International Systems and Domestic Politics: Linking Complex Interactions with Empirical Models in International Relations (Empirical Appendix)

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This appendix describes the supplementary empirical analysis for "International Systems and Domestic Politics: Linking Complex Interactions with Empirical Models in International Relations." It contains the algorithm for estimating the model with time varying ρ, the algorithm for estimating a multilevel model with a time-varying ρ, a description of the nonconvergence diagnostics run on each model, and a description of the stationarity checks ran on the model with time-invariant ρ.

1 Estimation Algorithm for Time-Varying ρ and W

This section describes the MCMC algorithm for estimating a spatial model with time varying ρ and $\mathbf{W}$, as in Section 2.5 of the paper. The model combining multilevel and spatial analyses is estimated using an algorithm combining the following algorithm and the algorithm proposed in [author information removed].

- A Spatial Model with time-varying ρ and $\mathbf{W}$

$$y_{it} = \rho_t \mathbf{w}_{it} \mathbf{y}_t + \mathbf{x}_{it} \boldsymbol{\beta} + \epsilon_{it}$$

Matrix notation

$$\mathbf{y}_t = \rho_t \mathbf{W}_t \mathbf{y}_t + \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\epsilon}_t, \quad t = 1, 2, ..., T$$

where $\mathbf{y}_t = \{y_{1t}, y_{2t}, ..., y_{nt}\}$, and $\boldsymbol{\rho}_t$ and $\mathbf{W}_t$ is a $N \times N$ weight matrix with each row summed up to 1 and all diagonal elements equal to 0. Assume that $\mathrm{E}(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \sigma_e^2 \mathbf{I}_n,\ \forall t$. This is a simultaneous model and all y's are determined at the same time. Rearrange the model to get

$$\mathbf{A}(\rho_t) \mathbf{y}_t = \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\epsilon}_t, \quad t = 1, 2, ..., T$$

where $\mathbf{A}(\rho_t) = (\mathbf{I} - \rho_t \mathbf{W}_t)$.

- Likelihood

$$f(\mathbf{Y}|\boldsymbol{\beta}, \mathbf{W}, \boldsymbol{\rho}, \sigma_e^2) \propto \prod_{t=1}^{T} |\mathbf{A}(\rho_t)| \sigma_e^{N_t} \exp\left[-\frac{1}{2\sigma_e^2}\{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\}'\{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\}\right]$$

- Priors

$$\boldsymbol{\beta} \sim \mathcal{N}_k(\boldsymbol{\beta}_0, \mathbf{B}_0), \quad \rho_t \sim \mathcal{U}(-1, 1), \quad \sigma_e^{-2} \sim \mathcal{G}(a_0, b_0)$$

- Posterior

(a) $\boldsymbol{\beta}$

$$\boldsymbol{\beta}|\boldsymbol{\rho}, \sigma_e^{-2} \sim \mathcal{N}(\bar{\boldsymbol{\beta}}, \mathbf{B}_1),$$

$$\text{where,} \quad \mathbf{B}_1 = \left(\mathbf{B}_0 + \sigma_e^{-2}\sum_{t=1}^{T}\mathbf{X}_t'\mathbf{X}_t\right)^{-1}$$
$$\bar{\boldsymbol{\beta}} = \mathbf{B}_1\left(\mathbf{B}_0^{-1}\boldsymbol{\beta}_0 + \sigma_e^{-2}\sum_{t=1}^{T}\mathbf{X}_t'\mathbf{A}(\rho_t)\mathbf{y}_t\right)$$

(b) σ_e^{-2}

$$\sigma_e^{-2}|\boldsymbol{\beta}, \boldsymbol{\rho} \sim \mathcal{G}(\alpha_1, \delta_1),$$

$$\text{where,} \quad \alpha_1 = a_0 + \sum_{t=1}^{T} N_t$$
$$\delta_1 = b_0 + \sum_{t=1}^{T}\left(\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\right)'\left(\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\right)$$

