Costly Inspection and Money Burning in Internal Capital Markets

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Abstract

Bureaucracy and influence activities consume a great deal of managers' time and effort in an organization. These activities are surplus destroying in the sense that they produce no direct output or information. This paper suggests a positive role for these activities. We develop a model for allocation in internal capital markets that takes a mechanism design perspective and incorporates both costly inspection and money burning (e.g. bureaucracy, influence activities) as tools for the headquarters to pursue optimal allocations. We find that the optimal mechanism deploys both the instruments of costly inspection and money burning, often at the same time on an agent.

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1 Introduction

Consider a situation in which the headquarters of a multidivisional firm must select one investment to make from a set of proposals from its division managers who are privately informed about their proposals' values. The objective of headquarters is to choose the highest value proposal, and the objective of the division managers is to get their own proposal selected. Headquarters is unable to condition monetary transfers on the proposals the managers make, but there are two instruments available to elicit information from them. First, the managers are able to engage in money-burning activities to bolster their claims, and this money burning is costly both to the managers and to the firm. Moreover, as in [32] or [16], headquarters can hire, at a cost to itself, an external firm to conduct an assessment of any division manager's proposal and find out its true value. Prior literature has considered the optimal use of money burning and costly state verification in isolation. This paper explores whether and how they should be optimally combined.

When only costly verification is available, the structure of the optimal mechanism is governed by feasibility for higher value projects. If a project is sufficiently valuable the headquarters would not want to decrease the probability of implementing such a project. However if all proposals are given the highest feasible probabilities the incentive costs associated with inspecting claims might add up quickly. Thus, proposals of low values are clustered together and treated identically with no verification and low implementation probabilities, and higher types have increasing probabilities of verification and implementation. However, in such a mechanism we show that the types that does have costly verification have no local incentive conditions. Instead all the lower types want to mimic all the higher types and the incentive costs for deterring low types from announcing higher types might be large and might require a larger than ideal cluster of lesser types. Thus when both money burning and costly verification are available to the mechanism designer, the mechanism designer might choose to engage in both the activities simultaneously. When distribution of types favors a separating equilibrium, money burning becomes a cheap and effective way of disincentivizing the very low types to deviate and announce a higher type.

We show that when both costly inspection and money burning are available instruments, the mechanism still shares some of the characteristics of a mechanism with only costly inspection. Namely, we show that the mechanism admits monotonicity in the ultimate probability of implementation with regards to the project value, and takes a threshold form where all the projects below a certain threshold are not subject to inspection, or money burning. Monotonicity does not follow immediately from the incentive compatibility constraints, since the presence of two instruments make non-monotonous mechanisms feasible; the optimal mechanism however is monotone. We show that the optimal mechanism admits certain features observed in corporations[1], and that the money burning activities in corporations may be result of optimal behavior on part of the headquarters.

Besides withholding our mechanism has two instruments thus our problem requires a novel approach. Instead of relying on envelope arguments we interpret our mechanism as a directed graph. Such an approach enables us to characterize incentive constraints regardless of whether money burning, costly verification or a combination of the two is used to tighten the incentive constraint. We explore the properties of the graph and deliver our results based on these properties. Using these techniques we show that any optimal mechanism must contain a *web* among the lower types.[2] These types are pooled together and have lower implementation probabilities, but no money burning or verification. We then show that above the web there is allocative efficiency, that is withholding is not used. This is crucial, along with some of the other properties we identify enables us to show that for any web finding the optimal mechanism can be reduced to a standard linear programming problem. Then finding the optimal mechanism is straightforward, the linear program can be solved for any given web, so the mechanism reduces to finding the optimal web.

The remainder of this paper is organized as follows: Section 1.1 provides a review of the related literature and discusses the contribution of the paper. Section 2 presents the model and the analysis. All proofs can be found in the appendix.

1.1 Literature review

Research on costly verification in mechanism design is not new. [32] models optimal contracts with costly verification. [8], [26] and [12] further study problems on costly state verification. [4] studies allocation problem with multiple agents in presence of costly verification techniques and [33] consid-

[1]Such as monotonicity of money burning in implementation without inspection, and the threshold nature of the mechanism. A more detailed discussion follows later.

[2]A web in a graph of incentives corresponds to tight constraints holding both ways across all types in the web.

ers a discrete version of the same problem. Taking a different approach in costly verification problems [21] considers mechanism design when there is limited punishments available to the designer. These papers however study the effects of costly verification in isolation to money burning, which is a key feature of internal capital markets.

There is also considerable research on money burning in mechanism design. [23], [17], [34], and [9] all study optimal allocation problems with money burning. Again, much of the literature in money burning deals with the problem in isolation. Our model is most closely related to [4] and builds upon their model by introducing money burning.

The usege of money burning in our environment is essentialy costly signalling and has an established role in many strands of economic literature. At a stretch, the costly signalling model of [30] can be considered as one of the earliest works for utilizing costly actions for signalling. [3] considers taking costly actions in the presence of forward induction, and show that once more than one player can do so the dynamics start to change. Besides the aforementioned earlier works, delegation literature in particular considers problems that share some features of our problem. [2] explores a contracting problem between a single agent and a principal where money burning is used to provide incentives to an agent when transfers are limited. Even though delation can be considered as allocation, the fundamental problem we address rests in selecting which agent to delegate and any rents are purely informational, rather than characterizing the optimal value of a delegation problem. Another paper in delegation literature is [1], where the delegation problem is between multiple agents, however coordination between multiple agents and the benefits from incentivizing the agents is the main concern in that research, whereas in our setting agents have no incentive to coordinate and there are no gains to be made from coordination.

Our main motivation for modelling the tools in our mechanism design problem rests in the literature about internal capital markets and the costly influence activities in them. Empirical evidence that diversification destroys shareholder value was provided in early studies by [20] and [5]. They show that the market values diversified firms less than a comparable portfolio of single segment firms.[3]. An important factor affecting the performance of a diversified firm is the efficiency of its internal capital markets; there is considerable empirical evidence to support this hypothesis. [27] show that the

[3]Also see, [18], [29], [6], [10], [22], [11] and [19] for additional evidence supporting the view that diversification destroys value.

discount at which the diversified firms trade is positively related to the capital mis-allocation. An internal capital market is an environment without transfers and the optimal allocation must be achieved through other means at the disposal of the headquarters. There is an informational asymmetry where the divisional managers are better informed about the true value of the project. The headquarters can ask the managers for the true value, but the managers' self-interest often does not align with that of the headquarters. For example, the managers might have an interest in securing funding for their own project regardless of the value of another project in a different division. This is likely to lead the managers to make exaggerated claims about the value of the project, and engage in activities that can influence the headquarters to approve their project at the cost of another. The headquarters often has the means to check the validity of claims of the managers; this can however be a costly exercise. [15], [16] and [31], for instance, explore the capital allocation problem in organizations where the headquarters can inspect the value of the projects in some sense. Combining these insights in internal capital markets, we explore the properties of a mechanism for allocation, where costly verification and money burning activities are the tools and we try to see whether these money burning practices have a place at all in an optimally performing allocation market.

2 The Model

A diversified firm has D divisions. Each of the division managers have an investment proposal with value v_{t_i}, where t_i is the type of the division i and $t_i \in \{1, 2, \ldots, n\}$ $\forall i$ without loss of generality. We assume that the division types, i.e., the value of the projects are i.i.d draws from the type-space following the type distribution f such that a given proposal is of type t with probability $f(t)$. We have $\sum_t f(t) = 1$, and with each type t is associated a value of the project, v_t. Since v_t can be arbitrary, we can assume that $f(t)$ follows a uniform distribution over $\{1, 2, \ldots, n\}$.[4] We will use V to denote both the sets, $\{1, 2, \ldots, n\}$ and $\{v_1, v_2, \ldots, v_n\}$ with a little abuse of notation. The objective of the headquarters is to maximize the expected value of the investment, minus the costs of any assessments done and the costs to the firm due to the money burning activities. The division managers and

[4] This can be done by replicating the types sufficient number of times, similar to taking the inverse of the cumulative distribution function as the type space in the case of continuous types, and can be used to approximate any distribution with a minor loss, any distribution with rational probabilities can be matched exactly.

the headquarters are all assumed to be risk neutral. The division manager's utility is monotone in the expected value generated by the investment minus the money burning cost. The cost of hiring an external firm to assess a division to discover the value of its investment is k; the firm charges a fixed cost irrespective of the division where $v_1 < k < v_n$. We are going to normalize v_i's and k so that $v_1 = 1$.[5] The headquarters must device a mechanism for optimal allocation of the resources. All agents are assumed to be risk neutral, and we will be looking at a Bayes Nash Equilibrium of the mechanism proposed by the headquarters.

The mechanism in its most general form can be complex. It can have multiple rounds of, assessments and eliminations of projects, and money burning activities, finally ending in an investment. We appeal to the revelation principle and invoke arguments in [4] to demonstrate that we need only to look at the truthful mechanisms to cover all such possible scenarios. A truthful mechanism is one where the truth telling strategy on part of the managers is a bayesian nash equilibrium of the game. The basic idea is simply to design a mechanism that compensates the managers for telling the truth exactly as much as they can gain by lying about their project values in the mechanism. Further following the arguments in [4] we also note that we need only look at mechanisms where a division manager's claim is inspected only if her project is being considered for implementation. After all, there is no point in spending money on inspecting a project's value if one has no intention of implementing it, especially since this is a setting where transfers are not permitted and no payments can be extracted from the division managers for their exaggerated claims. For maximum penalty, the project of the lying manager must be shelved. This suggests that we can restrict our attention to mechanisms where the turn of events is as follows. We assume that a single entity, called the principal, is in charge of designing and implementing the optimal mechanism.

Stage 1: The principal announces a policy that specifies a selection policy, an inspection policy and a money burning policy
Stage 2: Agents report their types to the principal
Stage 3: Following the selection policy, a project is selected for Stage 4
Stage 4: The selected agent burns the required money
Stage 5: The selected project is inspected depending on the inspection policy
Stage 6: The selected project is implemented if there is no inspection, or if

[5] This can be done by simply dividing all the values and k by v_1.

the agent is found truthful upon inspection

Note that this specific structure on how headquarters can choose to implement projects is not something selected by us for analysis; it is the result stemming from the mechanism design literature – that of all the possible ways that the firm can chose to implement its policies, none can do better than doing it in the aforementioned 6 Stage plan.

Following these arguments, a bayesian incentive compatible mechanism can be fully characterized by the three functions $p_i(t_i, t_{-i}) : V^n \mapsto \mathbb{R}_+$, $a_i(t_i, t_{-i}) : V^n \mapsto [0, 1]$, and $c_i(t_i, t_{-i}) : V^n \mapsto [0, 1]$. Here $p_i(t_i, t_{-i})$ is the amount of money burning by the division manager i in Stage 2, when the announced project values by the division managers are $\boldsymbol{v} = (v_i, v_{-i}) \in V^n$ corresponding to the type announcement $\boldsymbol{t} = \{t_i, t_{-i}\}$. We further abuse the notation to use $\boldsymbol{t}$ and $\boldsymbol{v}$ interchangeably, to mean the appropriate thing, in order to further ease the expositional burden. Since the amount of money burned cannot be negative, we must impose a constraint $p_i(\boldsymbol{t}) \geq 0 \ \forall i \in D, \forall \boldsymbol{t} \in V^n$. $a_i(t_i, t_{-i})$ is the probability of the i^{th} project being selected in Stage 3 for further review given the announced types $\boldsymbol{t} = (t_i, t_{-i})$, thus, we must have $\sum_{i=1}^{D} a_i(\boldsymbol{t}) \leq 1 \ \forall \boldsymbol{t} \in V^n$ and $a_i(\boldsymbol{t}) \geq 0 \ \forall \boldsymbol{t} \in V^n$. We have $(1 - c_i(t_i, t_{-i}))$ as the probability that a costly inspection of the selected manager's claims is done in Stage 4, hence $0 \leq c_i(\boldsymbol{t}) \leq 1 \ \forall \boldsymbol{t} \in V^n$.

