

Polynomial Norms

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Abstract

In this paper, we study *polynomial norms*, i.e. norms that are the d^{th} root of a degree- d homogeneous polynomial f . We first show that a necessary and sufficient condition for $f^{1/d}$ to be a norm is for f to be strictly convex, or equivalently, convex and positive definite. Though not all norms come from d^{th} roots of polynomials, we prove that any norm can be approximated arbitrarily well by a polynomial norm. We then investigate the computational problem of testing whether a form gives a polynomial norm. We show that this problem is strongly NP-hard already when the degree of the form is 4, but can always be answered by testing feasibility of a semidefinite program (of possibly large size). We further study the problem of optimizing over the set of polynomial norms using semidefinite programming. To do this, we introduce the notion of *r-sos-convexity* and extend a result of Reznick on sum of squares representation of positive definite forms to positive definite biforms. We conclude with some applications of polynomial norms to statistics and dynamical systems.

Keywords: polynomial norms, sum of squares polynomials, convex polynomials, semidefinite programming

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1 Introduction

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a *norm* if it satisfies the following three properties:

- (i) positive definiteness: $f(x) > 0$, $\forall x \neq 0$, and $f(0) = 0$.
- (ii) 1-homogeneity: $f(\lambda x) = |\lambda|f(x)$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}$.
- (iii) triangle inequality: $f(x + y) \leq f(x) + f(y)$, $\forall x, y \in \mathbb{R}^n$.

Some well-known examples of norms include the 1-norm, $f(x) = \sum_{i=1}^n |x_i|$, the 2-norm, $f(x) = \sqrt{\sum_{i=1}^n x_i^2}$, and the ∞ -norm, $f(x) = \max_i |x_i|$. Our focus throughout this paper is on norms that can be derived from multivariate polynomials. More specifically, we are interested in establishing conditions under which the d^{th} root of a homogeneous polynomial of degree d is a norm, where d is an even number. We refer to the norm obtained when these conditions are met as a *polynomial norm*. It is easy to see why we restrict ourselves to d^{th} roots of degree- d homogeneous polynomials. Indeed, nonhomogeneous polynomials cannot hope to satisfy the homogeneity condition and homogeneous

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polynomials of degree $d > 1$ are not 1-homogeneous unless we take their d^{th} root. The question of when the square root of a homogeneous quadratic polynomial is a norm (i.e., when $d = 2$) has a well-known answer (see, e.g., [14, Appendix A]): a function $f(x) = \sqrt{x^T Q x}$ is a norm if and only if the symmetric $n \times n$ matrix Q is positive definite. In the particular case where Q is the identity matrix, one recovers the 2-norm. Positive definiteness of Q can be checked in polynomial time using for example Sylvester's criterion (positivity of the n leading principal minors of Q). This means that testing whether the square root of a quadratic form is a norm can be done in polynomial time. A similar characterization in terms of conditions on the coefficients are not known for polynomial norms generated by forms of degree greater than 2. In particular, it is not known whether one can efficiently test membership or optimize over the set of polynomial norms.

Outline and contributions. In this paper, we study polynomial norms from a computational perspective. In Section 2, we give two different necessary and sufficient conditions under which the d^{th} root of a degree- d form f will be a polynomial norm: namely, that f be strictly convex (Theorem 2.2), or (equivalently) that f be convex and positive definite (Theorem 2.1). Section 3 investigates the relationship between general norms and polynomial norms: while many norms are polynomial norms (including all p -norms with p even), some norms are not (consider, e.g., the 1-norm). We show, however, that any norm can be approximated to arbitrary precision by a polynomial norm (Theorem 3.1). In Section 4, we move on to complexity results and show that simply testing whether the d^{th} root of a quartic form is a norm is strongly NP-hard (Theorem 4.1). We then provide a semidefinite programming-based test for checking whether the d^{th} root of a degree d form is a norm (Theorem 4.4) and a semidefinite programming-based hierarchy to optimize over a subset of the set of polynomial norms (Theorem 4.20). The latter is done by introducing the concept of r -sum of squares-convexity (see Definition 4.6). We show that any form with a positive definite Hessian is r -sos-convex for some value of r , and present a lower bound on that value (Theorem 4.7). We also show that the level r of the semidefinite programming hierarchy cannot be bounded as a function of the number of variables and the degree only (Theorem 4.18). Finally, we cover a few applications of polynomial norms in statistics and dynamical systems in Section 5. In Section 5.1, we compute approximations of two different types of norms, polytopic gauge norms and p -norms with p noneven, using polynomial norms. The techniques described in this section can be applied to norm regression. In Section 5.2, we use polynomial norms to prove stability of a switched linear system, a task which is equivalent to computing an upperbound on the joint spectral radius of a family of matrices.

2 Two equivalent characterizations of polynomial norms

We start this section with two theorems that provide conditions under which the d^{th} root of a degree- d form is a norm. These will be useful in Section 4 to establish semidefinite programming-based approximations of polynomial norms. Note that throughout this paper, d is taken to be an even positive integer.

Theorem 2.1. *The d^{th} root of a degree- d form f is a norm if and only if f is convex and positive definite.*

Proof. If $f^{1/d}$ is a norm, then $f^{1/d}$ is positive definite, and so is f . Furthermore, any norm is convex and the d^{th} power of a nonnegative convex function remains convex.

Assume now that f is convex and positive definite. We show that $f^{1/d}$ is a norm. Positivity and homogeneity are immediate. It remains to prove the triangle inequality. Let $g := f^{1/d}$. Denote by S_f and S_g the 1-sublevel sets of f and g respectively. It is clear that

$$S_g = \{x \mid f^{1/d}(x) \leq 1\} = \{x \mid f(x) \leq 1\} = S_f,$$

and as f is convex, S_f is convex and so is S_g . Let $x, y \in \mathbb{R}^n$. We have that $\frac{x}{g(x)} \in S_g$ and $\frac{y}{g(y)} \in S_g$. From convexity of S_g ,

$$g\left(\frac{g(x)}{g(x) + g(y)} \cdot \frac{x}{g(x)} + \frac{g(y)}{g(x) + g(y)} \cdot \frac{y}{g(y)}\right) \leq 1.$$

Homogeneity of g then gives us

$$\frac{1}{g(x) + g(y)} g(x + y) \leq 1$$

which shows that triangle inequality holds. \square

Theorem 2.2. *The d^{th} root of a degree- d form f is a norm if and only if f is strictly convex, i.e.,*

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y), \forall x \neq y, \forall \lambda \in (0, 1).$$

Proof. We will show that a degree- d form f is strictly convex if and only if f is convex and positive definite. The result will then follow from Theorem 2.1.

Suppose f is strictly convex, then the first-order characterization of strict convexity gives us that

$$f(y) > f(x) + \nabla f(x)^T (y - x), \forall y \neq x.$$

For $x = 0$, the inequality becomes $f(y) > 0, \forall y \neq 0$, as $f(0) = 0$ and $\nabla f(0) = 0$. Hence, f is positive definite. Of course, a strictly convex function is also convex.