(c) $\{\rho_t\}$ Metropolis-Hastings algorithm. The proposal density is

$$\rho_t^* \sim \mathcal{N}(\phi, \Phi)$$

$$\text{where,} \quad \Phi = \left(\sigma_e^{-2} \sum_{i=1}^{N_t} (\mathbf{w}_{it}\mathbf{y}_t)'(\mathbf{w}_{it}\mathbf{y}_t) \right)^{-1}$$

$$\phi = \Phi\sigma^{-2} \sum_{i=1}^{N_t} (\mathbf{w}_{it}\mathbf{y}_t)'(y_{it} - \mathbf{x}_{it}\boldsymbol{\beta})$$

$$\alpha = \frac{|\mathbf{A}(\rho_t^*)| \exp\left[-\frac{1}{2\sigma_e^2} \{\mathbf{A}(\rho_t^*)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\}'\{\mathbf{A}(\rho_t^*)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\} \right]}{|\mathbf{A}(\rho_t)| \exp\left[-\frac{1}{2\sigma_e^2} \{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\}'\{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\} \right]}$$

The determinant $|\mathbf{A}(\rho_t)| = \prod_{i=1}^{m}(1-\rho_t\lambda_i)$ where λ_i is the ith eigenvalue of the m eigenvalues of $\mathbf{W}_t$. Then

$$\log(\alpha) = \sum_{i=1}^{m} \log(1 - \rho_1^*\lambda_i) - \sum_{i=1}^{m} \log(1 - \rho_1\lambda_i) - \frac{1}{2\sigma_e^2}(\{\mathbf{A}(\rho_t^*)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\}' \\ \{\mathbf{A}(\rho_t^*)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\} - \{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\}'\{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta}\})$$

if $\log(\alpha) > 0$ accept ρ_t^*; otherwise, reject the proposal and keep ρ_t as the updated value.

2 Multilevel Spatial Modeling

This section describes the MCMC algorithm for estimating a multilivel spatial model with time varying ρ and $\mathbf{W}$, as in Section 2.6 of the paper.

- Model specification as in section 2.6 of the paper.

$$Y_{it} = \beta_0 + \beta_{0t} + b_i + \rho_t W_{it}\mathbf{Y}_t + \beta_{Dt}D_{it} + \epsilon_{it} \tag{1}$$

$$\beta_{0t} = \beta_S S_t + c_t \tag{2}$$

$$\beta_{Dt} = \gamma_0 + \gamma_S S_t + \zeta_t, \tag{3}$$

The reduced form of the model (plug the last two equations into the first one, and reaggrange the function)

$$Y_{it} = \beta_0 + \rho_t W_{it}\mathbf{Y}_t + \gamma_0 D_{it} + \beta_S S_t + \gamma_S S_t D_{it} + \zeta_t D_{it} + b_i + c_t + \epsilon_{it} \tag{4}$$

The composite error is $\varepsilon_{it} = \zeta_t D_{it} + \beta_{0i} + c_t + \epsilon_{it}$, which is apparently correlated with two terms of the observed for sure. To solve this serious endogeneity problem, the Bayesian approach takes all the three terms $\zeta_t D_{it}, \beta_{0i}, c_t$ out of the error term and treats ζ_t, β_{0i} and c_t as model parameters to be estimated based on data, though those parameters may not be interesting.

If the first column of the matrix of $\mathbf{D}_t$ are all 1, then the matrix expression of the reduced model is as following:

$$\begin{aligned}\mathbf{Y}_t =& \rho_t \mathbf{W}_t \mathbf{Y}_t + \mathbf{D}_t \boldsymbol{\gamma}_0 + S_t \boldsymbol{\beta}_s + S_t \mathbf{D}_t \boldsymbol{\gamma}_s + \mathbf{D}_t \mathbf{c}_t + \mathbf{b}_i + \boldsymbol{\epsilon}_t, & (5)\\ =& \rho_t \mathbf{W}_t \mathbf{Y}_t + \mathbf{X}_t \boldsymbol{\beta} + \mathbf{D}_t \mathbf{c}_t + \mathbf{b}_i + \boldsymbol{\epsilon}_t & (6)\end{aligned}$$