At this point, it becomes necessary for us to clarify further what these money burning activities are, and in what fashion is the money burned. Attentive readers will notice that the probability of project selection and the probability of inspection are not allowed to depend on the level of money burning activities, the way we have formulated the mechanism. This is a restriction but it is without loss of generality. Money burning activities can be of many forms – the headquarters can introduce a policy of extensive documentation for these proposals, or require approval and support of multiple offices which would consume the manager's time and burn resources. This type of workplace bureaucracy is costly and does not have any direct output. It is for the headquarters to decide how much bureaucracy each manager must go through to submit a certain project of specific value to be considered for implementation. Because the choice of how much money is being burned by the managers in this case rests with the headquarters, we need not have the money burned as an argument for the probability of project selection or inspection. If the manager fails to go through the required bureaucratic procedures then they are automatically out of the running, and we have no need for additional notation to accommodate for that.

Another form of money burning can be influence activities as described by [25] and [24], where the managers spend time and effort trying to affect the decisions of the headquarters. If the managers engage in activities trying to persuade the decision makers of the merits of their proposal, they do so at the cost of time that could have otherwise been used to carry out their responsibilities in the firm more efficiently. These influence activities are examples of money burning where the level of money burned is determined by the managers. In such a scenario, the headquarters should actually condition the project selection and inspection probabilities on the level of money burning. We must thus have the probability of selection for manager i as $a_i(\boldsymbol{t}, \boldsymbol{p})$ and similarly the probability of inspection as $c_i(\boldsymbol{t}, \boldsymbol{p})$. We will argue that the two formulations are one and the same.

It is clear that the headquarters can do no better in a mechanism where managers decide the money burned than in the one where the headquarters make the call. To see that the two are equivalent, consider a scheme where the managers pick the level of money burned $a_i(\boldsymbol{t}, \boldsymbol{p}) = a_i(\boldsymbol{t})$ when $\boldsymbol{p} = \langle p_i(\boldsymbol{v}) \rangle$ and zero otherwise, and define the inspection probability similarly. This mechanism is essentially the same as in the case when the headquarters decide the level of money burning. Basically, if the managers pick any level of money burning other than what the headquarters want, their project is shelved and they gain nothing. This is to say that if a certain divisional manager schedules more than, say ten, meetings to discuss and elaborate on their project, the headquarters can threaten to automatically disqualify their proposal so as to discourage an undesirable level of money burning. Thus, the amount of money burned can be interpreted as either the choice of money burned by the managers, or that burned through bureaucratic means by the headquarters, or some combination of the two. It is noteworthy that our choice of Bayesian-Nash equilibria as the solution concept plays a critical role in establishing this equivalence.

We are now in a position to clearly outline the optimization problem with the objective function of the headquarters and the incentive constraints of various agents. The headquarters wish to maximize:

$$
\begin{aligned}
& E_{\boldsymbol{t}}\left[\sum_{i=1}^{D}\left[v_{t_i} a_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) - k\left(1 - c_i(t_i, t_{-i})\right) a_i(t_i, t_{-i})\right]\right] \\
=& \sum_{i=1}^{D}\left[E_{t_i}\left[E_{t_{-i}}\left[v_{t_i} a_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) - k\left(1 - c_i(t_i, t_{-i})\right) a_i(t_i, t_{-i})\right]\right]\right] \\
=& \sum_{i=1}^{D}\Bigg[E_{t_i}\Bigg[v_{t_i} E_{t_{-i}}\left[a_i(t_i, t_{-i})\right] - E_{t_{-i}}\left[p_i(t_i, t_{-i})\right] \\
& \qquad + k E_{t_{-i}}\left[c_i(t_i, t_{-i}) a_i(t_i, t_{-i})\right] - k E_{t_{-i}}\left[a_i(t_i, t_{-i})\right]\Bigg]\Bigg]
\end{aligned}
$$

The incentive constraints of the division managers are:

$$
E_{t_{-i}}\left[v_{t_i} a_i(t_i, t_{-i}) - p_i(t_i, t_{-i})\right] \geq E_{t_{-i}}\left[v_{t_i} a_i(\hat{t}_i, t_{-i}) c_i(\hat{t}_i, t_{-i}) - p_i(\hat{t}_i, t_{-i})\right] \;\; \forall t_i, \hat{t}_i \in V
$$

Now we note that for any asymmetric mechanism there must exist a symmetric mechanism (an anonymous mechanism that does not favor any agent ex-ante) that does just as well, which can be achieved by permuting the indexes on the types in all possible ways and then implementing any one of these mechanisms randomly. Thus, we will now focus our attention on symmetric mechanisms. Symmetric mechanisms are also more desirable in the real world due to legal and practical reasons. The headquarters do not want to be perceived as preferring some division over another when there is nothing to be gained by it.[6] We can thus look at mechanisms where $p_i(t_i, t_{-i}) = p_j(t_j, t_{-j})$, $a_i(t_i, t_{-i}) = a_j(t_j, t_{-j})$, and $c_i(t_i, t_{-i}) = c_j(t_j, t_{-j})$ if $t_i = t_j$ and $t_{-i} = t_{-j}$. This gives us $E_{t_{-i}}\left[a_i(t_i, t_{-i})\right] = E_{t_{-j}}\left[a_j(t_j, t_{-j})\right]$ if $t_i = t_j$ and we will abuse the notation to define $\mathcal{A}_t = E_{t_{-i}}\left[a_i(t_i, t_{-i})\right]$ when $t_i = t$. Similarly we define $p_t = E_{t_{-i}}\left[p_i(t_i, t_{-i})\right]$ when $t_i = t$. Also, define $c_t = \frac{E_{t_{-i}}[c_i(t_i, t_{-i}) a_i(t_i, t_{-i})]}{E_{t_{-i}}[a_i(t_i, t_{-i})]}$. We can say that $0 \leq c_t \leq 1$ since $0 \leq c_i(t_i, t_{-i}) \leq 1 \; \forall \boldsymbol{t} \in V^n$ thus implying $E_{t_{-i}}\left[c_i(t_i, t_{-i}) a_i(t_i, t_{-i})\right] \leq E_{t_{-i}}\left[a_i(t_i, t_{-i})\right]$. Of course, this definition imposes constraints on the interim probabilities, $\mathcal{A}_t$, as well. [7] provides us with a characterization of the set of feasible interim allocation probabilities that can be achieved using an allocation rule. We use $\langle \mathcal{A} \rangle$ to

[6]On the other hand, it is possible that the headquarters actually have a real preference for some divisions. Such favoritism can be modeled through, say, breaks in money burning activities or higher returns for their project implementation. This is a possibility we will not explore in the present work.

denote the collection $\langle \mathcal{A}_t \rangle_{t \in V}$. The inequalities state that $\langle \mathcal{A} \rangle \in \mathbb{R}^n_+$ is feasible if and only if,

$$\sum_{t \in S} f_t \mathcal{A}_t \leq \frac{1 - \left(\sum_{t \notin S} f_t\right)^D}{D} \qquad \forall S \subseteq V$$

where f_t is the probability of an agent being type t. Since f follows the uniform distribution, this can be written as:

$$\frac{D}{n} \sum_{t \in S} \mathcal{A}_t \leq 1 - \left(1 - \frac{|S|}{n}\right)^D \qquad \forall S \subseteq V$$

For any S the inequality can be written as $\sum_{t \in S} \mathcal{A}_t \leq g(S)$ where we define $g : \mathcal{P}(V) \mapsto \mathbb{R}^+$ as $g(S) = \frac{n}{D} - \frac{n}{D}\left(1 - \frac{|S|}{n}\right)^D$; $\mathcal{P}(X)$ is the set of all subsets of a set X. We note that since $g(S)$ is essentially a function of the cardinality of the set S, we will abuse the notation to mean $g(S) = g(|S|)$ where g will take as argument either sets or positive integers. One of the properties of the function g that will be useful to us is that it is strictly submodular, proved as Lemma 1. Furthermore, we have $p_t \geq 0 \ \forall t \in V$ since $p_i(t_i, t_{-i}) \geq 0 \ \forall i \in D, \forall \boldsymbol{t} \in V^n$. With these constraints being necessary and sufficient on $\mathcal{A}_t, c_t, p_t$ we can rewrite the equations and the problem.

Let us denote by the set $\mathscr{A}$ the set of all feasible intermediate allocations that satisfy Border's inequalities. A mechanism can now be characterized as a tuple $M = \{\langle \mathcal{A} \rangle, \langle c \rangle, \langle p \rangle\}$ where,

$$\begin{aligned} \mathcal{A}_t &: \text{interim probability of selection of type } t \\ 1 - c_t &: \text{probability of assessment if type } t \text{ is the selected type} \\ p_t &: \text{amount of money burning required for announcing type } t \end{aligned}$$

The objective is to maximize

$$\begin{aligned} &\sum_{i=1}^{D} \Bigg[E_{t_i} \Bigg[v_{t_i} E_{t_{-i}} \left[a_i(t_i, t_{-i})\right] - E_{t_{-i}} \left[p_i(t_i, t_{-i})\right] \\ &\qquad\qquad + k E_{t_{-i}} \left[c_i(t_i, t_{-i}) a_i(t_i, t_{-i})\right] - k E_{t_{-i}} \left[a_i(t_i, t_{-i})\right] \Bigg] \Bigg] \\ &= \sum_{i=1}^{D} \left[E_{t_i} \left[v_{t_i} \mathcal{A}_{t_i} - p_{t_i} + k \mathcal{A}_{t_i} c_{t_i} - k \mathcal{A}_{t_i} \right] \right] \end{aligned}$$

However, since the mechanism is symmetric and the expectation is the same for all the agents, we can drop the sum and write the objective function for a mechanism M as:

$$Obj(M) = E_t \left[v_t \mathcal{A}_t - p_t + k\mathcal{A}_t c_t - k\mathcal{A}_t \right] \tag{1}$$