Suppose now that f is convex, positive definite, but not strictly convex, i.e., there exists $\bar{x}, \bar{y} \in \mathbb{R}^n$ with $\bar{x} \neq \bar{y}$, and $\gamma \in (0, 1)$ such that

$$f(\gamma \bar{x} + (1 - \gamma)\bar{y}) = \gamma f(\bar{x}) + (1 - \gamma)f(\bar{y}).$$

Let $g(\alpha) := f(\bar{x} + \alpha(\bar{y} - \bar{x}))$. Note that g is a restriction of f to a line and, consequently, g is a convex, positive definite, univariate polynomial in α . We now define

$$h(\alpha) := g(\alpha) - (g(1) - g(0))\alpha - g(0). \quad (1)$$

Similarly to g , h is a convex univariate polynomial as it is the sum of two convex univariate polynomials. We also know that $h(\alpha) \geq 0, \forall \alpha \in (0, 1)$. Indeed, by convexity of g , we have that $g(\alpha x + (1 - \alpha)y) \geq \alpha g(x) + (1 - \alpha)g(y), \forall x, y \in \mathbb{R}$ and $\alpha \in (0, 1)$. This inequality holds in particular for $x = 1$ and $y = 0$, which proves the claim. Observe now that $h(0) = h(1) = 0$. By convexity of h and its nonnegativity over $(0, 1)$, we have that $h(\alpha) = 0$ on $(0, 1)$ which further implies that $h = 0$. Hence, from (1), g is an affine function. As g is positive definite, it cannot be that g has a nonzero slope, so g has to be a constant. But this contradicts that $\lim_{\alpha \rightarrow \infty} g(\alpha) = \infty$. To see why this limit must be infinite, we show that $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$. As $\lim_{\alpha \rightarrow \infty} \|\bar{x} + \alpha(\bar{y} - \bar{x})\| = \infty$ and $g(\alpha) = f(\bar{x} + \alpha(\bar{y} - \bar{x}))$, this implies that $\lim_{\alpha \rightarrow \infty} g(\alpha) = \infty$. To show that $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$, let

$$x^* = \operatorname{argmin}_{\|x\|=1} f(x).$$

By positive definiteness of f , $f(x^*) > 0$. Let M be any positive scalar and define $R := (M/f(x^*))^{1/d}$. Then for any x such that $\|x\| = R$, we have

$$f(x) \geq \min_{\|x\|=R} f(x) \geq R^d f(x^*) = M,$$

where the second inequality holds by homogeneity of f . Thus $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$. \square

3 Approximating norms by polynomial norms

It is easy to see that not all norms are polynomial norms. For example, the 1-norm $\|x\|_1 = \sum_{i=1}^n |x_i|$ is not a polynomial norm. Indeed, all polynomial norms are differentiable at all but one point (the origin) whereas the 1-norm is nondifferentiable whenever one of the components of x is equal to zero. In this section, we show that, though not every norm is a polynomial norm, any norm can be approximated to arbitrary precision by a polynomial norm (Theorem 3.1). The proof of this theorem is inspired from a proof by Ahmadi and Jungers in [1, 4]. A related result is given by Barvinok in [11]. In that paper, he shows that any norm can be approximated by the d -th root of a nonnegative

degree- d form, and quantifies the quality of the approximation as a function of n and d . The form he obtains however is not shown to be convex. In fact, in a later work [12, Section 2.4], Barvinok points out that it would be an interesting question to know whether any norm can be approximated by the d^{th} root of a convex form with the same quality of approximation as for d -th roots of nonnegative forms. The result below is a step in that direction though no quantitative result on the quality of approximation is given. Throughout, S^{n-1} denotes the unit sphere in \mathbb{R}^n .

Theorem 3.1. *Let $\|\cdot\|$ be any norm on \mathbb{R}^n . For any $\epsilon > 0$, there exist an even integer d and a convex positive definite form f of degree d such that*

$$\max_{x \in S^{n-1}} |f^{1/d}(x) - \|x\|| \leq \epsilon.$$

Note that, from Theorem 2.1, $f^{1/d}$ is a polynomial norm as f is a convex positive definite form. To show this result, we start with the following lemma.

Lemma 3.2. *Let $\|\cdot\|$ be any norm on \mathbb{R}^n . For any $\epsilon > 0$, there exist an even integer d and an n -variate convex positive definite form f of degree d such that*

$$\|x\| \leq f^{1/d}(x) \leq (1 + \epsilon)\|x\|, \forall x \in \mathbb{R}^n. \quad (2)$$

Proof. Throughout, we let $B_\alpha := \{x \mid \|x\| \leq \alpha\}$. When $\alpha = 1$, we drop the subscript and simply denote by B the unit ball of $\|\cdot\|$. We will also use the notation ∂S to denote the boundary of a set S and $\text{int}(S)$ to denote its interior. Let $\bar{\epsilon} := \frac{\epsilon}{1+\epsilon}$. The crux of the proof lies in proving that there exists an integer d and a positive definite convex form f of degree d such that

$$B_{1-\bar{\epsilon}} \subseteq \{x \mid f(x) \leq 1\} \subseteq B. \quad (3)$$

If we prove this, then Lemma 3.2 can be obtained as follows. Let $x \in \mathbb{R}^n$. To show the first inequality in (2), we proceed by contradiction. Suppose that $\|x\| > f^{1/d}(x)$. If $f^{1/d}(x) \neq 0$, then $\|x/f^{1/d}(x)\| > 1$ while $f(x/f^{1/d}(x)) = 1$. (If $f^{1/d}(x) = 0$ then $x = 0$ and the inequality holds.) Hence,

$$x/f^{1/d}(x) \in \{x \mid f(x) \leq 1\}$$

but $x/f^{1/d}(x) \notin B$ which contradicts (3). To prove the second inequality in (2), note that the first inclusion of (3) gives us $f^{1/d}((1-\bar{\epsilon})x/\|x\|) \leq 1$, which is equivalent to $f^{1/d}(x/\|x\|) \leq \frac{1}{1-\bar{\epsilon}} = 1 + \epsilon$. Multiplying by $\|x\|$ on both sides gives us the result.

We now focus on showing the existence of a positive definite convex form f that satisfies (3). The proof is a simplification of the proof of Theorem 3.2. in [1, 4] with some modifications.

Let $x \in \partial B_{1-\bar{\epsilon}/2}$. To any such x , we associate a dual vector $v(x)$ orthogonal to a supporting hyperplane of $B_{1-\bar{\epsilon}/2}$ at x . By definition of a supporting hyperplane, we have that $v(x)^T y \leq v(x)^T x, \forall y \in B_{1-\bar{\epsilon}/2}$, and, as $B_{1-\bar{\epsilon}} \subset B_{1-\bar{\epsilon}/2}$, we have

$$v(x)^T y < v(x)^T x, \forall y \in B_{1-\bar{\epsilon}}. \quad (4)$$

Let

$$S(x) := \{y \mid v(x)^T y > v(x)^T x \text{ and } \|y\| = 1\}.$$

It is easy to see that $S(x)$ is an open subset of the boundary ∂B of B . Furthermore, since $x \in \text{int}(B)$, $x/\|x\| \in S(x)$ which implies that $S(x)$ is nonempty and that the family of sets $S(x)$ (as x ranges over $\partial B_{1-\bar{\epsilon}/2}$) is a covering of ∂B . As $\{S(x)\}_{x \in \partial B_{1-\bar{\epsilon}/2}}$ is an open covering of the compact set ∂B , there exists a finite covering of ∂B , i.e., one can choose $x_1, \dots, x_N \in \partial B_{1-\bar{\epsilon}/2}$ in such a way that $\cup_{i=1}^N S(x_i) = \partial B$.

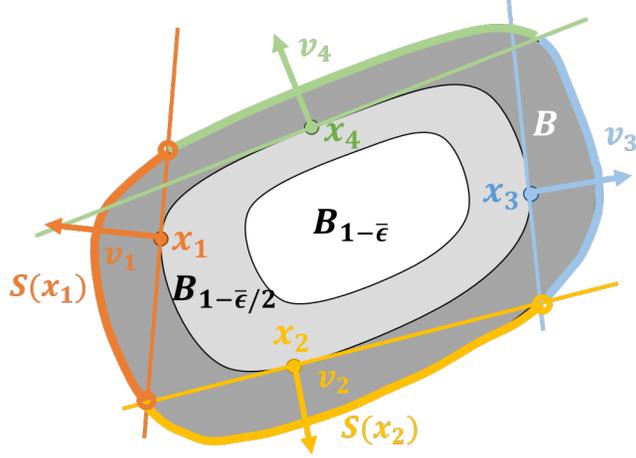


Figure 1: An illustration of the construction of the open covering of ∂B .

For ease of notation, we now let $v_i := v(x_i)$ for all $i = 1, \dots, N$. From (4), we have that $\frac{v_i^T y}{v_i^T x_i} < 1$ for any i and for any $y \in B_{1-\bar{\epsilon}}^1$. Since $B_{1-\bar{\epsilon}}$ is compact, we get $\max_i \max_{y \in B_{1-\bar{\epsilon}}} \left(\frac{v_i^T y}{v_i^T x_i} \right) < 1$. Hence, there exists an integer d such that

$$\left(\max_i \max_{y \in B_{1-\bar{\epsilon}}} \left(\frac{v_i^T y}{v_i^T x_i} \right) \right)^d < \frac{1}{N}. \quad (5)$$

We now define

$$f(y) := \sum_{i=1}^N \left(\frac{v_i^T y}{v_i^T x_i} \right)^d, \quad (6)$$

where d is any integer satisfying (5). The form f is convex as a sum of even powers of linear forms. Let

$$\mathcal{L} := \{y \mid f(y) \leq 1\}.$$

By (5), it is straightforward to see that $B_{1-\bar{\epsilon}} \subseteq \mathcal{L}$.