where $\mathbf{Y}_t = \{Y_{1t}, Y_{2t}, ..., Y_{nt}\}$, $\mathbf{b}_i = \{b_1, b_2, ..., b_n\}$, $\mathbf{c} = \{\zeta_t, c_t\}$, $\boldsymbol{\beta} = \{\boldsymbol{\gamma}_0, \boldsymbol{\beta}_s, \boldsymbol{\gamma}_s\}$, $\mathbf{X}_t = \{\mathbf{D}_t, S_t, S_t\mathbf{D}_t\}$and $\mathbf{W}_t$ is a $N \times N$ weight matrix with each row summed up to 1 and all diagonal elements equal to 0. Assume that $\mathrm{E}(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \sigma^2 \mathbf{I}_n, \ \forall t$. This is a simultaneous model and all y's are determined at the same time. Rearrange the model to get

$$\mathbf{A}(\rho_t)\mathbf{Y}_t = \mathbf{X}_t \boldsymbol{\beta} + \mathbf{D}_t \mathbf{c}_t + \mathbf{b}_i + \boldsymbol{\epsilon}_t, \tag{7}$$

where $\mathbf{A}(\rho_t) = (\mathbf{I} - \rho_t \mathbf{W}_t)$.

- Priors: priors are required for a Bayesian model. The parameters are assigned with priors which assume the following distributive forms:

$$\begin{aligned}&\boldsymbol{\beta} \sim \mathcal{N}_{K_1}(\boldsymbol{\beta}_0, \mathbf{B}_0), \qquad \{\mathbf{b}_i\} \sim \mathcal{N}_{K_2}(\mathbf{0}, \mathbf{D}), \qquad \mathbf{D}^{-1} \sim \mathcal{W}(\nu_0, \mathbf{D}_0),\\ &\{\mathbf{c}_t\} \sim \mathcal{N}(\mathbf{0}, \mathbf{E}), \qquad \mathbf{E}^{-1} \sim \mathcal{W}(\boldsymbol{\eta}_0, \mathbf{E}_0), \qquad \rho_t \sim \mathcal{U}(\rho : \rho \in S_\rho),\\ &\epsilon_t \sim \mathcal{N}(0, \sigma^2), \qquad \sigma^{-2} \sim \mathcal{G}(a_0, b_0),\end{aligned}$$

where $\mathcal{N}_k$ denotes a k-dimensional multivariate normal distribution, $\mathcal{W}$ denotes a Wishart distribution, $\mathcal{U}$ denotes a uniform distribution, and $\mathcal{G}$ denotes a Gamma distribution.

- The posterior is as follows:

$$\pi(\boldsymbol{\Theta}|\mathbf{Y}) \propto f(\mathbf{Y}|\boldsymbol{\Theta})\pi(\boldsymbol{\Theta}) \tag{8}$$

$$\begin{aligned}\pi(\boldsymbol{\beta}, \{\rho_t\}, \{\mathbf{b}_i\}, \{\mathbf{c}_t\}, \mathbf{D}, \mathbf{E}, \sigma^2|\mathbf{Y}) \propto & \prod_{t=1}^{T} |\mathbf{A}(\rho_t)|\sigma^{N_t} \\ & \times \exp\left[-\frac{1}{2\sigma^2}\{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\}'\{\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\}\right] \\ & \times \pi(\boldsymbol{\beta})\pi(\{\rho_t\})\pi(\{\mathbf{b}_i\})\pi(\{\mathbf{c}_t)\pi(\mathbf{D})\pi(\mathbf{E})\pi(\sigma^2)\end{aligned} \tag{9}$$