The incentive constraints are:

$$v_{t_i}\mathcal{A}_{t_i} - p_{t_i} \geq v_{t_i}\mathcal{A}_{\hat{t}_i} c_{\hat{t}_i} - p_{\hat{t}_i} \qquad \forall t_i, \hat{t}_i \in V \tag{2}$$

or

$$v_t\mathcal{A}_t - p_t \geq v_t\mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \qquad \forall t, \hat{t} \in V \tag{3}$$

after dropping the unnecessary subscript denoting the player. This gives us the optimization problem as follows:

$$\begin{aligned} \max_{\langle \mathcal{A} \rangle, \langle c \rangle, \langle p \rangle} \; & E_t \left[v_t \mathcal{A}_t - p_t + k\mathcal{A}_t c_t - k\mathcal{A}_t \right] \\ & s.t. \\ & \langle \mathcal{A} \rangle \in \mathscr{A}; \quad 0 \leq c_t p_t \leq p_t \;\; \forall t \in V; \;\; c_t, p_t \in \mathbb{R}^+ \\ & v_t\mathcal{A}_t - p_t \geq v_t\mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \quad \forall t, \hat{t} \in V \\ & v_t\mathcal{A}_t - p_t \geq 0 \qquad\qquad\qquad \forall t \in V \end{aligned}$$

where the last set of constraints are the individual rationality constraints of the managers. Having outlined the objective and the IC constraints in terms of the intermediate probability, we can now use Border's inequalities to search for an optimal mechanism. Let us define a transform function, which defines a new mechanism from any given mechanism M. We write $M^* = T_{M,r,s}(\mathcal{A}'_r, \mathcal{A}'_s, c'_r, c'_s, p'_r, p'_s)$ to mean that M^* is a transform of the mechanism M where if $M^* = \{\langle \mathcal{A}^* \rangle, \langle c^* \rangle, \langle p^* \rangle\}$ then $\mathcal{A}^*_r = \mathcal{A}'_r$; $\mathcal{A}^*_s = \mathcal{A}'_s$; $c^*_r = c'_r$; $c^*_s = c'_s$; $p^*_r = p'_r$; $p^*_s = p'_s$; $\mathcal{A}^*_t = \mathcal{A}_t \; \forall t \notin \{r, s\}$; $c^*_t = c_t \; \forall t \notin \{r, s\}$; $p^*_t = p_t \; \forall t \notin \{r, s\}$. We say that a mechanism $M = \{\langle \mathcal{A} \rangle, \langle c \rangle, \langle p \rangle\}$ is feasible if $\langle \mathcal{A} \rangle \in \mathscr{A}$ and $0 \leq c_t p_t \leq p_t \; \forall t \in V$; $c_t, p_t \in \mathbb{R}^+$. A mechanism is said to be incentive compatible if it satisfies all the IC constraints.

We will interpret a mechanism M as a directed graph. Let G_M be the directed graph that represents the mechanism M. Let $V(G_M)$ be the set of vertices of the graph and $E(G_M)$ be the set of edges where $(s, t) \in E(G_M)$ means that there is a directed edge originating at s and pointing towards t in G_M. For any mechanism M, we define $V(G_M) = V$. Since all graphs we consider will have the same vertex set, we will simply use V to denote the set of vertices instead of $V(G_M)$. We say that $(s, t) \in E(G_M)$ if the incentive constraint for type s pretending to be type t is tight. More specifically $(s,t) \in E(G_M)$ if and only if

$$v_s \mathcal{A}_s - p_s = v_s \mathcal{A}_t c_t - p_t$$

For any $t \in V$ let $deg_M^-(t)$ denote the indegree of t and let $deg_M^+(t)$ denote the outdegree of t in G_M. Let $i_M(t) = \{r : (r,t) \in E(G_M)\}$ and $o_M(t) = \{r : (t,r) \in E(G_M)\}$. Also let $I_M(t)$ be the set of all types t' such that there is a directed path leading from t' to t in G_M, and let $O_M(t)$ be the set of types t' such that there is a directed path leading from t to t' in G_M. We have $i_M(t) \subseteq I_M(t)$ and $o_M(t) \subseteq O_M(t)$. Similarly, for any $S \subseteq V$ define

$$\begin{aligned} i_M(S) &= \{t \in V \backslash S : (t,s) \in E(G_M) \text{ for some } s \in S\} \\ o_M(S) &= \{t \in V \backslash S : (s,t) \in E(G_M) \text{ for some } s \in S\} \\ I_M(S) &= \{t \in V \backslash S : t \in I_M(s) \text{ for some } s \in S\} \\ O_M(S) &= \{t \in V \backslash S : t \in O_M(s) \text{ for some } s \in S\} \end{aligned}$$

We will call a graph G_M associated with an optimal mechanism M an optimal graph. We will study the properties of this graph. We start with the following definition

Definition 1. *We say that $W \subseteq V$ is a* web *in G_M iff $(s,t) \in E(G_M)\ \forall s,t \in W$ and there do not exist r,s such that $r \in V \backslash W;\ s \in W$ and $\{(r,s),(s,r)\} \subseteq E(G_M)$. A web W is* non-trivial *if $|W| > 1$.*

In essence, a web is a set of types where every node points to every other node in G_M, and no more nodes can be added to it while still having every node pointing to every other node. While this doesn't directly follow from the definition, we will later see that in an optimal mechanism this is in-fact the case. It follows from the definition that any $t \in V$ such that $deg_M^-(t) = 0$ is a web. Moreover, every singleton set in V either belongs in a web, or is a web itself. In the appendix we show some properties of the optimal mechanism and how it relates to webs. Lemma 6 shows that a directed cycle in an optimal mechanism should be part of a web, and that in a web containing more than one type, none of the proposals may undergo assessment. Lemma 7 shows that an optimal graph G_M must be weakly connected and must contain exactly one web W^* - the proposals(nodes) in the web requiring no money burning. We will need these results along with Lemma 8 to deliver our first result regarding incentives.

Our first result says that the managers are strictly better off by announcing the true value of the project than any other lower value of the project, in the optimal mechanism. While incentive compatibility already implies that the managers can do no better by lying, this result says that we can drop all the downward facing constraints from the problem and it remains unchanged – that we can assume no manager will ever devaluate their project to the headquarters. This is very much in line with what is observed in an organization. The proof follows previous results and uses the fact that if a manager is just as well off announcing a different type than the true type, then it must be the case that this different type project has a higher or the same probability of implementation. This means that if (s,t) is tight then we must have $\mathcal{A}_s \leq \mathcal{A}_t$. However, if $v_s > v_t$ then we can implement s with a greater probability and t with a lower probability to improve the objective function overall. The detailed proof can be found in the appendix.

Proposition 1. *In an optimal mechanism, if $(s,t) \in E(G_M)$ then either $s < t$ or there exists a web W such that $\{s,t\} \subseteq W$.*

The goal of the influence activities carried out by the managers is to further the cause of their project. When spending time and effort in influencing the decision maker, the manager must believe that this will increase the probability of implementation for their project. This is often perceived to be an unnecessary distortion in allocation and popular wisdom seem to suggest that the probability of project implementation must remain unaffected, or even decrease, in the amount of money burned by the manager. However, it still remains the case that these influence activities persist, and increase the probability of the manager's project implementation.[7] Why do the leaders of these firms remain susceptible to these influence activities across the board? We posit that they are in fact acting in the best interests of the firm, and that in the optimal mechanism higher influence activities should correspond to higher chances of project implementation. Moreover, it stands to reason that if the project is inspected, that takes away any incentive for the manager to engage in money burning activities to influence the decision maker since they already know the true value of the project. This suggests that higher influence activities must be related to higher probability of project implementation, but without assessment. The next result reinforces exactly the intuition that the headquarters is acting optimally when we observe that influence activities in the firm work for the intended purpose.

Lemma 9. *In an optimal mechanism, the amount of money burning is strictly monotonic in the probability of project implementation without an*

[7]It is almost tautological that influence activities must improve the chances of project activities, for there would be no influence activities if they didn't.

assessment, i.e., in an optimal M*,* $\mathcal{A}_t c_t > \mathcal{A}_s c_s \iff p_t > p_s$ *and* $\mathcal{A}_s c_s = \mathcal{A}_t c_t \iff p_s = p_t$*.*

In a full information setting, the goal of the headquarters would be to undertake the highest value investment. While this is not possible when the information on project value is known only to the division manager, if a mechanism implements low value projects with higher probability as compared to the high value projects, we can be reasonably sure that the job is not done well. Indeed, we see that in an optimal mechanism the probability of selection in stage one must be higher for the high value investments. This, coupled with the fact that the mechanisms are truthful, implies that the overall probability of implementation for the high value projects remain higher than the low value projects. We have:

Proposition 2. *In an optimal mechanism high value projects are implemented with a higher probability, i.e.,* $v_t > v_{\hat{t}} \implies \mathcal{A}_t \geq \mathcal{A}_{\hat{t}}$*.*

The proof is done in multiple steps. If $t > \hat{t}$ with $\mathcal{A}_t < \mathcal{A}_{\hat{t}}$, we first show that the higher type must have positive probability of inspection and the lower type must have positive money burning. We note that at least one of the types must be inspected, for if not, then the two IC constraints cannot hold simultaneously. It can further be shown that t must be inspected. Moreover, it must also be true that $\mathcal{A}_{\hat{t}} c_{\hat{t}} > A_t c_t$ since otherwise we can change the mechanism to offer $\hat{t}$ the same deal as type t and save on the cost of inspection. However, this means that the money burning for $\hat{t}$ must be strictly higher than type t, as we have seen with Lemma 9. Thus we have that $c_t < 1$ and $p_{\hat{t}} > 0$.

We will now construct a mechanism M^* that is an improvement. We need to show three things to prove the result, that the mechanism is feasible, that it is incentive compatible and that it is an improvement over any existing mechanism. The mechanism is constructed by choosing $\mathcal{A}, c$ and p values such that the new values of $p_t, p_{\hat{t}}$ are linear combinations of the original values, similarly the changed values of $\mathcal{A}_t c_t$ and $\mathcal{A}_{\hat{t}} c_{\hat{t}}$ are linear combinations of the ones in the original mechanism, while slightly decreasing $\mathcal{A}_{\hat{t}}$ and slightly increasing $\mathcal{A}_t$. We use Lemma 2 and the fact that $c_t < 1$; $p_{\hat{t}} > 0$ to establish feasibility of interim probabilities. Thus for a specific choice of δ, c and p we have:

$$\begin{aligned} p^*_{\hat{t}} &= (1-\delta)p_{\hat{t}} + \delta p_t \\ p^*_t &= (1-\delta)p_t + \delta p_{\hat{t}} \\ \mathcal{A}^*_{\hat{t}} c^*_{\hat{t}} &= (1-\delta)\mathcal{A}_{\hat{t}} c_{\hat{t}} - \delta \mathcal{A}_t c_t \\ \mathcal{A}^*_t c^*_t &= (1-\delta)\mathcal{A}_t c_t - \delta \mathcal{A}_{\hat{t}} c_{\hat{t}} \end{aligned}$$

This ensures incentive compatibility of the mechanism M^* since for any arbitrary type r:

$$\begin{aligned} v_r \mathcal{A}^*_t c^*_t - p^*_t &= v_r \left((1-\delta)\mathcal{A}_t c_t + \delta \mathcal{A}_{\hat{t}} c_{\hat{t}}\right) - \left((1-\delta)p_t + \delta p_{\hat{t}}\right) \\ &= (1-\delta)\left[v_r \mathcal{A}_t c_t - p_t\right] + \delta \left[v_r \mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}}\right] \\ &\leq v_r \mathcal{A}_r - p_r \end{aligned}$$

It is noteworthy that the result on monotonicity of interim allocation probability follows from incentive compatibility in much of the classical mechanism design literature. However, in our setting we can implement almost any arbitrary feasible vector of interim allocation probabilities. Thus, again, we need to use the optimality of the mechanism to achieve the result. These results now allow us to shed some light on the structure of the optimal mechanism.

