$$

The first order condition for the optimal portfolio weight is

$$
-\lambda \frac{e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)} qrw}{(\xi - qrw)^2} + \frac{\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)}}{\xi} - \frac{\lambda e^{1+W\left(\frac{\eta^J-\lambda}{e\lambda}\right)}}{\xi - qrw} - \sqrt{2\eta}\sigma - qr\sigma^2 w + R = 0 \tag{74}
$$

[15] The integral with respect to the jump measure in Equation (70) is only convergent if $\xi - qrw > 0$ which is very likely to be satisfied given standard parameter values for risk aversion $q \in \{1, \ldots, 10\}$, $r \in [0, 0.05]$ and $\xi > 10$ which would correspond to a jump size of 10% or less.

which is a cubic polynomial in the amount invested in the risky asset. Let

$$A(\xi) = -\xi q^3 r^3 \sigma^2, \qquad B^{DJ}(\xi) = q^2 r^2 \left(\lambda e^{1+W\left(\frac{\eta^J - \lambda}{e\lambda}\right)} + \xi \left(R - \sqrt{2\eta^D} J\sigma + 2\xi\sigma^2 \right) \right),$$

$$C^{DJ}(\xi) = -\xi qr \left(2e^{1+W\left(\frac{\eta^J - \lambda}{e\lambda}\right)} \lambda + \xi \left(2(R - \sqrt{2\eta^D}\sigma) + \xi\sigma^2 \right) \right), \qquad D^{DJ}(\xi) = \xi^3 \left(\sqrt{2\eta^D}\sigma - R \right)$$

Then Equation (74) can be written as $A(\xi)x^3 + B^{DI}(\xi)x^2 + C^{DI}(\xi)x + D^{DI}(\xi)$. Next, we define $a^{DI}(\xi) = B^{DI}(\xi)/A(\xi)$, $b^{DI}(\xi) = C^{DI(\xi)}/A(\xi)$, $c^{DI}(\xi) = D^{DI}(\xi)/A(\xi)$ and $G^{DI}(\xi) = \left(a^{DI}(\xi)^2 - 3b^{DI}(\xi) \right)/9$, $H^{DI}(\xi) = \left(2a^{DI}(\xi)^2 - 9a^{DI}(\xi)b^{DI}(\xi) + 27c^{DI}(\xi) \right)/54$ such that the solution to Equation (74) subject to the integral convergence condition $\xi - qrw > 0$ is given by

$$w_E^* = -2\sqrt{G^{DJ}(\xi)} \cos \left(\operatorname{acos} \left(\frac{1}{3} H^{DJ}(\xi) / \sqrt{G^{DJ}(\xi)} - \frac{2\pi}{3} \right) \right) - a^{DJ}(\xi)/3$$

where w_E^* refers to both drift and jump intensity compensated portfolio weights when the investor is assumed to have exponential utility. In Figure 5 below we plot the portfolio weights for different levels of jump sizes ξ. A first important observation from Panel A to D is that increasing the size of the jumps ($\xi \downarrow$) reduces the absolute amount invested into the risky asset w_E^* as $\mathbb{E}^{\vartheta}[z] = \mathbb{E}[z] = 1/\xi$. A tenfold increase in the coefficient of risk aversion (q) leads to a proportional decrease in w_E^*. On the other hand, increasing the frequency of jumps (λ) by a factor of ten, shows that the optimal investment into the risky asset decreases over-proportionally.