We now show that $\mathcal{L} \subseteq \text{int}(B)$. Let $y \in \mathcal{L}$, then $f(y) \leq 1$. If a sum of nonnegative terms is less than or equal to 1, then each term has to be less than or equal to 1, which implies that $\frac{v_i^T y}{v_i^T x_i} \leq 1$, for all $i = 1, \dots, N$. From this, we deduce that $y \notin \partial B$. Indeed, if $y \in \partial B$, there exists $i \in \{1, \dots, N\}$ such that $y \in S(x_i)$ as $\{S(x_i)\}_i$ is a cover of ∂B . But, by definition of $S(x_i)$, we would have $v_i^T y > v_i^T x_i$ which contradicts the previous statement. We have that $\partial B \cap \mathcal{L} = \emptyset$ as a consequence. However, as \mathcal{L} and B both contain the zero vector, this implies that $\mathcal{L} \subseteq \text{int}(B)$. Note that the previous inclusion guarantees positive definiteness of f . Indeed, if f were not positive definite, \mathcal{L} would be unbounded and could not be a subset of B (which is bounded). \square

Proof of Theorem 3.1. Let $\epsilon > 0$ and denote by $\alpha := \max_{x \in S^{n-1}} \|x\|$. By Lemma 3.2, there exists an integer d and a convex form f such that

$$\|x\| \leq f^{1/d}(x) \leq \left(1 + \frac{\epsilon}{\alpha}\right) \|x\|, \forall x.$$

This is equivalent to

$$0 \leq f^{1/d}(x) - \|x\| \leq \frac{\epsilon}{\alpha} \|x\|, \forall x.$$

¹Note that $v_i^T x_i \neq 0, \forall i$. In fact, we have $v_i^T x_i > 0, \forall i$. To see this, recall that by definition of a supporting hyperplane, $v_i \neq 0$ and $v_i^T x_i \geq v_i^T y$, for all $y \in B_{1-\bar{\epsilon}/2}$. In particular, there exists $\alpha_i > 0$ such that $\alpha_i v_i \in B_{1-\bar{\epsilon}/2}$. Hence, $v_i^T x_i \geq \alpha \|v_i\|_2^2 > 0$.

For $x \in S^{n-1}$, as $\|x\|/\alpha \leq 1$, this inequality becomes

$$0 \leq f^{1/d}(x) - \|x\| \leq \epsilon.$$

□

Remark 3.3. We remark that the polynomial norm constructed in Theorem 3.1 is the d^{th} -root of an sos-convex polynomial. Hence, one can approximate any norm on \mathbb{R}^n by searching for a polynomial norm using semidefinite programming. To see why the polynomial f in (6) is sos-convex, observe that linear forms are sos-convex and that an even power of an sos-convex form is sos-convex.

4 Semidefinite programming-based approximations of polynomial norms

4.1 Complexity

It is natural to ask whether testing if the d^{th} root of a given degree- d form is a norm can be done in polynomial time. In the next theorem, we show that, unless $P = NP$, this is not the case even when $d = 4$.

Theorem 4.1. *Deciding whether the 4th root of a quartic form is a norm is strongly NP-hard.*

Proof. The proof of this result is adapted from a proof in [5]. Recall that the CLIQUE problem can be described thus: given a graph $G = (V, E)$ and a positive integer k , decide whether G contains a clique of size at least k . The CLIQUE problem is known to be NP-hard [16]. We will give a reduction from CLIQUE to the problem of testing convexity and positive definiteness of a quartic form. The result then follows from Theorem 2.1. Let $\omega(G)$ be the clique number of the graph at hand, i.e., the number of vertices in a maximum clique of G . Consider the following quartic form

$$b(x; y) := -2k \sum_{i,j \in E} x_i x_j y_i y_j - (1-k) \left(\sum_i x_i^2 \right) \left(\sum_i y_i^2 \right).$$

In [5], using in part a result in [20], it is shown that

$$\omega(G) \leq k \Leftrightarrow b(x; y) + \frac{n^2 \gamma}{2} \left(\sum_{i=1}^n x_i^4 + \sum_{i=1}^n y_i^4 + \sum_{1 \leq i < j \leq n} (x_i^2 x_j^2 + y_i^2 y_j^2) \right) \quad (7)$$

is convex and $b(x; y)$ is positive semidefinite. Here, γ is a positive constant defined as the largest coefficient in absolute value of any monomial present in some entry of the matrix $\left[\frac{\partial^2 b(x; y)}{\partial x_i \partial y_j} \right]_{i,j}$. As $\sum_i x_i^4 + \sum_i y_i^4$ is positive definite and as we are adding this term to a positive semidefinite expression, the resulting polynomial is positive definite. Hence, the equivalence holds if and only if the quartic on the righthandside of the equivalence in (7) is convex and positive definite. □

Note that this also shows that strict convexity is hard to test for quartic forms (this is a consequence of Theorem 2.2). A related result is Proposition 3.5. in [5], which shows that testing strict convexity of a polynomial of even degree $d \geq 4$ is hard. However, this result is not shown there for *forms*, hence the relevance of the previous theorem.

Theorem 4.1 motivates the study of tractable sufficient conditions to be a polynomial norm. The sufficient conditions we consider next are based on semidefinite programming.

4.2 Sum of squares polynomials and semidefinite programming review

We start this section by reviewing the notion of *sum of squares polynomials* and related concepts such as *sum of squares-convexity*. We say that a polynomial f is a *sum of squares* (sos) if $f(x) = \sum_i q_i^2(x)$, for some polynomials q_i . Being a sum of squares is a sufficient condition for being nonnegative. The converse however is not true, as is exemplified by the Motzkin polynomial

$$M(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \quad (8)$$

which is nonnegative but not a sum of squares [22]. The sum of squares condition is a popular surrogate for nonnegativity due to its tractability. Indeed, while testing nonnegativity of a polynomial of degree greater or equal to 4 is a hard problem, testing whether a polynomial is a sum of squares can be done using *semidefinite programming*. This comes from the fact that a polynomial p of degree d is a sum of squares if and only if there exists a positive semidefinite matrix Q such that $f(x) = z(x)^T Q z(x)$, where $z(x)$ is the standard vector of monomials of degree up to d (see, e.g., [23]). As a consequence, any optimization problem over the coefficients of a set of polynomials which includes a combination of affine constraints and sos constraints on these polynomials, together with a linear objective can be recast as a semidefinite program. These type of optimization problems are known as *sos programs*.

Though not all nonnegative polynomials can be written as sums of squares, the following theorem by Artin [9] circumvents this problem using sos multipliers.

Theorem 4.2 (Artin [9]). *For any nonnegative polynomial f , there exists an sos polynomial q such that $q \cdot f$ is sos.*

This theorem in particular implies that if we are given a polynomial f , then we can always check its nonnegativity using an sos program that searches for q (of a fixed degree). However, the condition does not allow us to optimize over the set of nonnegative polynomials using an sos program (as far as we know). This is because, in that setting, products of decision variables arise from multiplying polynomials f and q , whose coefficients are decision variables.

By adding further assumptions on f , Reznick showed in [24] that one could further pick q to be a power of $\sum_i x_i^2$.

Theorem 4.3 (Reznick [24]). *Let f be a positive definite form of degree d in n variables and define*

$$\epsilon(f) := \frac{\min\{f(u) \mid u \in S^{n-1}\}}{\max\{f(u) \mid u \in S^{n-1}\}}.$$

If $r \geq \frac{nd(d-1)}{4 \log(2)\epsilon(f)} - \frac{n+d}{2}$, then $(\sum_{i=1}^n x_i^2)^r \cdot f$ is a sum of squares.