Since higher valued projects have a higher chance of implementation we do not expect very low value projects typically to attract an assessment or a lot of money burning. This is supported by the following result that says that there is a threshold and projects below that have a fixed probability of implementations. No money burning is required for these low value projects, furthermore, no assessment is done on these projects.

Lemma 10. *Every optimal mechanism M is of the form where $\exists\ \underline{t} \in V$ such that for all projects with $v_t \leq v_{\underline{t}}$ there is no assessment and no money burning. Furthermore, for all the types above this threshold, i.e. $v_t > v_{\underline{t}}$, there is either a positive probability of assessment, or some money burning effort, or both. Also, all the projects below this threshold have the lowest probability of implementation, i.e., $\mathcal{A}_t = \min_{r \in V} \mathcal{A}_r\ \forall t \leq \underline{t}$.*

Seen through the lens of alternative interpretation of interim allocation probabilities as the expected amount of capital allocation, the optimal mechanism is in line with the actual practices in the industry where the divisional managers have the power to approve projects below a certain budget, and require approval for bigger projects. In the former case, there is no inspection and no money burning, however a combination of the two is likely for approval of the bigger projects, guided by the earlier results.

Lemma 11. *The gains realized by the division managers are non-decreasing in the value of the project, i.e.* $v_t\mathcal{A}_t - p_t \geq v_{\hat{t}}\mathcal{A}_{\hat{t}} - p_{\hat{t}}$ *if* $t \geq \hat{t}$.

This result gives us one of the properties that is strongly desirable in an optimal mechanism from a corporate standpoint – it must be fair. What we mean by this is that a division manager proposing a high value project must not achieve a lower surplus than another manager proposing a lower value project. If the headquarters adopt a mechanism that punishes managers for proposing better projects, it might be a cause for resentment. Lemma **??** shows that this is not the case in the optimal mechanism. Note that it is not entirely obvious from the way the problem is set. One might be tempted to think that since a manager can always announce a lower value, she must always be able to ensure herself the gains associated with the lower value project. However, it can be shown fairly easily that if the higher value project has higher money burning than another lower value project, and if the lower value project has a high chance of inspection, then the surplus of the latter might be higher than the former in an incentive compatible mechanism. In the appendix we show that such an arrangement, although feasible, contradicts the monotonicity properties of an optimal mechanism.

Lemma 12. *In every optimal mechanism* M *where there are two types that require costly verification but no money burning, these types do not want to mimic each other. That is, in an optimal mechanism* M*, if* $t, \hat{t}$ *have* $c_t < 1$, $c_{\hat{t}} < 1$ *and* $p_t = p_{\hat{t}} = 0$*, then* $t \notin i_M(\hat{t})$ *and* $\hat{t} \notin i_M(t)$.

An important implication of lemma 12 is that if there is no money burning in the entire mechanism, the allocation probabilities are driven by the feasibility conditions as the incentives are rather simple, for any type only the constraint regarding the web is binding. This is exactly the solution in [33] and can be fully characterized using a greedy algorithm. The mechanism gives the highest feasible probability of implementation to the highest type, second highest to the second highest type and so on until some lower types are grouped together with no verification in order to balance the verification costs and probabilities of implementation. In other words, when only costly verification and withholding are the available tools to the headquarters, allocative inefficiency only happens for the pooled types. Our next result shows that this result remains true even when money burning is allowed.

Proposition 3. *In any optimal mechanism, there exists a cutoff type* $\underline{t}$*, the* *probability of implementation for all types above $\underline{t}$ is given by*

$$\mathcal{A}_n = g(1), \mathcal{A}_{n-1} = g(2) - g(1), \ldots, \mathcal{A}_{\underline{t}+1} = g(n-\underline{t}) - g(n-\underline{t}-1)$$
$$\mathcal{A}_{\underline{t}} = \mathcal{A}_{\underline{t}-1} \ldots = \mathcal{A}_1 = \frac{g(n) - g(\underline{t})}{\underline{t}}$$

Once $\underline{t}$ is known, proposition 3 reduces our problem to a fairly standard linear optimization problem. In particular for any given $\underline{t}$, using propositions 1, 3 and lemma 11 we can reduce our problem significantly.

$$\begin{aligned}
\max_{\langle c\rangle,\langle p\rangle} E_t\left[v_t\mathcal{A}_t - p_t + k\mathcal{A}_t c_t - k\mathcal{A}_t\right]& \\
& s.t. \\
& 0 \leq c_t p_t \leq p_t \; \forall t \in V; \;\; c_t, p_t \in \mathbb{R}^+ \\
& v_t\mathcal{A}_t - p_t \geq v_t\mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \quad \forall t, \hat{t} > t \in V
\end{aligned}$$

Notice we dropped the constraints for incentives to pretend to be lower types since we know they are not binding and the IR constraints since we know the utilities are increasing and the types in the web have positive utility. Additionally we must have $\mathcal{A}_t \geq \mathcal{A}_{t-k}$ for any k and we know c_t, p_t must be 0 for all the types in the web so we can drop them from the problem as well except the constraints involving $\underline{t}$ pretending to be some higher type. Since $\mathcal{A}$'s are given we can equivalently write the above problem as follows:

$$\begin{aligned}
\max_{\langle c\rangle,\langle p\rangle} \sum_{t=\underline{t}+1}^{n} & \left[-p_t + k\mathcal{A}_t c_t\right] \\
& s.t. \\
& c_t \leq 1 \; \forall t \in V; \;\; c_t, p_t \in \mathbb{R}^+ \\
& v_t\mathcal{A}_t \geq v_t\mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} + p_t \quad \forall t, \hat{t} \in \{\underline{t}, n\}, \hat{t} > t
\end{aligned}$$

The last formulation is a fairly standard linear programming problem now that the polymatroid part is resolved and all choice variables are weakly positive. In the appendix lemma 15 that if $\underline{t} > k$, the solution is even simpler, there is no money burning in the mechanism at all, therefore from [33] and $c_i = \mathcal{A}_i/\mathcal{A}_1$ for all types $i > \underline{t}$. On the contrary if $\underline{t} < k$, then lemma 16 money burning must be a part of the mechanism, and simplex method can be used solve the optimal $\langle c\rangle, \langle p\rangle$. This last observation also suggests a an algorithm to solve problem completely:

1. Set $\underline{t} = t'$ calculate $\mathcal{A}$ via proposition 3, solve the resulting LP via simplex method and calculate the resulting value $V(t')$.

2. Choose $\underline{t} \in \{1 \ldots, n\}$ to maximize $V(\underline{t})$.

Given the methodology above, we will present an example that demonstrates that both money burning and inspection can occur at the same time in an optimal mechanism, even for the same type. Consider three divisions that can take any one of the three values $V = \{1, 6, 10\}$ and let the cost of inspection be $k = 2$. Table 1 represents an optimal mechanism.

v_{t_i}	$\mathcal{A}_i$	$1 - c_i$	p_i
1	0.04	0	0
6	0.26	0	0.22
10	0.70	0.63	0.22

Table 1: An Optimal Mechanism

Consider the type i such that $t_i = 10$. We note that the type is inspected in many states of the world, and yet must burn money at the same time. This happens because the two tools act as substitutes for each other when enforcing the incentive compatibility constraints. If only money burning were to be used, a great deal of money will need to be burned to deter the closer type $t_i = 6$ from announcing it's type as type $t_i = 10$. Inspection is a much cheaper alternative when trying to deter $t_i = 6$ from announcing $t_i = 10$. However, a very small amount of money burned can deter $t_i = 1$ from announcing $t_i = 10$ and using only inspection for the purpose will prove costly. Therefore, a combination of the two proves to be the most efficient option.

Finally figure 1 shows the percentage loss of optimal value of the mechanism when money burning is excluded from the mechanism, as a function of the cost of inspection. We see that if inspection is costless, money burning is unnecessary, as should be the case. However, money burning becomes a relevant tool when inspection comes at a cost. Moreover, the greater the costs of inspection, the bigger are the downsides of curtailing any money burning activities.

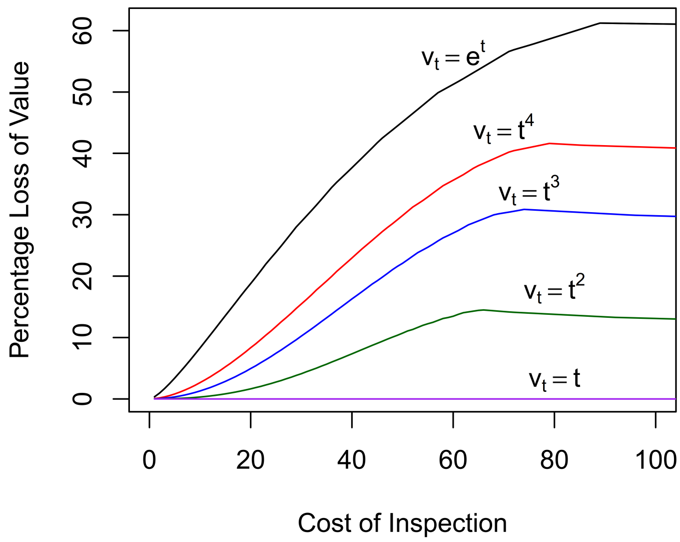

Figure 1: Percentage Loss of Optimal Value due to Lack of Money Burning

3 Conclusions and Future Work

3.1 Conclusions

We develop a model with minimal assumptions - multiple projects are proposed with different values and the managers act in self-interest while the headquarters try to device a mechanism to implement the best project. We allow the headquarters to use costly inspection and observable money burning activities to allocate optimally. From a mechanism design perspective, we see that when money burning activities are not available there are tight incentive conditions between the lowest value projects and the highest value projects. Minimizing the cost of inspection regarding these conditions might cause some projects to be implemented with less than ideal probabilities. Money burning activities enable headquarters to disincentivize the lowest types from mimicking very high types thus might help the headquarters to avoid unnecessarily low probabilities of implementation for valuable projects. Going back to our motivation the perception that these money burning activities are just that - money burning - and somehow have managed to persist due to the possible ineptitude of the executive officers appears to be unjustified even though money burning is destroying surplus, it might be a useful instrument in lowering incentive rents.

We also show that the optimal mechanism can be calculated and has some particularly desirable properties, there is no allocative inefficiency on the top, no manager wants to deflate the value of their project, and money burning activities are closely tied to allocation without inspection.

3.2 Future Work

Our analysis is flawed in many ways. Money burning activities can be of many more types than the ones outlined by us. Even within the space of influence activities, it is possible that some of those activities are not observable. While we have modeled the observable activities, it is likely that these unobservable money burning activities are the cause of many efficiencies and it will take greater analysis to understand the impact of these activities on organizations.

On the technical front we tackle a multi-dimensional mechanism design problem via a graph theoretic approach. The applicability of a similar approach to other mechanism design problems is of particular interest. Our results relied on no assumptions on the distributions of types, thus the optimal mechanism requires a two step approach. One interesting question that remains is if there are additional assumptions on the distribution whether sharper results can be achieved. Our translation to the uniform distribution makes the exposition of our proofs relatively simple and in this direction, readers can easily see that adding additional assumptions on v_i's which would translate would yield even sharper results. In turn such translating such assumptions on the values with uniform and its implications to the underlying distribution that it maps to is a difficult but promising task that we hope to revisit in future works.