2.1 MCMC Algorithm

Besides the spatial autoregressive coefficient, all model parameters can be directly sampled from their full conditional posteriors using the Gibbs Sampler. For $\{\rho_t\}$, a Metropolis-Hastings algorithm with a normal proposal distribution is applied to update the parameters. The iterative simulation scheme is as follows:

(a) Sample $\boldsymbol{\beta}$ from the multivariate distribution below conditional on the most updated values of other parameters and the data:

$$\boldsymbol{\beta} \sim \mathcal{N}(\bar{\boldsymbol{\beta}}, \mathbf{B}_1), \tag{10}$$

$$\text{where,} \quad \mathbf{B}_1 = \left(\mathbf{B}_0 + \sigma^{-2}\sum_{t=1}^{T}\mathbf{X}_t'\mathbf{X}_t\right)^{-1} \tag{11}$$

$$\bar{\boldsymbol{\beta}} = \mathbf{B}_1\left(\mathbf{B}_0^{-1}\boldsymbol{\beta}_0 + \sigma^{-2}\sum_{t=1}^{T}\mathbf{X}_t'\mathbf{A}(\rho_t)(\mathbf{Y}_t - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i)\right) \tag{12}$$

(b) Sample σ^{-2} from the gamma distribution below conditional on the most updated values of other parameters and the data

$$\sigma^{-2} \sim \mathcal{G}(\alpha_1, \delta_1), \tag{13}$$

$$\text{where,} \quad \alpha_1 = a_0 + \sum_{t=1}^{T} N_t \tag{14}$$

$$\delta_1 = b_0 + \sum_{t=1}^{T} \left(\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\right)' \left(\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\right) \tag{15}$$

(c) Update $\mathbf{b}_i$ one by one based on the following conditional posterior distribution:

$$\mathbf{b}_i \sim \mathcal{N}(\bar{\mathbf{b}}_i, \mathbf{D}_{1i}), \tag{16}$$

where

$$\mathbf{D}_{1i} =(\boldsymbol{D}^{-1} + \sigma^{-2}\mathbf{I})^{-1} \tag{17}$$

$$\bar{\mathbf{b}}_i =\mathbf{D}_{1i}\sigma^{-2}(\sum_{t=1}^{T}(Y_{it} - \rho_t W_{it}\mathbf{Y}_t - X_{it}\boldsymbol{\beta} - D_{it}\mathbf{c}_t) \tag{18}$$

(d) Draw $\mathbf{c}_t$ one by one from their respective conditional posterior distribution conditional on the most updated values of other parameters and the data:

$$\mathbf{c}_t \sim \mathcal{N}(\bar{\mathbf{c}}_t, \mathbf{E}_{1i}), \tag{19}$$

where

$$\mathbf{E}_{1i} =(\mathbf{E}^{-1} + +\sigma^{-2}\mathbf{D}_t'\mathbf{D}_t)^{-1} \tag{20}$$

$$\bar{\mathbf{c}}_t =\mathbf{E}_{1i}\sigma^{-2}\mathbf{D}_t'(\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{b}_i) \tag{21}$$

(e) Update the variance-covariance matrices of $\mathbf{D}$ in the following way:

$$\mathbf{D}^{-1} \sim \mathcal{W}(\nu_1, \boldsymbol{D}_1), \tag{22}$$

where $\nu_1 = \nu_0 + N$, and $\boldsymbol{D}_1 = (\mathbf{D}_0^{-1} + \sum_{i=1}^{N}\boldsymbol{\beta}_i\boldsymbol{\beta}_i')^{-1}$,

(f) Similarly, $\mathbf{E}$ is updated using its conditional posterior:

$$\mathbf{E}^{-1}|\{\mathbf{c}_t\} \sim \mathcal{W}(\boldsymbol{\eta}_1, \mathbf{E}_1), \tag{23}$$

where, $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_0 + T$, and $\mathbf{E}_1 = (\mathbf{E}_0^{-1} + \sum_{i=1}^{T} \mathbf{c}_t\mathbf{c}_t')^{-1}$