Motivated by this theorem, the notion of r -sos polynomials can be defined: a polynomial f is said to be r -sos if $(\sum_i x_i^2)^r \cdot f$ is sos. Note that it is clear that any r -sos polynomial is nonnegative and that the set of r -sos polynomials is included in the set of $(r+1)$ -sos polynomials. The Motzkin polynomial in (8) for example is 1-sos although not sos.

To end our review, we briefly touch upon the concept of sum of squares-convexity (sos-convexity), which we will build upon in the rest of the section. Let H_f denote the Hessian matrix of a polynomial f . We say that f is *sos-convex* if $y^T H_f(x) y$ is a sum of squares (as a polynomial in x and y). As before, optimizing over the set of sos-convex polynomials can be cast as a semidefinite program. Sum of squares-convexity is obviously a sufficient condition for convexity via the second-order characterization of convexity. However, there are convex polynomials which are not sos-convex (see, e.g., [6]). For a more detailed overview of sos-convexity including equivalent characterizations and settings in which sos-convexity and convexity are equivalent, refer to [7].

4.2.1 Notation

Throughout, we will use the notation $H_{n,d}$ (resp. $P_{n,d}$) to denote the set of forms (resp. positive semidefinite, aka nonnegative, forms) in n variables and of degree d . We will furthermore use the falling factorial notation $(t)_0 = 1$ and $(t)_k = t(t-1) \dots (t-(k-1))$ for a positive integer k .

4.3 A test for validity of polynomial norms

In this subsection, we assume that we are given a form f of degree d and we would like to test whether $f^{1/d}$ is a norm using semidefinite programming.

Theorem 4.4. *Let f be a degree- d form. Then $f^{1/d}$ is a polynomial norm if and only if there exist $c > 0$, $r \in \mathbb{N}$, and an sos form $q(x, y)$ such that $q(x, y) \cdot y^T H_f(x, y)y$ is sos and $(f(x) - c(\sum_i x_i^2)^{d/2}) (\sum_i x_i^2)^r$ is sos. Furthermore, this condition can be checked using semidefinite programming.*

Proof. It is immediate to see that if there exist such a c , r , and q , then f is convex and positive definite. From Theorem 2.1, this means that $f^{1/d}$ is a polynomial norm.

Conversely, if $f^{1/d}$ is a polynomial norm, then, by Theorem 2.1, f is convex and positive definite. As f is convex, the polynomial $y^T H_f(x, y)y$ is nonnegative. Using Theorem 4.2, we conclude that there exists an sos polynomial $q(x, y)$ such that $q(x, y) \cdot y^T H_f(x, y)y$ is sos. We now show that, as f is positive definite, there exist $c > 0$ and $r \in \mathbb{N}$ such that $(f(x) - c(\sum_i x_i^2)^{d/2}) (\sum_i x_i^2)^r$ is sos. Let f_{min} denote the minimum of f on the sphere. As f is positive definite, $f_{min} > 0$. We take $c := \frac{f_{min}}{2}$ and consider $g(x) := f(x) - c(\sum_i x_i^2)^{d/2}$. We have that g is a positive definite form: indeed, if x is a nonzero vector in \mathbb{R}^n , then

$$\frac{g(x)}{\|x\|^d} = \frac{f(x)}{\|x\|^d} - c = f\left(\frac{x}{\|x\|}\right) - c > 0,$$

by homogeneity of f and definition of c . Using Theorem 4.3, $\exists r \in \mathbb{N}$ such that $g(x)(\sum_i x_i^2)^r$ is sos.

For fixed r , a given form f , and a fixed degree d , one can search for $c > 0$ and an sos form q of degree d such that $q(x, y) \cdot y^T H_f(x, y)y$ is sos and $(f(x) - c(\sum_i x_i^2)^{d/2}) (\sum_i x_i^2)^r$ is sos using semidefinite programming. This is done by solving the following semidefinite feasibility problem:

$$\begin{aligned} & q(x, y) \text{ sos} \\ & c \geq 0 \\ & q(x, y) \cdot y^T H_f(x, y)y \text{ sos} \\ & \left(f(x) - c \left(\sum_i x_i^2 \right)^{d/2} \right) \left(\sum_i x_i^2 \right)^r \text{ sos,} \end{aligned} \tag{9}$$

where the unknowns are the coefficients of q and the real number c . □

Remark 4.5. *We remark that we are not imposing $c > 0$ in the semidefinite program above. This is because, in practice, especially if the semidefinite program is solved with interior point methods, the solution returned by the solver will be in the interior of the feasible set, and hence c will automatically be positive. One can slightly modify (9) however to take the constraint $c > 0$ into consideration explicitly. Indeed, consider the following semidefinite feasibility problem where both the degree of q and the integer r are fixed:*

$$\begin{aligned} & q(x, y) \text{ sos} \\ & \gamma \geq 0 \\ & q(x, y) \cdot y^T H_f(x, y)y \text{ sos} \\ & \left(\gamma f(x) - \left(\sum_i x_i^2 \right)^{d/2} \right) \left(\sum_i x_i^2 \right)^r \text{ sos.} \end{aligned} \tag{10}$$

It is easy to check that (10) is feasible with $\gamma \geq 0$ if and only if the last constraint of (9) is feasible with $c > 0$. To see this, take $c = 1/\gamma$ and note that γ can never be zero.

To the best of our knowledge, we cannot use the approach described in Theorem 4.4 to optimize over the set of polynomial norms with a semidefinite program. This is because of the product of decision variables in the coefficients of f and q . The next subsection will address this issue.

4.4 Optimizing over the set of polynomial norms

In this subsection, we consider the problem of optimizing over the set of polynomial norms. To do this, we introduce the concept of r -sos-convexity. Recall that the notation H_f references the Hessian matrix of a form f .

4.4.1 Positive definite biforms and r -sos-convexity

Definition 4.6. For an integer r , we say that a polynomial f is r -sos-convex if $y^T H_f(x)y \cdot (\sum_i x_i^2)^r$ is sos.

Observe that, for fixed r , the property of r -sos-convexity can be checked using semidefinite programming (though the size of this SDP gets larger as r increases). Any polynomial that is r -sos-convex is convex. Note that the set of r -sos-convex polynomials is a subset of the set of $(r+1)$ -sos-convex polynomials and that the case $r=0$ corresponds to the set of sos-convex polynomials.

It is natural to ask whether any convex polynomial is r -sos-convex for some r . Our next theorem shows that this is the case under a mild assumption.

Theorem 4.7. Let f be a form of degree d such that $y^T H_f(x)y > 0$ for $(x, y) \in S^{n-1} \times S^{n-1}$. Let

$$\eta(f) := \frac{\min\{y^T H_f(x)y \mid (x, y) \in S^{n-1} \times S^{n-1}\}}{\max\{y^T H_f(x)y \mid (x, y) \in S^{n-1} \times S^{n-1}\}}.$$

If $r \geq \frac{n(d-2)(d-3)}{4 \log(2)\eta(f)} - \frac{n+d-2}{2} - d$, then f is r -sos-convex.

Remark 4.8. Note that $\eta(f)$ can also be interpreted as

$$\eta(f) = \frac{\min_{x \in S^{n-1}} \lambda_{\min}(H_f(x))}{\max_{x \in S^{n-1}} \lambda_{\max}(H_f(x))} = \frac{1}{\max_{x \in S^{n-1}} \|H_f^{-1}(x)\|_2 \cdot \max_{x \in S^{n-1}} \|H_f(x)\|_2}.$$

Remark 4.9. Theorem 4.7 is a generalization of Theorem 4.3 by Reznick. Note though that this is not an immediate generalization. First, $y^T H_f(x)y$ is not a positive definite form (consider, e.g., $y=0$ and any nonzero x). Secondly, note that the multiplier is $(\sum_i x_i^2)^r$ and does not involve the y variables. (As we will see in the proof, this is essentially because $y^T H_f(x)y$ is quadratic in y .)

Remark 4.10. Theorem 4.7 can easily be adapted to biforms of the type $\sum_j f_j(x)g_j(y)$ where f_j 's are forms of degree d in x and g_j 's are forms of degree \tilde{d} in y . In this case, there exist integers s, r such that

$$\sum_j f_j(x)g_j(y) \cdot (\sum_i x_i^2)^r \cdot (\sum_i y_i^2)^s$$

is sos. For the purposes of this paper however and the connection to polynomial norms, we will show the result in the particular case where the biform of interest is $y^T H_f(x)y$.