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Appendices

Appendix

Lemma 1. *$g(S)$ is strictly submodular, i.e. for any two sets S, S' where $S \not\subset S'$ and $S' \not\subset S$ we have $g(S \cup S') + g(S \cap S') < g(S) + g(S')$.*

Proof. Let $h(S) = 1 - g(S)$, and we abuse the notation similarly with h to write $h(|S|) = h(S)$. We will first show that if $x > y$ then

$$h(x-1) - h(x) < h(y-1) - h(y) \tag{4}$$

From the definition we have $h(i) = \left(1 - \frac{i}{n}\right)^n$ and using binomial theorem we can write

$$h(i-1) - h(i) = \sum_{j=1}^{n} \left(1 - \frac{i}{n}\right)^{n-j} \left(\frac{1}{n}\right)^j$$

And since this is a decreasing function of i, we have (4). Let $|S \cap S'| = y$ and let $|S| = x$ then $x > y$. Also let $|S \cup S'| - |S| = |S| - |S \cap S'| = z$. We need to show,

$$\begin{aligned}
& g(S \cup S') + g(S \cap S') < g(S) + g(S') \\
\text{or} \quad & g(S \cup S') - g(S) < g(S') - g(S \cap S') \\
\text{or} \quad & g(x+z) - g(x) < g(y+z) - g(y) \\
\text{or} \quad & h(x) - h(x+z) < h(y) - h(y+z) \\
\text{or} \quad & \sum_{i=1}^{z} h(x+i-1) - h(x+i) < \sum_{i=1}^{z} h(y+i-1) - h(y+i)
\end{aligned}$$

And since for each i the term in the summation on the left side is smaller than the corresponding term one on the right, we have strict submodularity of g. □

Lemma 2. *Let $\langle \mathcal{A} \rangle \in \mathscr{A}$ and $\mathcal{A}_t \geq \mathcal{A}_s$ for some s, t, then $\exists\ \delta > 0$ such that $\langle \mathcal{A}' \rangle \in \mathscr{A}$, where $\mathcal{A}'_t = \mathcal{A}_t - \delta$, $\mathcal{A}'_s = \mathcal{A}_s + \delta$ and $\mathcal{A}'_r = \mathcal{A}_r\ \forall r \notin \{s, t\}$.*

Proof. Consider an arbitrary set $S \subseteq V \backslash \{s, t\}$. Let $S_t \equiv S \cup \{t\}$, $S_s \equiv S \cup \{s\}$. Since $\mathcal{A}_s \leq \mathcal{A}_t$ we can write

$$\sum_{r \in S_s} \mathcal{A}_r \leq \frac{1}{2} \left[\sum_{r \in S} \mathcal{A}_r + \sum_{r \in S_t \cup S_s} \mathcal{A}_r \right]$$

We need to show,

$$\sum_{r \in S'} \mathcal{A}'_r \leq g(S') \qquad \forall S' \subseteq V$$

For the sets that do not contain s, or contain both s, t the inequality holds since the left hand side has not increased. Since S is arbitrary and δ can be arbitrarily small, we need only to show:

$$\sum_{r \in S_s} \mathcal{A}_r < g(S_s)$$

However, from Lemma 1 we have

$$\begin{aligned}
& g(S_s \cup S_t) + g(S) < g(S_s) + g(S_t) \\
\implies \quad & \sum_{r \in S} \mathcal{A}_r + \sum_{r \in S_t \cup S_s} \mathcal{A}_r < g(S_s) + g(S_t) \\
\implies \quad & 2 \sum_{r \in S_s} \mathcal{A}_r < g(S_s) + g(S_t) \\
\implies \quad & \sum_{r \in S_s} \mathcal{A}_r < g(S_s) \qquad \because g(S_s) = g(S_t)
\end{aligned}$$

Thus concluding the proof. □

Lemma 3. *In an optimal M if $deg^-_M(t) = 0$, then $p_t = 0$; $c_t = 1$ and $\mathcal{A}_t = \min_{r \in V} \mathcal{A}_r$*

Proof. If $p_t > 0$ then define $M' = T_{M,s,t}(\mathcal{A}_s, \mathcal{A}_t, c_s, c_t, p_s, p_t - \delta)$ for small $\delta > 0$ then M' is feasible and is an improvement over M of δ in the objective function. If $c_t < 1$ then define $M' = T_{M,s,t}(\mathcal{A}_s, \mathcal{A}_t, c_s, c_t + \delta, p_s, p_t)$ for small $\delta > 0$ then M' is feasible, incentive compatible and an improvement in the objective function of $k\mathcal{A}_t\delta$. Also, if $\mathcal{A}_t > \mathcal{A}_s$ for some s we have $v_s\mathcal{A}_s - p_s < v_s\mathcal{A}_t$ since $p_s \geq 0$, thus violating the IC constraint. □

Lemma 4. *In an optimal M,* $\min_s \mathcal{A}_s c_s \geq \min_s \mathcal{A}_s$

Proof. Suppose not. Consider r, t be such that $\mathcal{A}_r c_r = \min_s \mathcal{A}_s c_s$ and $\mathcal{A}_t = \min_s \mathcal{A}_s$ then $\mathcal{A}_r c_r < \mathcal{A}_t$, and we must have $c_r < 1$, otherwise $\mathcal{A}_r < \mathcal{A}_t$ which is not possible. Now, we must have $deg^-_M(r) = 0$ otherwise there must be an r' such that $(r', r) \in E(G_M)$ but then $v_{r'}\mathcal{A}_t > v_{r'}\mathcal{A}_r c_r - p_r = v_{r'}\mathcal{A}_{r'} - p_{r'}$ violates the IC constraint. However, if $deg^-_M(r) = 0$ then $c_r = 1$ due to Lemma 3 giving us a contradiction. □

Lemma 5. *In an optimal mechanism M if $(t, \hat{t}) \in E(G_M)$ then $\mathcal{A}_{\hat{t}} c_{\hat{t}} \geq \mathcal{A}_t$*

Proof. Suppose not, and let $\mathcal{A}_{\hat{t}} c_{\hat{t}} < \mathcal{A}_t$, then it must be the case that $p_t > p_{\hat{t}} \geq 0$. Now, consider the mechanism

$$M' = T_{M,t,\hat{t}}(\mathcal{A}_{\hat{t}} c_{\hat{t}}, \mathcal{A}_{\hat{t}}, c_t, c_{\hat{t}}, p_{\hat{t}}, p_{\hat{t}})$$

M' is clearly feasible. We now show that this new mechanism still satisfies all the incentive constraints. All incentive constraints not involving the type t remain unaffected. For the incentive constraints involving type t pretending to be any other type r, we have

$$\begin{aligned} v_t \mathcal{A}'_t - p'_t &= v_t \mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \\ &= v_t \mathcal{A}_t - p_t && \because (t, \hat{t}) \in E(G_M) \\ &\geq v_t \mathcal{A}_r c_r - p_r \quad \forall r \neq t \\ &= v_t \mathcal{A}'_r c'_r - p'_r \end{aligned}$$

And for all the constraints where any other type r can pretend to be type t we have

$$v_r \mathcal{A}'_r - p'_r = v_r \mathcal{A}_r - p_r \geq v_r \mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \geq v_r \mathcal{A}_{\hat{t}} c_{\hat{t}} c_t - p_{\hat{t}} = v_r \mathcal{A}'_t c'_t - p'_t$$

And so all the incentive constraints are satisfied. Now, looking at the increase in the objective function due to this change,

$$Obj(M') - Obj(M) = k(1 - c_t)(\mathcal{A}_t - \mathcal{A}_{\hat{t}} c_{\hat{t}})$$

Which is positive if $c_t < 1$ giving a contradiction. If $c_t = 1$ then we must have $deg^-_M(t) > 0$ otherwise Lemma 3 and Lemma 4 jointly give us a contradiction since $\mathcal{A}_t > \mathcal{A}_{\hat{t}} c_{\hat{t}}$. Since $deg^-_M(t) > 0$ let $r' \in i_M(t)$ and $r' < t$ then

$$\begin{aligned} & v_t \mathcal{A}_t - p_t = v_t \mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \\ \implies \quad & v_{r'}(\mathcal{A}_t - \mathcal{A}_{\hat{t}} c_{\hat{t}}) < p_t - p_{\hat{t}} && \because \mathcal{A}_t > \mathcal{A}_{\hat{t}} c_{\hat{t}};\ t > r' \\ \implies \quad & v_{r'} \mathcal{A}_{r'} - p_{r'} < v_{r'} \mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} && \because c_t = 1;\ (r, t) \in E(G_M) \end{aligned}$$

Again, giving us a contradiction. So we must have $r' \geq t$ for all r' such that $(r', t) \in E(G_M)$.

Again, define $M' = T_{M,t,\hat{t}}(\mathcal{A}_t - \delta, \mathcal{A}_{\hat{t}}, c_t, c_{\hat{t}}, p_t - v_t \delta, p_{\hat{t}})$ then this mechanism is feasible for small enough δ since $p_t > 0$. We see that for any r' such that

$(r', t) \in E(G_M)$ we have

$$\begin{aligned} & v_{r'}\mathcal{A}_{r'} - p_{r'} = v_{r'}\mathcal{A}_t - p_t \\ \implies \quad & v_{r'}\mathcal{A}_{r'} - p_{r'} \geq v_{r'}\mathcal{A}_t - v_{r'}\delta - p_t + v_t\delta \\ \implies \quad & v_{r'}\mathcal{A}_{r'} - p_{r'} \geq v_{r'}(\mathcal{A}_t - \delta) - (p_t - v_t\delta) \\ \implies \quad & v_{r'}\mathcal{A}_{r'} - p_{r'} \geq v_{r'}\mathcal{A}'_t - p'_t \end{aligned}$$

And hence M' is incentive compatible. Moreover, we see that $deg^-_{M'}(t) = 0$ and hence we must have $\mathcal{A}'_t \leq \mathcal{A}_{\hat{t}}c_{\hat{t}}$ due to lemma 3 and lemma 4, since M' is an optimal mechanism as well, but for δ small enough we still must have $\mathcal{A}'_t \leq \mathcal{A}_{\hat{t}}c_{\hat{t}} < \mathcal{A}'_t = \mathcal{A}_t - \delta$ which is a contradiction. □

Lemma 6. *In an optimal M, if W' is a directed cycle then all the projects in W' must have same probability of selection, and the same amount of money burning with no chance of assessment. Equivalently, $W' \subseteq W$ for some web W.*

Proof. Let $W' = \{t_1, t_2, \ldots, t_n\}$ be the directed cycle then from Lemma 5 we have $\mathcal{A}_{t_1} \leq \mathcal{A}_{t_2}c_{t_2} \leq \mathcal{A}_{t_2} \leq \mathcal{A}_{t_3}c_{t_3} \leq \cdots \leq \mathcal{A}_{t_1}c_{t_1}$ thus giving us an equality throughout. However, from IC constraints we have $v_{t_1}\mathcal{A}_{t_1} - p_{t_1} = v_{t_1}\mathcal{A}_{t_2}c_{t_2} - p_{t_2} \implies p_{t_1} - p_{t_2} = 0$, hence all the agents in the cycle must have the same persuasion requirements. Since $\mathcal{A}_{t_i} = \mathcal{A}_{t_j}$; $p_{t_i} = p_{t_j}$; $c_{t_i} = c_{t_j} = 1\ \forall t_i, t_j \in W'$ we have $t_i \in i_M(t_j)\ \forall t_i, t_j \in W'$ and W' must be part of a web. □

Lemma 7. *Every optimal G_M is weakly connected and contains exactly one web W^* with $i_M(W^*) = \emptyset$.*

Proof. Let G_M be optimal, we will first show that there must exist a web W in G_M with $i_M(W) = \emptyset$. If $\exists t \in V$ such that $i_M(t) = \emptyset$ then $\{t\}$ is the desired web. So, $deg^-_M(t) > 0\ \forall t \in V$, and G_M must contain directed cycles. Due to Lemma 6 these cycles must belong in a web and thus G_M must contain webs. Let $\mathscr{W}$ be the set of webs in G_M.

