DC decompositions via algebraic techniques
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Win my poster’s heart!

Write the heart polynomial as a difference of two convex polynomials to win a prize!

Given a polynomial $f$, how to find a difference of convex (dc) decomposition of $f$, i.e., convex polynomials $g$ and $h$ such that $f = g - h$?

Motivation

Difference of convex programming:

$$\min g_0(x) - h_0(x) \quad \text{s.t. } g_i(x) - h_i(x) \leq 0 \quad \forall i \in I$$

where $g_i, h_i, i = 0, \ldots, m$ are all convex for $I = \{0, \ldots, m\}$. Applications in Machine Learning, Statistical Physics, and MINLP.

Our question: what if we don’t have such a decomposition?

Does a dc decomposition always exist?

**Theorem:** Any polynomial can be written as the difference of two convex polynomials.

**Proof:** Let $K$ be a full dimensional cone in a vector space $E$. Then any $v \in E$ can be written as $v = k_1 - k_2$, where $k_1, k_2 \in K$.

Let $k \in \text{int}(K)$:

$$3 \sigma < 1 \text{ such that } (1 - \sigma)v + \sigma k \in K$$

$$\begin{align*}
\sigma v &= 1 - \sigma k - \sigma\sigma k = \sigma v - \sigma k \\
&= k_1 - k_2 \in K
\end{align*}$$

Here:

$K$ is the set of polynomials of degree $d$ in $n$ variables.

Why is $K$ full dimensional?

$\begin{align*}
p(x_1, \ldots, x_n) &= \{y^2\}^d \\
&\text{is in the interior of } K
\end{align*}$

Is the decomposition unique?

No:

Initial decomposition $f(x) = g_0(x) - h_0(x)$

Alternative decomposition $f(x) = (g_0(x) + p(x)) - (h_0(x) + p(x))$ for $p$ convex

What is the “best” decomposition?

We want to make the Convex-Concave Procedure (CCP) faster:

- CCP is a popular heuristic for solving difference of convex programs.

Input

Output

Converge by checking $h$ is a feasible solution:

Take $x_0$ to be the solution of $\min f(x)$.

Solve convex subproblem

$$\begin{align*}
h(x) &= f(x) - R(x) \\
&\text{(where } R(x) \text{ is the regularizer)}
\end{align*}$$

Best decomposition at a point (or iterate)

Best decomposition globally

<table>
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<th>Idea</th>
<th>Example</th>
<th>Notation</th>
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</thead>
<tbody>
<tr>
<td>Pick $h_0(x)$ with lowest curvature around $x_0$</td>
<td>$h_0(x)$</td>
<td>$h_0(x)$ with lowest curvature overall</td>
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Mathematical translation

$\min \{ f(x) - R(x) \}$

Find an undominated decomposition $(g, h)$ of $f$.

Mathematically, such that no other decomposition of $f$ can be obtained by restricting a convex function from $h$.

Optimization problem

$$\begin{align*}
\mathbb{R}_+^{n+1} &\to \mathbb{R} \quad (x, y, z) &\mapsto f(x) - R(x) \\
\text{subject to} & & & &
\end{align*}$$

Theorem: Given a polynomial $f$, consider

$$\min_{x, y, z} f(x) - R(x) - \sum_{i=1}^n z_i y_i$$

An optimal solution strongly exists and is an undominated dc decomposition of $f$.

But:

Theorem: If $f$ has degree 4, it is strongly NP-hard to solve (2).

Are dc decompositions computable efficiently?

**Definition:** A polynomial $p$ is sos convex if $y^T H_p(x) y$ is a sum of squares of polynomials in $x$ and $y$, i.e.,

$$y^T H_p(x) y = \sum_{i=1}^n q_i(x, y)^2$$

Alternative characterization: $p$ is sos convex if and only if $\exists Q \succ 0$ s.t.

$$y^T H_p(x) y = (x, y)^T (Q(x, y) Q(x, y))^T (x, y)^T$$

where $Q(x, y)$ is the standard vector of monomials in $x$ and $y$.

Find $g, h$ such that:

$$\begin{align*}
\min g(x) - h(x) \\
\text{subject to } & & & &
\end{align*}$$

What did we learn?

- We studied the question of decomposing a polynomial into the difference of two convex polynomials.
- This decomposition always exists and is not unique.
- Choice of decomposition can impact convergence rate of the CCP algorithm.
- DC decompositions can be efficiently obtained using convex relaxations based on sos-convexity (LP), sos-convexity (SDC), and sos-convexity (SDP).
- Dc-convex and sos-convex relaxations scale to a large number of variables.

Want to know more? [http://scholar.princeton.edu/shall]