We associate to any form $f \in H_{n,d}$, the d -th order differential operator $f(D)$, defined by replacing each occurrence of x_j with $\frac{\partial}{\partial x_j}$. For example, if $f(x_1, \dots, x_n) := \sum_i c_i x_1^{a_1^i} \dots x_n^{a_n^i}$ where $c_i \in \mathbb{R}$ and $a_j^i \in \mathbb{N}$, then its differential operator will be

$$f(D) = \sum_i c_i \frac{\partial^{a_1^i}}{\partial x_1^{a_1^i}} \dots \frac{\partial^{a_n^i}}{\partial x_n^{a_n^i}}.$$

Our proof will follow the structure of the proof of Theorem 4.3 given in [24] and reutilize some of the results given in the paper which we quote here for clarity of exposition.

Proposition 4.11 ([24], see Proposition 2.6). *For any nonnegative integer r , there exist nonnegative rationals λ_k and integers α_{kl} such that*

$$(x_1^2 + \dots + x_n^2)^r = \sum_k \lambda_k (\alpha_{k1}x_1 + \dots + \alpha_{kn}x_n)^{2r}.$$

For simplicity of notation, we will let $\alpha_k := (\alpha_{k1}, \dots, \alpha_{kn})^T$ and $x := (x_1, \dots, x_n)^T$. Hence, we will write $\sum_k \lambda_k (\alpha_k^T x)^{2r}$ to mean $\sum_k \lambda_k (a_{k1}x_1 + \dots + a_{kn}x_n)^{2r}$.

Proposition 4.12 ([24], see Proposition 2.8). *If $g \in H_{n,e}$ and $h = \sum_k \lambda_k (\alpha_k^T x)^{d+e} \in H_{n,d+e}$, then*

$$g(D)h = (d+e)_e \sum_k \lambda_k g(\alpha_k) (\alpha_k^T x)^d.$$

Proposition 4.13 ([24], see Theorem 3.7 and 3.9). *For $f \in H_{n,d}$ and $s \geq d$, we define $\Phi_s(f) \in H_{n,d}$ by*

$$f(D)(x_1^2 + \dots + x_n^2)^s =: \Phi_s(f)(x_1^2 + \dots + x_n^2)^{s-d}. \quad (11)$$

The inverse $\Phi_s^{-1}(f)$ of $\Phi_s(f)$ exists and this is a map verifying $\Phi_s(\Phi_s^{-1}(f)) = f$.

Proposition 4.14 ([24], see Theorem 3.12). *Suppose f is a positive definite form in n variables and of degree d and let*

$$\epsilon(f) = \frac{\min\{f(u) \mid u \in S^{n-1}\}}{\max\{f(u) \mid u \in S^{n-1}\}}.$$

If $s \geq \frac{nd(d-1)}{4 \log(2)\epsilon(f)} - \frac{n-d}{2}$, then $\Phi_s^{-1}(f) \in P_{n,d}$.

We will focus throughout the proof on biforms of the following structure

$$F(x; y) := \sum_{1 \leq i, j \leq n} y_i y_j p_{ij}(x), \quad (12)$$

where $p_{ij}(x) \in H_{n,d}$, for all i, j , and some even integer d . Note that the polynomial $y^T H_f(x) y$ (where f is some form) has this structure. We next present three lemmas which we will then build on to give the proof of Theorem 4.7.

Lemma 4.15. *For a biform $F(x; y)$ of the structure in (12), define the operator $F(D; y)$ as*

$$F(D; y) = \sum_{i,j} y_i y_j p_{ij}(D).$$

If $F(x; y)$ is positive semidefinite (i.e., $F(x; y) \geq 0, \forall x, y$), then, for any $s \geq 0$, the biform

$$F(D; y)(x_1^2 + \dots + x_n^2)^s$$

is a sum of squares.

Proof. Using Proposition 4.11, we have

$$(x_1^2 + \dots + x_n^2)^s = \sum_l \lambda_l (\alpha_{l1}x_1 + \dots + \alpha_{ln}x_n)^{2s},$$

where $\lambda_l \geq 0$ and $\alpha_l \in \mathbb{Z}^n$. Hence, applying Proposition 4.12, we get

$$\begin{aligned} F(D; y)(x_1^2 + \dots + x_n^2)^s &= \sum_{i,j} y_i y_j (p_{ij}(D)(x_1^2 + \dots + x_n^2)^s) \\ &= \sum_{i,j} y_i y_j \left((2s)_d \sum_l \lambda_l p_{ij}(\alpha_l) (\alpha_l^T x)^{2s-d} \right) \\ &= (2s)_d \sum_l \lambda_l (\alpha_l^T x)^{2s-d} \sum_{i,j} y_i y_j p_{ij}(\alpha_l). \end{aligned} \quad (13)$$

Notice that $\sum_{i,j} y_i y_j p_{ij}(\alpha_l)$ is a quadratic form in y which is positive semidefinite by assumption, which implies that it is a sum of squares (as a polynomial in y). Furthermore, as $\lambda_l \geq 0 \forall l$ and $(\alpha_l^T x)^{2s-d}$ is an even power of a linear form, we have that $\lambda_l (\alpha_l^T x)^{2s-d}$ is a sum of squares (as a polynomial in x). Combining both results, we get that (13) is a sum of squares. \square

We now extend the concept introduced by Reznick in Proposition 4.13 to biforms.

Lemma 4.16. *For a biform $F(x; y)$ of the structure as in (12), we define the biform $\Psi_{s,x}(F(x; y))$ as*

$$\Psi_{s,x}(F(x; y)) := \sum_{i,j} y_i y_j \Phi_s(p_{ij}(x)),$$

where Φ_s is as in (11). Define

$$\Psi_{s,x}^{-1}(F(x; y)) := \sum_{i,j} y_i y_j \Phi_s^{-1}(p_{ij}(x)),$$

where Φ_s^{-1} is the inverse of Φ_s . Then, we have

$$F(D; y)(x_1^2 + \dots + x_n^2)^s = \Psi_{s,x}(F)(x_1^2 + \dots + x_n^2)^{s-d} \quad (14)$$

and

$$\Psi_{s,x}(\Psi_{s,x}^{-1}(F)) = F. \quad (15)$$

Proof. We start by showing that (14) holds:

$$\begin{aligned} F(D; y)(x_1^2 + \dots + x_n^2)^s &= \sum_{i,j} y_i y_j p_{ij}(D)(x_1^2 + \dots + x_n^2)^s \\ &= \sum_{i,j} y_i y_j \Phi_s(p_{ij}(x))(x_1^2 + \dots + x_n^2)^{s-d} \\ &\stackrel{\text{using (11)}}{=} \sum_{i,j} y_i y_j \Phi_s(p_{ij}(x))(x_1^2 + \dots + x_n^2)^{s-d} \\ &= \Psi_{s,x}(F)(x_1^2 + \dots + x_n^2)^{s-d}. \end{aligned}$$

We now show that (15) holds:

$$\Psi_{s,x}(\Psi_{s,x}^{-1}(F)) = \Psi_{s,x} \left(\sum_{i,j} y_i y_j \Phi_s^{-1}(p_{ij}(x)) \right) = \sum_{i,j} y_i y_j \Phi_s \Phi_s^{-1}(p_{ij}) = \sum_{i,j} y_i y_j p_{ij} = F.$$

\square

Lemma 4.17. *For a biform $F(x; y)$ of the structure in (12), which is positive on the bisphere, let*

$$\eta(F) := \frac{\min\{F(x; y) \mid (x, y) \in S^{n-1} \times S^{n-1}\}}{\max\{F(x; y) \mid (x, y) \in S^{n-1} \times S^{n-1}\}}.$$

If $s \geq \frac{nd(d-1)}{4 \log(2)\eta(F)} - \frac{n-d}{2}$, then $\Psi_{s,x}^{-1}(F)$ is positive semidefinite.