Let $W^* = \arg\min_{W \in \mathscr{W}}\{\mathcal{A}_t : t \in W\}$, so that for any $t^* \in W^*$ and any $t \in W \in \mathscr{W}$, $\mathcal{A}_{t^*} < \mathcal{A}_t$. They cannot be equal ($\mathcal{A}_{t^*} \neq \mathcal{A}_t$) since that would mean that both the webs have the same money burning as well (since there are no assessments in webs, IC constraints would require same money burning) thus contradicting the fact that they are webs.

It must be the case that $i_M(W^*) = \emptyset$, for otherwise let $(t_1, t^*) \in E(G_M)$; $t^* \in W^*$; $t_1 \in V \backslash W^*$ then $\mathcal{A}_{t^*} \geq \mathcal{A}_{t_1}$ due to Lemma 5. Now, since $deg^-_M(t_1) > 0$

we can find a t_2 such that $\mathcal{A}_{t_1} \geq \mathcal{A}_{t_2}$ and so on until we end up in another cycle with $\mathcal{A}_{t_k} \leq \mathcal{A}_{t^*}$ which is a contradiction to $W^* = \arg\min_{W \in \mathscr{W}}\{\mathcal{A}_t : t \in W\}$.

We need to show now that W^* is the only web with $i_M(W^*) = \emptyset$. Let W be another such web. $deg_M^-(W) = 0 \implies p_t = 0\ \forall t \in W$, for if not then we can decrease money burning for all the types in the web by small $\delta > 0$ and improve the objective function without violating any IC constraints. Similarly, $deg_M^-(W^*)$ mean that there is no money burning for projects in W^*. However, let $t \in W, t^* \in W^*$, since $p_t = p_{t^*} = 0$ and $\mathcal{A}_t > \mathcal{A}_{t^*}$, $v_{t^*}\mathcal{A}_{t^*} < v_{t^*}\mathcal{A}_t$ thus violating the IC constraint and giving a contradiction.

Now suppose the graph is not connected, then one of the components must contain W^*. Consider any other component, repeating the argument above, it follows that this component must contain a web W with $deg_M^-(W^*) = 0$. However, this gives us a contradiction since there can only be one such web. □

Lemma 8. *If M is optimal, $(s,t) \in E(G_M)$ and $\mathcal{A}_s = \mathcal{A}_t c_t$ then $c_s = 1$.*

Proof. If s has indegree zero then we have the result using Lemma 3 and so let us assume there is an r such that (r,s) bind, and let $c_s < 1$. Since $\mathcal{A}_s = \mathcal{A}_t c_t$ and $v_s\mathcal{A}_s - p_s = v_s\mathcal{A}_t c_t - p_t$ we have $p_s = p_t$. Now, since (r,s) bind,

$$\begin{aligned} v_r\mathcal{A}_r - p_r &= v_r\mathcal{A}_s c_s - p_s \\ &= v_r\mathcal{A}_t c_t c_s - p_s \\ &< v_r\mathcal{A}_t c_t - p_t \end{aligned}$$

Contrary to the IC constraint, and hence we must have $c_s = 1$. □

Proposition 1. *In an optimal mechanism, if $(s,t) \in E(G_M)$ then either $s < t$ or there exists a web W such that $\{s,t\} \subseteq W$.*

Proof. Suppose not and let $s > t$. Note that Lemma 5 gives us $\mathcal{A}_t \geq \mathcal{A}_s$. We consider four cases,

Case I: $c_t < 1$; $o_M(t) \neq \emptyset$

Let $r \in o_M(t)$. We have,

$$
\begin{aligned}
& v_t\mathcal{A}_t - p_t = v_t\mathcal{A}_r c_r - p_r \\
\implies \quad & p_r - p_t \le v_s\left(\mathcal{A}_r c_r - \mathcal{A}_t\right) \quad \because v_s > v_t;\ \mathcal{A}_r c_r \ge \mathcal{A}_t \text{ due to Lemma 5} \\
\implies \quad & p_r - p_t < v_s\left(\mathcal{A}_r c_r - \mathcal{A}_t c_t\right) \quad \because c_t < 1 \\
\implies \quad & v_s\mathcal{A}_s - p_s < v_s\mathcal{A}_r c_r - p_r \quad \because (s,t) \in E(G_M)
\end{aligned}
$$

Violating the IC constraint.

Case II: $c_t < 1;\ o_M(t) = \emptyset$
Due to Lemma 5 we have $\mathcal{A}_t \ge \mathcal{A}_s$. Now, consider the mechanism

$$
M' = T_{M,s,t}\left(\mathcal{A}_s + \delta, \mathcal{A}_t - \delta, \frac{\mathcal{A}_s c_s}{\mathcal{A}_s + \delta}, \frac{\mathcal{A}_t c_t}{\mathcal{A}_t - \delta}, p_s, p_t\right)
$$

then M' is feasible due to Lemma 2 and the fact that $c_t < 1$. All incentive constraints not involving s, t remain unchanged. $\mathcal{A}'_t c'_t = \mathcal{A}_t c_t$ and $\mathcal{A}'_s c'_s = \mathcal{A}_s c_s$ and so no other type has an incentive to pretend to be s or t. Surplus of s has increased so it has no incentive to deviate in M' and since t has outdegree zero, for small enough δ, t will not deviate. M', however, is an improvement of $\delta(v_s - v_t)$ in the objective function.

Case III: $c_t = 1;\ \mathcal{A}_t = \mathcal{A}_s$
From Lemma 8 we have that $c_s = 1$. Since $v_s\mathcal{A}_s - p_s = v_s\mathcal{A}_t - p_t$ we get $p_s = p_t$ and so $s \in i_M(t);\ t \in i_M(s)$ and they must belong in a web.

Case IV: $c_t = 1;\ \mathcal{A}_t > \mathcal{A}_s$
We have,

$$
\begin{aligned}
& p_t - p_s = v_s(\mathcal{A}_t - \mathcal{A}_s) \\
\implies \quad & p_t - p_s > v_t(\mathcal{A}_t - \mathcal{A}_s) \quad \because v_t < v_s \text{ and } \mathcal{A}_s < \mathcal{A}_t \\
\implies \quad & v_t\mathcal{A}_s - p_s > v_t\mathcal{A}_t - p_t
\end{aligned}
$$

Now, consider the mechanism $M' = T_{M,s,t}\left(\mathcal{A}_t, \mathcal{A}_s, c_t, c_s, p_t, p_s\right)$ then it is clearly feasible and we can see that all the IC constraints not involving t, s remain unchanged. s, t have no incentive to deviate in M' since their surplus is not decreasing. No other type has any incentive to pretend to be s or t either since they would otherwise have done that in M. M' is however an improvement in the objective function of $(v_t\mathcal{A}_s - p_s) - (v_t\mathcal{A}_t - p_t)$ which is strictly positive as seen above. □

Lemma 9. *In an optimal mechanism, the amount of money burning is strictly monotonic in the probability of project implementation without an assessment, i.e., in an optimal* M*,* $\mathcal{A}_t c_t > \mathcal{A}_s c_s \iff p_t > p_s$ *and* $\mathcal{A}_s c_s = \mathcal{A}_t c_t \iff p_s = p_t$*.*

Proof. If $deg_M^-(s) \cdot deg_M^-(t) = 0$ then the result follows from Lemma 3 and Lemma 8. Let s', t' be such that $(s', s) \in E(G_M)$ and $(t', t) \in E(G_M)$ then we have

$$v_{s'}\mathcal{A}_s c_s - p_s \geq v_{s'}\mathcal{A}_t c_t - p_t$$
$$v_{t'}\mathcal{A}_t c_t - p_t \geq v_{t'}\mathcal{A}_s c_s - p_s$$

Combining the two we get

$$v_{t'}(\mathcal{A}_t c_t - \mathcal{A}_s c_s) \geq p_t - p_s \geq v_{s'}(\mathcal{A}_t c_t - \mathcal{A}_s c_s)$$

Thus concluding the proof. □

Proposition 2. *In an optimal mechanism high value projects are implemented with a higher probability, i.e.,* $v_t > v_{\hat{t}} \implies \mathcal{A}_t \geq \mathcal{A}_{\hat{t}}$*.*

Proof. Let $\hat{t}, t$ be such that $t > \hat{t}$ and $\mathcal{A}_t < \mathcal{A}_{\hat{t}}$ in an optimal mechanism M. If $deg_M^-(\hat{t}) = 0$ then the proposition is trivially true due to Lemma 3, so $i_M(\hat{t}) \neq \emptyset$. The proof consists of two parts. First, we show that for this mechanism M we must have:

$$0 < c_t p_{\hat{t}} < p_{\hat{t}} \tag{5}$$

Once we have proved (5) we will construct a feasible and incentive compatible mechanism that is a strict improvement over M; the mechanism that violates the proposition statement.

Suppose (5) does not hold. If $c_{\hat{t}} = c_t = 1$ then from the IC constraints of $t, \hat{t}$ we have

$$v_{\hat{t}}(\mathcal{A}_{\hat{t}} - \mathcal{A}_t) \geq p_{\hat{t}} - p_t \geq v_t(\mathcal{A}_{\hat{t}} - \mathcal{A}_t)$$

which is not possible since $\mathcal{A}_{\hat{t}} > \mathcal{A}_t$ and $v_t > v_{\hat{t}}$, so:

$$c_{\hat{t}} c_t < 1 \tag{6}$$

If $deg_M^+(\hat{t}) = 0$ and $c_{\hat{t}} < 1$ then consider the mechanism M'' with

$$M' = T_{M,\hat{t},t}\left(\mathcal{A}_{\hat{t}} - \delta, \mathcal{A}_t + \delta, \frac{\mathcal{A}_{\hat{t}} c_{\hat{t}}}{\mathcal{A}_{\hat{t}} - \delta}, \frac{\mathcal{A}_t c_t}{\mathcal{A}_t + \delta}, p_{\hat{t}}, p_t\right)$$

This is a feasible allocation as dictated by Lemma 2, for a sufficiently small δ. All IC constraints can be verified to be satisfied for small δ and the objective function improvement is $\delta(v_t - v_{\hat{t}})$. And so if $deg^+_M(\hat{t}) = 0 \implies c_{\hat{t}} = 1$. However, this implies that $p_{\hat{t}} > 0$, since otherwise $\hat{t}$ must have lowest allocation probability and $\mathcal{A}_{\hat{t}} \leq \mathcal{A}_t$ contradicting the supposition. It also implies $c_t < 1$ by (6). Thus, we have $\neg(5) \implies deg^+_M(\hat{t}) > 0$.