(g) $\{\rho_t\}$ MH algorithm. The proposal density is

$$\rho_t^* \sim \mathcal{N}(\phi, \Phi) \tag{24}$$

$$\text{where,} \quad \Phi = \left(\sigma^{-2}\sum_{i=1}^{N}(W_{it}\mathbf{Y}_t)'(W_{it}\mathbf{Y}_t)\right)^{-1} \tag{25}$$

$$\phi = \Phi\sigma^{-2}\sum_{i=1}^{N_t}(W_{it}\mathbf{Y}_t)'(Y_{it} - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i) \tag{26}$$

$$\alpha = \frac{|\mathbf{A}(\rho_t^*)| \exp\left[-\frac{1}{2\sigma^2}\{\mathbf{A}(\rho_t^*)\mathbf{Y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\}'\{\mathbf{A}(\rho_t^*)\mathbf{Y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\}\right]}{|\mathbf{A}(\rho_t)| \exp\left[-\frac{1}{2\sigma^2}\{\mathbf{A}(\rho_t)\mathbf{Y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\}'\{\mathbf{A}(\rho_t)\mathbf{Y}_t - \mathbf{X}_t\boldsymbol{\beta} - \mathbf{D}_t\mathbf{c}_t - \mathbf{b}_i\}\right]} \tag{27}$$

The determinant $|\mathbf{A}(\rho_t)| = \prod_{i=1}^{m}(1 - \rho_t\lambda_i)$ where λ_i is an eigenvalue of $\mathbf{W}_t$. Then if $\alpha > 1$ accept ρ_t^*; otherwise, reject the proposal and keep ρ_t as the updated value

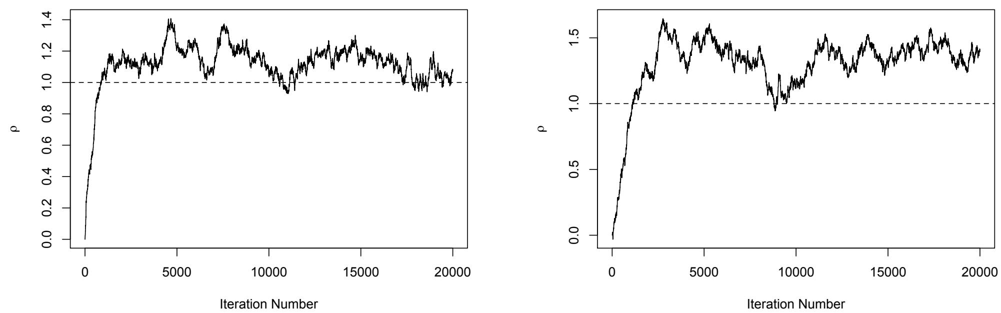
(h) Repeat the process from (a) to (g) till convergence.

3 Stationarity Check

To assess stationarity concerns with the time-invariant ρ model, we re-estimated time-invariant ρ model without placing any restrictions on the parameter space of ρ. We started the chain for ρ at zero. Figure 1 shows the trace plots of draws of ρ in two such MCMC simulations without a burn-in stage.

The plot on the left-side is based on a spatial model using mutual GATT/WTO membership to construct the matrix of spatial weights. After approximately one hundred iterations, the chain went out of the stationarity space, and stayed in the area above 1 for most of the time. There is no sign that the chain would return in the stationarity space by increasing the number of iterations.

Figure 1: System-Specific/Time-Specific Effect of Regime



Spatial Weight: Mutual GATT/WTO Membership

Spatial Weight: Geographical Distance

To further check this problem, we re-estimated the same model, but used a time-invariant matrix of spatial weights, where the weight for two countries consists of their geographical distance. The trace plot on the right-side hand shows that it is very unlikely that the process is stationary, either. While the chain for ρ moves above 1 slightly less quickly (though still pretty quickly), the chain is very rarely below 1.