Proof. Fix $y \in S^{n-1}$ and consider $F_y(x) = F(x; y)$, which is a positive definite form in x of degree d . From Proposition 4.14, if

$$s \geq \frac{nd(d-1)}{4 \log(2)\epsilon(F_y)} - \frac{n-d}{2},$$

then $\Phi_s^{-1}(F_y)$ is positive semidefinite. As $\eta(F) \leq \epsilon(F_y)$ for any $y \in S^{n-1}$, we have that if

$$s \geq \frac{nd(d-1)}{4\log(2)\eta(F)} - \frac{n-d}{2},$$

then $\Phi_s^{-1}(F_y)$ is positive semidefinite, regardless of the choice of y . Hence, $\Psi_{s,x}^{-1}(F)$ is positive semidefinite (as a function of x and y). □

Proof of Theorem 4.7. Let $F(x; y) = y^T H_f(x)y$, let $r \geq \frac{n(d-2)(d-3)}{4\log(2)\eta(f)} - \frac{n+d-2}{2} - d$, and let

$$G(x; y) = \Psi_{r+d,x}^{-1}(F).$$

We know by Lemma 4.17 that $G(x; y)$ is positive semidefinite. Hence, using Lemma 4.15, we get that

$$G(D, y)(x_1^2 + \dots + x_n^2)^{r+d}$$

is sos. Lemma 4.16 then gives us:

$$\begin{aligned} G(D; y)(x_1^2 + \dots + x_n^2)^{r+d} & \stackrel{\text{using (14)}}{=} \Psi_{r+d,x}(G)(x_1^2 + \dots + x_n^2)^r \\ & \stackrel{\text{using (15)}}{=} F(x; y)(x_1^2 + \dots + x_n^2)^r. \end{aligned}$$

As a consequence, $F(x; y)(x_1^2 + \dots + x_n^2)^r$ is sos. □

The last theorem of this section shows that one cannot bound the integer r in Theorem 4.7 as a function of n and d only.

Theorem 4.18. *For any integer $r \geq 0$, there exists a form f in 3 variables and of degree 8 such that $H_f(x) \succ 0, \forall x \neq 0$, but f is not r -sos-convex.*

Proof. Consider the trivariate octic:

$$\begin{aligned} f(x_1, x_2, x_3) &= 32x_1^8 + 118x_1^6x_2^2 + 40x_1^6x_3^2 + 25x_1^4x_2^2x_3^2 - 35x_1^4x_2^4 + 3x_1^2x_2^4x_3^2 - 16x_1^2x_2^2x_3^4 + 24x_1^2x_2^6 \\ &+ 16x_2^8 + 44x_2^6x_3^2 + 70x_2^4x_3^4 + 60x_2^2x_3^6 + 30x_3^8. \end{aligned}$$

It is shown in [6] that f has positive definite Hessian, and that the $(1, 1)$ entry of $H_f(x)$, which we will denote by $H_f^{(1,1)}(x)$, is 1-sos but not sos. We will show that for any $r \in \mathbb{N}$, one can find $s \in \mathbb{N} \setminus \{0\}$ such that

$$g_s(x_1, x_2, x_3) = f(x_1, sx_2, sx_3)$$

satisfies the conditions of the theorem.

We start by showing that for any s , g_s has positive definite Hessian. To see this, note that for any $(x_1, x_2, x_3) \neq 0, (y_1, y_2, y_3) \neq 0$, we have:

$$(y_1, y_2, y_3)H_{g_s}(x_1, x_2, x_3)(y_1, y_2, y_3)^T = (y_1, sy_2, sy_3)H_f(x_1, sx_2, sx_3)(y_1, sy_2, sy_3)^T.$$

As $y^T H_f(x)y > 0$ for any $x \neq 0, y \neq 0$, this is in particular true when $x = (x_1, sx_2, sx_3)$ and when $y = (y_1, sy_2, sy_3)$, which gives us that the Hessian of g_s is positive definite for any $s \in \mathbb{N} \setminus \{0\}$.

We now show that for a given $r \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that $(x_1^2 + x_2^2 + x_3^2)^r y^T H_{g_s}(x)y$ is not sos. We use the following result from [25, Theorem 1]: for any positive semidefinite form p which is not sos, and any $r \in \mathbb{N}$, there

exists $s \in \mathbb{N} \setminus \{0\}$ such that $(\sum_{i=1}^n x_i^2)^r \cdot p(x_1, sx_2, \dots, sx_n)$ is not sos. As $H_f^{(1,1)}(x)$ is 1-sos but not sos, we can apply the previous result. Hence, there exists a positive integer s such that

$$(x_1^2 + x_2^2 + x_3^2)^r \cdot H_f^{(1,1)}(x_1, sx_2, sx_3) = (x_1^2 + x_2^2 + x_3^2)^r \cdot H_{g_s}^{(1,1)}(x_1, x_2, x_3)$$

is not sos. This implies that $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ is not sos. Indeed, if $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ was sos, then $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ would be sos with $y = (1, 0, 0)^T$. But, we have

$$(x_1^2 + x_2^2 + x_3^2)^r \cdot (1, 0, 0) H_{g_s}(x) (1, 0, 0)^T = (x_1^2 + x_2^2 + x_3^2)^r \cdot H_{g_s}^{(1,1)}(x),$$

which is not sos. Hence, $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ is not sos, and g is not r -sos-convex. \square

Remark 4.19. Any form f with $H_f(x) \succ 0, \forall x \neq 0$ is strictly convex but the converse is not true.

To see this, note that any form f of degree d with a positive definite Hessian is convex (as $H_f(x) \succeq 0, \forall x$) and positive definite (as, from a recursive application of Euler's theorem on homogeneous functions, $f(x) = \frac{1}{d(d-1)} x^T H_f(x)x$). From the proof of Theorem 2.2, this implies that f is strictly convex.

To see that the converse statement is not true, consider the strictly convex form $f(x_1, x_2) := x_1^4 + x_2^4$. We have

$$H_f(x) = 12 \cdot \begin{bmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{bmatrix}$$

which is not positive definite e.g., when $x = (1, 0)^T$.

4.4.2 Optimizing over a subset of polynomial norms with r -sos-convexity

In the following theorem, we show how one can efficiently optimize over the set of forms f with $H_f(x) \succ 0, \forall x \neq 0$. Comparatively to Theorem 4.4, this theorem allows us to impose as a constraint that the d^{th} root of a form be a norm, rather than simply testing whether it is. This comes at a cost however: in view of Remark 4.19 and Theorem 2.2, we are no longer considering all polynomial norms, but a subset of them whose d^{th} power has a positive definite Hessian.

Theorem 4.20. Let f be a degree- d form. Then $H_f(x) \succ 0, \forall x \neq 0$ if and only if $\exists c > 0, r \in \mathbb{N}$ such that $f(x) - c(\sum_i x_i^2)^{d/2}$ is r -sos-convex. Furthermore, this condition can be imposed using semidefinite programming.

Proof. If there exist $c > 0, r \in \mathbb{N}$ such that $g(x) = f(x) - c(\sum_i x_i^2)^{d/2}$ is r -sos-convex, then $y^T H_g(x)y \geq 0, \forall x, y$. As the Hessian of $(\sum_i x_i^2)^{d/2}$ is positive definite for any nonzero x and as $c > 0$, we get $H_f(x) \succ 0, \forall x \neq 0$.

Conversely, if $H_f(x) \succ 0, \forall x \neq 0$, then $y^T H_f(x)y > 0$ on the bisphere (and conversely). Let

$$f_{\min} := \min_{\|x\|=\|y\|=1} y^T H_f(x)y.$$

We know that f_{\min} is attained and is positive. Take $c := \frac{f_{\min}}{2d(d-1)}$ and consider

$$g(x) := f(x) - c(\sum_i x_i^2)^{d/2}.$$

Then

$$y^T H_g(x)y = y^T H_f(x)y - c \cdot \left(d(d-2) \left(\sum_i x_i^2 \right)^{d/2-2} \left(\sum_i x_i y_i \right)^2 + d \sum_i (x_i^2)^{d/2-1} \left(\sum_i y_i^2 \right) \right).$$

Note that, by Cauchy-Schwarz, we have $(\sum_i x_i y_i)^2 \leq \|x\|^2 \|y\|^2$. If $\|x\| = \|y\| = 1$, we get

$$y^T H_g(x)y \geq y^T H_f(x)y - c(d(d-1)) > 0.$$

Hence, $H_g(x) \succ 0, \forall x \neq 0$ and there exists r such that g is r -sos-convex from Theorem 4.7.