Now, since $deg^+_M(\hat{t}) > 0$ (i.e. $o_M(\hat{t}) \neq \emptyset$), let $\hat{t}' \in o_M(\hat{t})$. We have:

$$\begin{aligned}
& v_t\mathcal{A}_t - p_t \geq v_t\mathcal{A}_{\hat{t}'}c_{\hat{t}'} - p_{\hat{t}'} \\
\implies \quad & p_{\hat{t}'} - p_t \geq v_t(\mathcal{A}_{\hat{t}'}c_{\hat{t}'} - \mathcal{A}_t) \\
\implies \quad & p_{\hat{t}'} - p_t > v_{\hat{t}}(\mathcal{A}_{\hat{t}'}c_{\hat{t}'} - \mathcal{A}_t) \qquad \because \text{Lemma 5} : \mathcal{A}_{\hat{t}'}c_{\hat{t}'} \geq \mathcal{A}_{\hat{t}} > \mathcal{A}_t \\
\implies \quad & v_{\hat{t}}\mathcal{A}_t - p_t > v_{\hat{t}}\mathcal{A}_{\hat{t}'}c_{\hat{t}'} - p_{\hat{t}'}
\end{aligned}$$

$$\implies \quad v_{\hat{t}}\mathcal{A}_t - p_t > v_{\hat{t}}\mathcal{A}_{\hat{t}} - p_{\hat{t}} \tag{7}$$

Consider the mechanism $M' = T_{M,\hat{t},t}(\mathcal{A}_t, \mathcal{A}_t, c_t, c_t, p_t, p_t)$ then there is no incentive for any type to deviate as shown by (7). The change in objective function however is

$$\begin{aligned}
& v_{\hat{t}}\mathcal{A}_t - p_t + k\mathcal{A}_tc_t - k\mathcal{A}_t - [v_{\hat{t}}\mathcal{A}_{\hat{t}} - p_{\hat{t}} + k\mathcal{A}_{\hat{t}}c_{\hat{t}} - k\mathcal{A}_{\hat{t}}] \\
&= (v_{\hat{t}}\mathcal{A}_t - p_t - v_{\hat{t}}\mathcal{A}_{\hat{t}} + p_{\hat{t}}) + k(\mathcal{A}_{\hat{t}} - \mathcal{A}_t) + k(\mathcal{A}_tc_t - \mathcal{A}_{\hat{t}}c_{\hat{t}})
\end{aligned}$$

Which would be strictly positive if $c_t = 1$, or $\mathcal{A}_tc_t \geq \mathcal{A}_{\hat{t}}c_{\hat{t}}$. Thus $c_t < 1$ and $\mathcal{A}_{\hat{t}}c_{\hat{t}} > \mathcal{A}_tc_t$ implying $p_{\hat{t}} > p_t \geq 0$ due to Lemma 9. This means (5) holds.

Now that we have established that (5) must hold, consider the following mechanism $M^* = T_{M,\hat{t},t}(\mathcal{A}^*_{\hat{t}}, \mathcal{A}^*_t, c^*_{\hat{t}}, c^*_t, p^*_{\hat{t}}, p^*_t)$ where,

$$\begin{aligned}
\mathcal{A}^*_{\hat{t}} = \mathcal{A}_{\hat{t}} - \varepsilon;\ c^*_{\hat{t}} = c_{\hat{t}} - \varepsilon\left(\frac{v_t(\mathcal{A}_{\hat{t}}c_{\hat{t}} - \mathcal{A}_tc_t) - c_{\hat{t}}(p_{\hat{t}} - p_t)}{(p_{\hat{t}} - p_t)(\mathcal{A}_{\hat{t}} - \varepsilon)}\right);\ p^*_{\hat{t}} = p_{\hat{t}} - v_t\varepsilon \\
\mathcal{A}^*_t = \mathcal{A}_t + \varepsilon;\ c^*_t = c_t + \varepsilon\left(\frac{v_t(\mathcal{A}_{\hat{t}}c_{\hat{t}} - \mathcal{A}_tc_t) - c_t(p_{\hat{t}} - p_t)}{(p_{\hat{t}} - p_t)(\mathcal{A}_t + \varepsilon)}\right);\ p^*_t = p_t + v_t\varepsilon
\end{aligned}$$

This part of the proof proceeds in 3 steps. First, we will show that M^* is a feasible mechanism and the values lie within their bounds. Then we will show that M^* is an incentive compatible mechanism; finally showing that M^* is an improvement over M.

Step 1 - M^* is feasible: Lemma 2 shows that $\mathcal{A}^*_{\hat{t}}$ and $\mathcal{A}^*_t$ are feasible for a small enough ε. We also know from (5) that $p_{\hat{t}} > 0$ and so $p_{\hat{t}}, p_t$ are feasible as well. That c^*_t is feasible can also be seen from (5), for small enough ε, since $c_t < 1$. It remains to be shown that $c^*_{\hat{t}}$ is feasible. To this end it will be shown that $0 < c^*_{\hat{t}} \leq c_{\hat{t}}$. The first part of the inequality will hold if ε is sufficiently small. For the second part, it must be that

$$\left(\frac{v_t(\mathcal{A}_{\hat{t}} c_{\hat{t}} - \mathcal{A}_t c_t) - c_{\hat{t}}(p_{\hat{t}} - p_t)}{(p_{\hat{t}} - p_t)(\mathcal{A}_{\hat{t}} - \varepsilon)} \right) \geq 0 \tag{8}$$

However consider $\underline{\hat{t}} = \inf i_{M^*}(\hat{t})$ then $t > \hat{t} > \underline{\hat{t}}$ due to Proposition 1. We have

$$\begin{aligned} & & v_{\underline{\hat{t}}} \mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} &\geq v_{\underline{\hat{t}}} \mathcal{A}_t c_t - p_t \\ &\Longrightarrow & v_{\underline{\hat{t}}}(\mathcal{A}_{\hat{t}} c_{\hat{t}} - \mathcal{A}_t c_t) &\geq p_{\hat{t}} - p_t \\ &\Longrightarrow & v_t(\mathcal{A}_{\hat{t}} c_{\hat{t}} - \mathcal{A}_t c_t) &\geq p_{\hat{t}} - p_t & \because \mathcal{A}_{\hat{t}} c_{\hat{t}} > \mathcal{A}_t c_t \\ &\Longrightarrow & v_t(\mathcal{A}_{\hat{t}} c_{\hat{t}} - \mathcal{A}_t c_t) - c_{\hat{t}}(p_{\hat{t}} - p_t) &\geq 0 \end{aligned}$$

Thus proving (8) and concluding Step 1.

Step 2 - M^* is incentive compatible: We must show that M^* satisfies all the IC constraints. The only changes are in the constraints relating to type $\hat{t}$ and t. We have $v_{\hat{t}} \mathcal{A}^*_{\hat{t}} - p^*_{\hat{t}} \geq v_{\hat{t}} \mathcal{A}_{\hat{t}} - p_{\hat{t}}$ and $v_t \mathcal{A}^*_t - p^*_t \geq v_t \mathcal{A}_t - p_t$ and so $\hat{t}, t$ have no incentive to deviate and pretend to be another type. Let us now define

$$\delta = \frac{v_t \varepsilon}{p_{\hat{t}} - p_t}$$

Then we can see that

$$\begin{aligned} p^*_{\hat{t}} &= (1 - \delta) p_{\hat{t}} + \delta p_t \\ p^*_t &= (1 - \delta) p_t + \delta p_{\hat{t}} \\ \mathcal{A}^*_{\hat{t}} c^*_{\hat{t}} &= (1 - \delta) \mathcal{A}_{\hat{t}} c_{\hat{t}} - \delta \mathcal{A}_t c_t \\ \mathcal{A}^*_t c^*_t &= (1 - \delta) \mathcal{A}_t c_t - \delta \mathcal{A}_{\hat{t}} c_{\hat{t}} \end{aligned}$$

Now, consider any arbitrary type r:

$$\begin{aligned} v_r \mathcal{A}^*_t c^*_t - p^*_t &= v_r \left((1 - \delta) \mathcal{A}_t c_t + \delta \mathcal{A}_{\hat{t}} c_{\hat{t}} \right) - \left((1 - \delta) p_t + \delta p_{\hat{t}} \right) \\ &= (1 - \delta) \left[v_r \mathcal{A}_t c_t - p_t \right] + \delta \left[v_r \mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \right] \\ &\leq v_r \mathcal{A}_r - p_r \end{aligned}$$

And so r has no incentive to deviate to type t. Similarly, we can show that no type has any incentive to deviate to type $\hat{t}$ either.

Step 3 - M^* is an improvement:

$$
\begin{aligned}
Obj(M^*) - Obj(M) &= v_{\hat{t}}\mathcal{A}^*_{\hat{t}} - p^*_{\hat{t}} + k\mathcal{A}^*_{\hat{t}}c^*_{\hat{t}} - k\mathcal{A}^*_{\hat{t}} - (v_{\hat{t}}\mathcal{A}_{\hat{t}} - p_{\hat{t}} + k\mathcal{A}_{\hat{t}}c_{\hat{t}} - k\mathcal{A}_{\hat{t}}) \\
&\quad + v_t\mathcal{A}^*_t - p^*_t + k\mathcal{A}^*_t c^*_t - k\mathcal{A}^*_t - (v_t\mathcal{A}_t - p_t + k\mathcal{A}_t c_t - k\mathcal{A}_t) \\
&= (v_t - v_{\hat{t}})\varepsilon - k\delta(\mathcal{A}_{\hat{t}}c_{\hat{t}} - \mathcal{A}_t c_t) + k\varepsilon \\
&\quad + k\delta(\mathcal{A}_{\hat{t}}c_{\hat{t}} - \mathcal{A}_t c_t) - k\varepsilon \\
&= (v_t - v_{\hat{t}})\varepsilon \\
&> 0
\end{aligned}
$$

□

Lemma 10. *Every optimal mechanism M is of the form where $\exists\ \underline{t} \in V$ such that for all projects with $v_t \leq v_{\underline{t}}$ there is no assessment and no money burning. Furthermore, for all the types above this threshold, i.e. $v_t > v_{\underline{t}}$, there is either a positive probability of assessment, or some money burning effort, or both. Also, all the projects below this threshold have the lowest probability of implementation, i.e., $\mathcal{A}_t = \min_{r\in V}\mathcal{A}_r\ \forall t \leq \underline{t}$.*

Proof. Let M^* be an optimal mechanism and consider the web W^* with $i_{M^*}(W^*) = \emptyset$. Let $t^* = \sup W^*$. Since $i_{M^*}(W^*) = \emptyset$ we must have $p_t = 0\ \forall t \in W^*$, otherwise the mechanism M' defined by decreasing money burning for all the projects in W^* by small ε is an improvement over M^* and is feasible and incentive compatible. Since $p_t = 0;\ c_t = 1\ \forall t \in W^*$ we must have $\mathcal{A}_t = \min_{r\in V}\mathcal{A}_r\ \forall t \in V$ for if not and let $\mathcal{A}_{\hat{t}} > \mathcal{A}_t$ then $v_{\hat{t}}\mathcal{A}_{\hat{t}}c_{\hat{t}} - p_{\hat{t}} < v_{\hat{t}}\mathcal{A}_t$, thus violating the IC constraint. Due to Proposition 2 we must have $t \in W^*\ \forall t < t^*$, since for all such $t, \mathcal{A}_t = \mathcal{A}_{t^*}$ thus implying $c_t = 1;\ p_t = 0$ and $\{(t, t^*), (t^*, t)\} \subseteq E(G_M)$. Moreover, for any $t > t^*$ we must have $\mathcal{A}_t > \mathcal{A}_{t^*}$ due to the definition of t^*. Thus the IC constraints dictate that every $t > t^*$ must either undergo assessment with a positive probability, or have positive money burning, or both. □