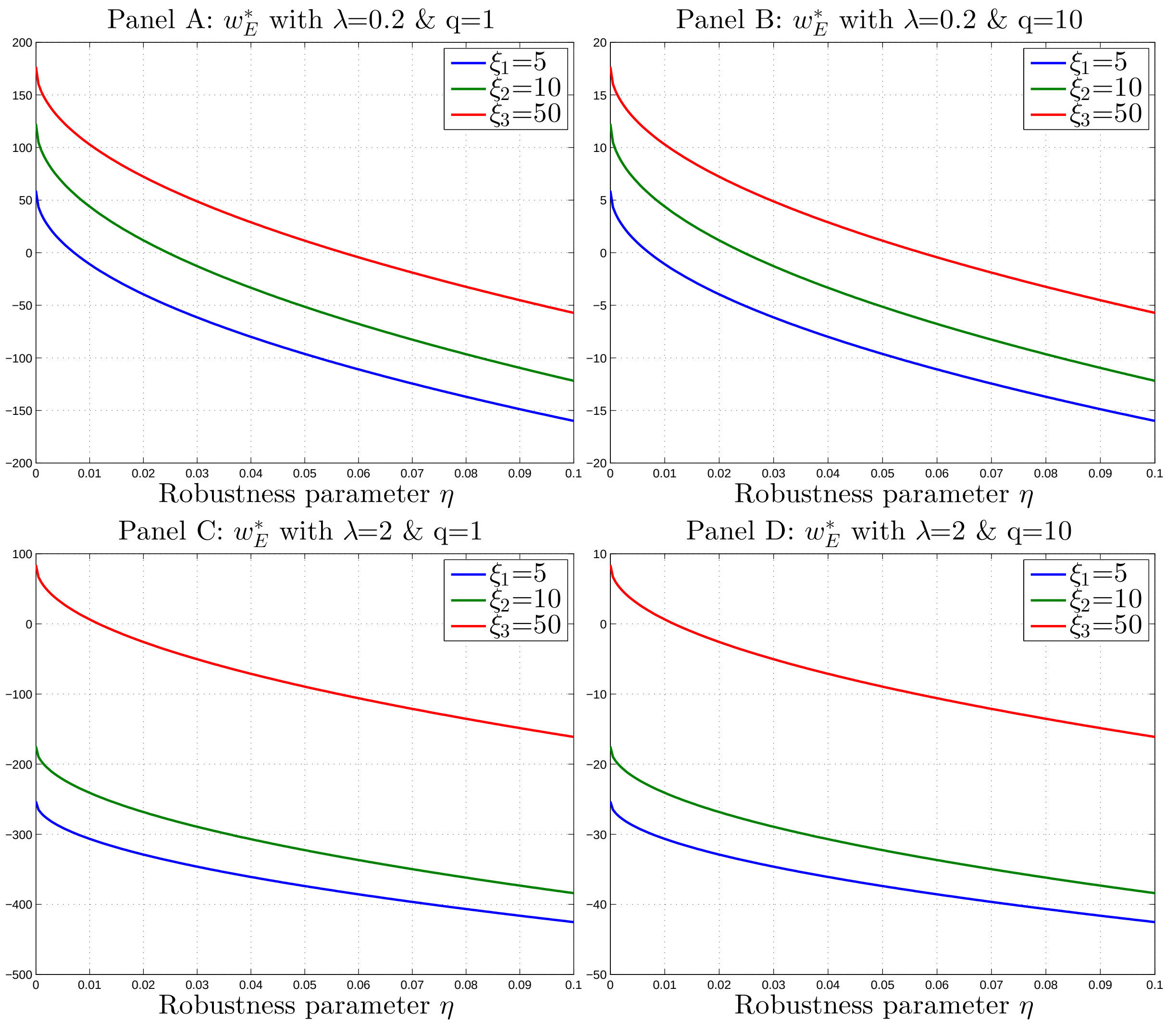

Figure 5: Optimal portfolio holdings w_E^* in the case of exponential utility and the underlying risky asset exhibits compensated Lévy dynamics as given in Equation (3) and both drift and jump intensity are perturbed: The benchmark parameters values are: $R = 0.1$, $\sigma = 0.2$, $J = 0.2$, $\lambda = 0.2, 2$, $q = 1, 10$ and $\eta \in [0, 0.1]$.

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