For fixed r , the condition that there be $c > 0$ such that $f(x) - c(\sum_i x_i^2)^{d/2}$ is r -sos-convex can be imposed using semidefinite programming. This is done by searching for coefficients of a polynomial f and a real number c such that

$$\begin{aligned} y^T H_{f - c(\sum_i x_i^2)^{d/2}} y \cdot \left(\sum_i x_i^2\right)^r \text{ sos} \\ c \geq 0. \end{aligned} \tag{16}$$

Note that both of these conditions can be imposed using semidefinite programming. □

Remark 4.21. *Note that we are not imposing $c > 0$ in the above semidefinite program. As mentioned in Section 4.3, this is because in practice the solution returned by interior point solvers will be in the interior of the feasible set.*

In the special case where f is completely free² (i.e., when there are no additional affine conditions on the coefficients of f), one can take $c \geq 1$ in (16) instead of $c \geq 0$. Indeed, if there exists $c > 0$, an integer r , and a polynomial f such that $f - c(\sum_i x_i^2)^{d/2}$ is r -sos-convex, then $\frac{1}{c}f$ will be a solution to (16) with $c \geq 1$ replacing $c \geq 0$.

5 Applications

5.1 Norm approximation and regression

In this section, we study the problem of approximating a (non-polynomial) norm by a polynomial norm. We consider two different types of norms: p -norms with p noneven (and greater than 1) and gauge norms with a polytopic unit ball. For p -norms, we use as an example $\|(x_1, x_2)^T\| = (|x_1|^{7.5} + |x_2|^{7.5})^{1/7.5}$. For our polytopic gauge norm, we randomly generate an origin-symmetric polytope and produce a norm whose 1-sublevel corresponds to that polytope. This allows us to determine the value of the norm at any other point by homogeneity (see [14, Exercise 3.34] for more information on gauge norms, i.e., norms defined by convex, full-dimensional, origin-symmetric sets). To obtain our approximations, we proceed in the same way in both cases. We first sample $N = 200$ points x_1, \dots, x_N on the sphere S that we denote by x_1, \dots, x_N . We then solve the following optimization problem with d fixed:

$$\begin{aligned} \min_{f \in H_{2,d}} \quad & \sum_{i=1}^N (\|x_i\|^d - f(x_i))^2 \\ \text{s.t.} \quad & f \text{ sos-convex.} \end{aligned} \tag{17}$$

Problem (17) can be written as a semidefinite program as the objective is a convex quadratic in the coefficients of f and the constraint has a semidefinite representation as discussed in Section 4.2. The solution f returned is guaranteed to be convex. Moreover, any sos-convex form is sos (see [17, Lemma 8]), which implies that f is nonnegative. One can numerically check to see if the optimal polynomial is in fact positive definite (for example, by checking the eigenvalues of the Gram matrix of a sum of squares decomposition of f). If that is the case, then, by Theorem 2.1, $f^{1/d}$ is a norm. Furthermore, note that we have

$$\begin{aligned} \left(\sum_{i=1}^N (\|x_i\|^d - f(x_i))^2 \right)^{1/d} & \geq \frac{N^{1/d}}{N} \sum_{i=1}^N (\|x_i\|^d - f(x_i))^{2/d} \\ & \geq \frac{N^{1/d}}{N} \sum_{i=1}^N (\|x_i\| - f^{1/d}(x_i))^2, \end{aligned}$$

where the first inequality is a consequence of concavity of $z \mapsto z^{1/d}$ and the second is a consequence of the inequality $|x - y|^{1/d} \geq ||x|^{1/d} - |y|^{1/d}|$. This implies that if the optimal value of (17) is equal to ϵ , then the sum of the squared differences between $\|x_i\|$ and $f^{1/d}(x_i)$ over the sample is less than or equal to $N \cdot (\frac{\epsilon}{N})^{1/d}$.

²This is the case of our two applications in Section 5.

It is worth noting that in our example, we are actually searching over the entire space of polynomial norms of a given degree. Indeed, as f is bivariate, it is convex if and only if it is sos-convex [7]. In Figure 2, we have drawn the 1-level sets of the initial norm (either the p -norm or the polytopic gauge norm) and the optimal polynomial norm obtained via (17) with varying degrees d . Note that when d increases, the approximation improves.

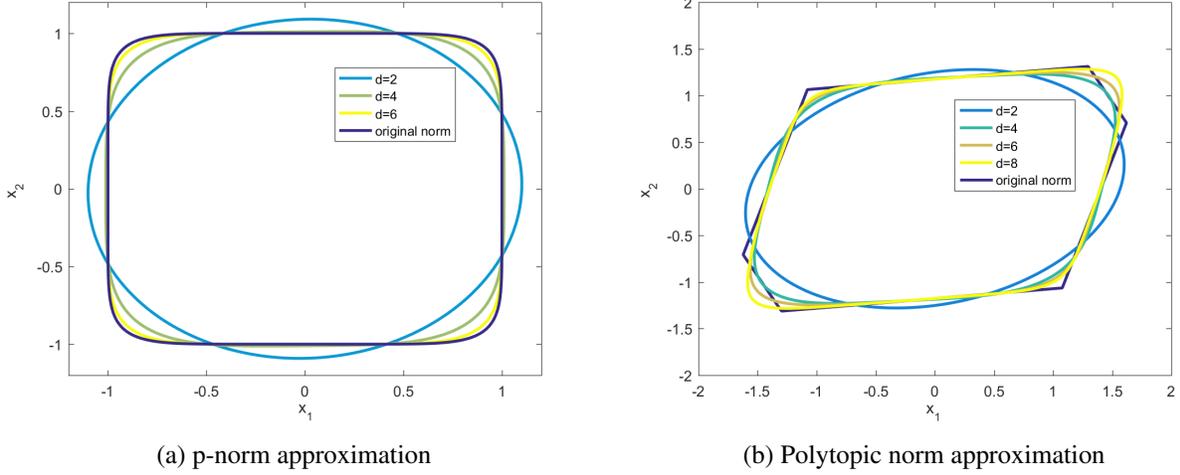


Figure 2: Approximation of non-polynomial norms by polynomial norms

A similar method could be used for *norm regression*. In this case, we would have access to data points x_1, \dots, x_N corresponding to noisy measurements of an underlying unknown norm function. We would then solve the same optimization problem as the one given in (17) to obtain a polynomial norm that most closely approximates the noisy data.

5.2 Joint spectral radius and stability of linear switched systems

As a second application, we revisit a result from Ahmadi and Jungers from [1, 4] on upperbounding the joint spectral radius of a finite set of matrices. We first review a few notions relating to dynamical systems and linear algebra. The spectral radius ρ of a matrix A is defined as

$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

The spectral radius happens to coincide with the eigenvalue of A of largest magnitude. Consider now the discrete-time linear system $x_{k+1} = Ax_k$, where x_k is the $n \times 1$ state vector of the system at time k . This system is said to be *asymptotically stable* if for any initial starting state $x_0 \in \mathbb{R}^n$, $x_k \rightarrow 0$, when $k \rightarrow \infty$. A well-known result connecting the spectral radius of a matrix to the stability of a linear system states that the system $x_{k+1} = Ax_k$ is asymptotically stable if and only if $\rho(A) < 1$.

In 1960, Rota and Strang introduced a generalization of the spectral radius to a *set* of matrices. The *joint spectral radius (JSR)* of a set of matrices $\mathcal{A} := \{A_1, \dots, A_m\}$ is defined as

$$\rho(\mathcal{A}) := \lim_{k \rightarrow \infty} \max_{\sigma \in \{1, \dots, m\}^k} \|A_{\sigma_k} \dots A_{\sigma_1}\|^{1/k}. \quad (18)$$

Analogously to the case where we have just one matrix, the value of the joint spectral radius can be used to determine stability of a certain type of system, called a *switched linear system*. A switched linear system models an uncertain and time-varying linear system, i.e., a system described by the dynamics

$$x_{k+1} = A_k x_k,$$

where the matrix A_k varies at each iteration within the set \mathcal{A} . As done previously, we say that a switched linear system is asymptotically stable if $x_k \rightarrow \infty$ when $k \rightarrow \infty$, for any starting state $x_0 \in \mathbb{R}^n$ and any sequence of products of matrices in \mathcal{A} . One can establish that the switched linear system $x_{k+1} = A_k x_k$ is asymptotically stable if and only if $\rho(\mathcal{A}) < 1$ [19].

Though they may seem similar on many points, a key difference between the spectral radius and the joint spectral radius lies in difficulty of computation: testing whether the spectral radius of a matrix A is less than equal (or strictly less) than 1 can be done in polynomial time. However, already when $m = 2$, the problem of testing whether $\rho(A_1, A_2) \leq 1$ is undecidable [13]. An active area of research has consequently been to obtain sufficient conditions for the JSR to be strictly less than one, which, for example, can be checked using semidefinite programming. The theorem that we revisit below is a result of this type. We start first by recalling a Theorem linked to stability of a linear system.

Theorem 5.1 (see, e.g., Theorem 8.4 in [18]). *Let $A \in \mathbb{R}^{n \times n}$. Then, $\rho(A) < 1$ if and only if there exists a contracting quadratic norm; i.e., a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form $V(x) = \sqrt{x^T Q x}$ with $Q \succ 0$, such that $V(Ax) < V(x), \forall x \neq 0$.*

The next theorem (from [1, 4]) can be viewed as an extension of Theorem 5.1 to the joint spectral radius of a finite set of matrices. It is known that the existence of a contracting quadratic norm is no longer necessary for stability in this case. This theorem show however that the existence of a contracting polynomial norm is.

Theorem 5.2 (adapted from [1, 4], Theorem 3.2). *Let $\mathcal{A} := \{A_1, \dots, A_m\}$ be a family of $n \times n$ matrices. Then, $\rho(A_1, \dots, A_m) < 1$ if and only if there exists a contracting polynomial norm; i.e., a function $V(x) = f^{1/d}(x)$, where f is an n -variate convex and positive definite form of degree d , such that $V(A_i x) < V(x), \forall x \neq 0$ and $\forall i = 1, \dots, m$.*

We remark that in [3], Ahmadi and Jungers show that the degree of f cannot be bounded as a function of m and n . This is expected from the undecidability result mentioned before.

Example 5.3. *We consider a modification of Example 5.4. in [2] as an illustration of the previous theorem. We would like to show that the joint spectral radius of the two matrices*

$$A_1 = \frac{1}{3.924} \begin{bmatrix} -1 & -1 \\ 4 & 0 \end{bmatrix}, \quad A_2 = \frac{1}{3.924} \begin{bmatrix} 3 & 3 \\ -2 & 1 \end{bmatrix}$$

is strictly less than one.

To do this, we search for a nonzero form f of degree d such that

$$\begin{aligned} f - \left(\sum_{i=1}^n x_i^2 \right)^{d/2} & \text{ sos-convex} \\ f(x) - f(A_i x) - \left(\sum_{i=1}^n x_i^2 \right)^{d/2} & \text{ sos, for } i = 1, 2. \end{aligned} \tag{19}$$

If problem (19) is feasible for some d , then $\rho(A_1, A_2) < 1$. A quick computation using the software package YALMIP [21] and the SDP solver MOSEK [8] reveals that, when $d = 2$ or $d = 4$, problem (19) is infeasible. When $d = 6$ however, the problem is feasible and we obtain a polynomial norm $V = f^{1/d}$ whose 1-sublevel set is the outer set plotted in Figure 3. We also plot on Figure 3 the images of this 1-sublevel set under A_1 and A_2 . Note that both sets are included in the 1-sublevel set of V as expected. From Theorem 5.2, the existence of a polynomial norm implies that $\rho(A_1, A_2) < 1$ and hence, the pair $\{A_1, A_2\}$ is asymptotically stable.

Remark 5.4. *As mentioned previously, problem (19) is infeasible for $d = 4$. Instead of pushing the degree of f up to 6, one could wonder whether the problem would have been feasible if we had asked that f of degree $d = 4$ be r -sos-convex for some fixed $r \geq 1$. As mentioned before, in the particular case where $n = 2$ (which is the case at hand here), the notions of convexity and sos-convexity coincide; see [7]. As a consequence, one can only hope to make problem (19) feasible by increasing the degree of f .*

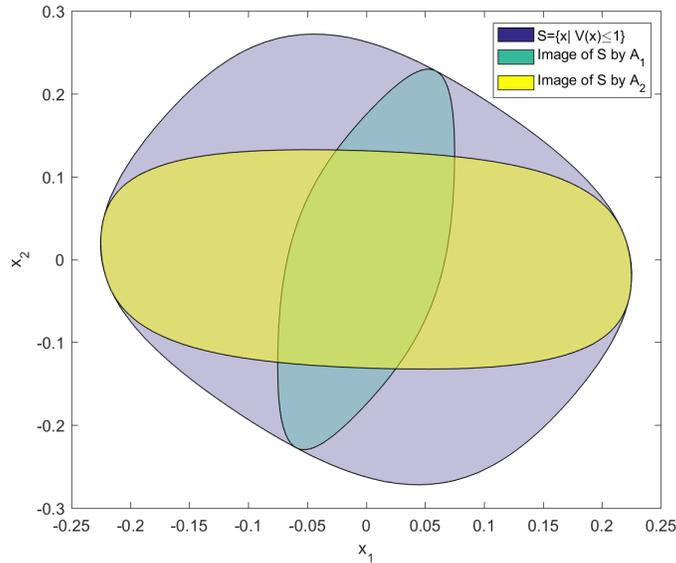


Figure 3: Image of the sublevel set of V under A_1 and A_2

6 Future directions

In this paper, we provided semidefinite programming-based conditions under which we could test whether the d^{th} root of a degree- d form is a polynomial norm (Section 4.3), and semidefinite programming-based conditions under which we could optimize over the set of forms with positive definite Hessians (Section 4.4). A clear gap emerged between forms which are strictly convex and those which have a positive definite Hessian, the latter being a sufficient (but not necessary) condition for the former. This leads us to consider the following two open problems.

Open Problem 1. *We have given a semidefinite programming hierarchy for optimizing over a subset of polynomial norms. Is there a semidefinite programming hierarchy that optimizes over all polynomial norms?*

Open Problem 2. *Helton and Nie have shown in [17] that sublevel sets of forms that have positive definite Hessians are SDP-representable. This means that we can optimize linear functions over these sets using semidefinite programming. Is the same true for sublevel sets of all polynomial norms?*

Another open problem that we think would be of interest relates to the quality of approximation of norms by polynomial norms.

Open Problem 3. *As mentioned previously, Barvinok has shown in [11] that for any norm $\|\cdot\|$, there exists a nonnegative form f of degree $2d$ such that*

$$f^{1/2d}(x) \leq \|x\| \leq \binom{n+d-1}{d}^{1/2d} f^{1/2d}(x).$$

We have shown in Section 3 that for any $\epsilon > 0$, there exists an sos-convex and positive definite polynomial f of degree d such that

$$f^{1/2d}(x) \leq \|x\| \leq (1 + \epsilon)f^{1/2d}(x).$$

Is it possible to quantify the degree d needed to obtain an approximation of precision ϵ as a function of ϵ and n only? How would the degree be impacted if we searched for an r -sos-convex polynomial instead? In this case, could we express d as a function of ϵ , n , and r only?

On the application side, it might be interesting to investigate how one can use polynomial norms to design *regularizers* in machine learning applications. Indeed, a very popular use of norms in optimization is as regularizers, with the goal of imposing additional structure (e.g., sparsity or low-rankness) on optimal solutions. One could imagine using polynomial norms to design regularizers that are based on the data at hand in place of more generic regularizers such as the 1-norm. Regularizer design is a problem that has already been considered (see, e.g., [10, 15]) but not using polynomial norms. This can be worth exploring as we have shown that polynomial norms can approximate any norm with arbitrary accuracy, while remaining differentiable everywhere (except at the origin), which can be beneficial for optimization purposes.

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