Lemma 11. *The gains realized by the division managers are non-decreasing in the value of the project, i.e. $v_t\mathcal{A}_t - p_t \geq v_{\hat{t}}\mathcal{A}_{\hat{t}} - p_{\hat{t}}$ if $t \geq \hat{t}$.*

Proof. Suppose not and let $\hat{t} < t$ with $v_{\hat{t}}\mathcal{A}_{\hat{t}} - p_{\hat{t}} > v_t\mathcal{A}_t - p_t$ then we must have $c_{\hat{t}} < 1$ otherwise $v_t\mathcal{A}_t - p_t < v_t\mathcal{A}_{\hat{t}} - p_{\hat{t}}$ violating the IC constraint. We must also have $o_M(\hat{t}) = \emptyset$ otherwise let $r \in o_M(\hat{t})$ then,

$$v_t\mathcal{A}_t - p_t < v_{\hat{t}}\mathcal{A}_{\hat{t}} - p_{\hat{t}} = v_{\hat{t}}\mathcal{A}_r c_r - p_r < v_t\mathcal{A}_r c_r - p_r$$

Violating the IC constraint. Now, consider the alternative mechanism

$$M' = T_{M,\hat{t},t}\left(\mathcal{A}_{\hat{t}} - \delta, \mathcal{A}_t + \delta, \frac{\mathcal{A}_{\hat{t}}c_{\hat{t}}}{\mathcal{A}_{\hat{t}} - \delta}, \frac{\mathcal{A}_t c_t}{\mathcal{A}_t + \delta}, p_{\hat{t}}, p_t\right)$$

then this is feasible due to Lemma 2 and the fact that $c_{\hat{t}} < 1$. That M' is also incentive compatible can be checked. The improvement in the objective function however is $\delta(v_t - v_{\hat{t}})$. □

Lemma 12. *In every optimal mechanism M where there are two types that require costly verification but no money burning, these types do not want to mimic each other. That is, in an optimal mechanism M, if $t, \hat{t}$ have $c_t < 1$, $c_{\hat{t}} < 1$ and $p_t = p_{\hat{t}} = 0$, then $t \notin i_M(\hat{t})$ and $\hat{t} \notin i_M(t)$.*

Proof. Without loss of generality assume $\hat{t} > t$, then by proposition 1 we must have $\hat{t} \notin i_M(t)$. For a contradiction suppose that $t \in i_M(\hat{t})$. Since $p_{\hat{t}} = 0$, by lemma 9 it must be the case that for any $l \in W^*$, $\mathcal{A}_l c_l = \mathcal{A}_{\hat{t}} c_{\hat{t}}$. Thus it must also be the case that $t \in i_M(l)$. Since $t \notin W^*$, we must have $t > l$, but by proposition 1 if M is optimal it must be the case that $t \notin i_M(l)$, a contradiction. □

Lemma 13. *Let M be an optimal mechanism and let W^* be the web in the optimal mechanism. Furthermore let $\hat{t}$ be such that $p_{\hat{t}} > 0$ and let $t \in W^*$. If $t \in i_M(\hat{t})$ then $\nexists t' \in W^*$ with $t' > t$.*

Proof. Let $\hat{t}$ be such that $p_{\hat{t}}$ and let $t \in W^*$. Let $t \in i_M(\hat{t})$ and for a contradiction suppose that $\exists t' \in W^*$ with $t' > t$. Since $t \in i_M(\hat{t})$ and $p_{\hat{t}} > 0$ and $t' \in W^*$. It must be the case that

$$\begin{aligned}
\mathcal{A}_t v_t &= \mathcal{A}_{\hat{t}} c_{\hat{t}} v_t - p_{\hat{t}} \\
\mathcal{A}_{t'} v_t &= \mathcal{A}_{\hat{t}} c_{\hat{t}} v_t - p_{\hat{t}} \\
p_{\hat{t}} &= \left(\mathcal{A}_{\hat{t}} c_{\hat{t}} - \mathcal{A}_{t'}\right) v_t
\end{aligned}$$

But since $t' > t$. It must be the case that

$$\begin{aligned}
p_{\hat{t}} &< \left(\mathcal{A}_{\hat{t}} c_{\hat{t}} - \mathcal{A}_{t'}\right) v_{t'} \\
\mathcal{A}_{t'} v_{t'} &< \mathcal{A}_{\hat{t}} c_{\hat{t}} v_{t'} - p_{\hat{t}}
\end{aligned}$$

Contradicting the incentive constraint of type t'. □

Lemma 14. *If $p_t > 0$ and $p_{\hat{t}} = 0$ then $(t, \hat{t}) \notin E(G_M)$.*

Proof. Suppose not, observe that by the IC condition of type t we must have $-p_t = (\mathcal{A}_{\hat{t}} c_{\hat{t}} - A_t)v_t$. But by lemma 5 the right hand side is weakly positive, whereas the left hand side is strictly negative, a contradiction. ☐

Proposition 3. *In any optimal mechanism, there exists a cutoff type $\underline{t}$, the probability of implementation for all types above $\underline{t}$ is given by*

$$\mathcal{A}_n = g(1), \mathcal{A}_{n-1} = g(2) - g(1), \dots, \mathcal{A}_{\underline{t}+1} = g(n - \underline{t}) - g(n - \underline{t} - 1)$$
$$\mathcal{A}_{\underline{t}} = \mathcal{A}_{\underline{t}-1} \dots = \mathcal{A}_1 = \frac{g(n) - g(\underline{t})}{\underline{t}}$$

Proof. An equivalent way to state the proposition is the following: In any optimal mechanism if $\underline{t}$ is the highest type in W^* and $k > \underline{t}$ then $\mathcal{A}_k = g(N - k + 1) - \sum_{j=k+1}^{N} \mathcal{A}_j$. In order to prove the result we formulate the following problem, if the required money burning and probability of allocation without inspection were known, what would be the optimal allocation probabilities that satisfy the Border inequalities. Namely we take $\mathcal{A}c, p$ as given and ask what the optimal $\mathcal{A}$'s are. Notice changing $\mathcal{A}$ while taking $\mathcal{A}c$ as given is equivalent to changing c. For any type t, consider the terms corresponding to the incentives denoted by $d_{t,t'} = \mathcal{A}_{t'} c_{t'} v_t - p_{t'} + p_t$ and let $d_t = \max\{\max t'\{d_{t,t'}\}, \mathcal{A}_t c_t\}$. Now we define the following polytope $\mathcal{P}(\mathcal{A}c, p) = \{\sum_{i \in S} \mathcal{A}_i \leq g(S), S \subseteq V; \mathcal{A}_i \geq d_i, \quad i \in S\}$. Our goal is to solve the following maximization problem

$$\max_{\langle \mathcal{A} \rangle \in P(\mathcal{A}c, p)} \sum_{t=1}^{n} v_t \mathcal{A}_t$$

This is a linear optimization over a polymatroid with side constraints d_i. Since we know the utilities of each type is weakly increasing we can restrict attention to d_i's that are weakly increasing and since no type can have higher allocation probability then $g(1)$, we assume all d_i's are less than $g(1)$. Clearly as shown by [28]if $(d_1, \dots, d_n) \notin \mathcal{P}(\mathcal{A}c, p)$ then the problem is not feasible. As noted earlier the function g only depends on the cardinality of the set, hence it identifies a so called "generalized symmetric" polymatroid (see [13]). By proposition 2.2 of [14] we can consider the contraction of the polymatroid with respect to d, which in turn also defines a polymatroid with the set function defined as $\rho(S) = \min_{S' \supseteq S}(g(S') - \sum_{i \in S'} d_i)$. Thus the maximization problem can be rewritten as a another polymatroid optimization problem which has no constraints, can solved by a greedy algorithm and has the following form.

$$\max_{\langle \mathcal{A} \rangle} \sum_{t=1}^{n} v_t(\mathcal{A}_t + d_t)$$
$$\text{s.t.} \sum_{s \in S} \mathcal{A}_s \leq \rho(S) \quad \forall S \subset V$$

Now since g is submodular and $d_i's$ are increasing, the ρ function can only be of two forms: either there exists some t such that $\rho(S)$ for all $S \subseteq \{n, n-1, n-2, \dots t\}$ or $\rho(\{n, n-1, \dots, k\}) = g(n-k) - \sum_{i=k}^{n} d_i$ for all $k \geq t$. In the former case, we have $d_{n-i} - d_{n-i+1}$ is larger than the decrease in $g(i+1) - g(i)$ so that the minimizing superset is shared across all top types. In the latter case this decrease matches $g(i+1) - g(i)$ exactly for all $i > t$. Now for a contradiction suppose there was a solution to the original problem with $(\mathcal{A}c, p)$ such that the resulting ρ was of the former type. Then the greedy algorithm would assign all types between $n-1$ to t at their constraint levels d, and they would all be slack. But in that case simply increasing p_{n-1} by $\varepsilon v_{\hat{t}} > 0$, where $\hat{t} = \max\{t' : t' \in i_M(n-1)\}$ and increasing $\mathcal{A}_{n-1}$ by ε would be a feasible improvement. Therefore the constraints $d_(n-i) - d_{n-i+1}$'s must be increasing at the exact rate $g(i+1) - g(i)$ until t, which in turn implies that $\mathcal{A}_t = g(n-t) - g(n-t+1)$ for all types above the web. □

Lemma 15. *Let M be an optimal mechanism and let W^* be the web in the optimal mechanism and let $\underline{t}$ be the highest type in the web. If $v_{\underline{t}} > k$ then there is no money burning in the mechanism.*

Proof. Suppose not, let t be such that $p_t > 0$. By lemma 13 any type in $i_M(t)$ must be weakly larger than $\underline{t}$. Let t' be the smallest type in $i_M(t)$, and consider the alternative mechanism where p_t is reduced by ε and c_t is reduced by $\frac{\varepsilon}{\mathcal{A}_t v_{t'}}$. The incentive constraint for type t' continues to hold with equality, whereas the incentive constraint for all other types in $i_M(t)$ are now holding strictly. The increase in the objective is given by $1/n\varepsilon\mathcal{A}_t(t'-k) > 0$, leading to the desired contradiction. □

Lemma 16. *Let M be an optimal mechanism and let W^* be the web in the optimal mechanism and let $\underline{t}$ be the highest type in the web. If $v_{\underline{t}} < k$ then $p_{\underline{t}+1} > 0$ and $c_{\underline{t}+1} = 1$.*

Proof. Let $\mathcal{A}$ be given by the optimal mechanism, then p and c must solve the following simplified problem.

$$\max_{\langle c\rangle,\langle p\rangle} E_t\left[v_t\mathcal{A}_t - p_t + k\mathcal{A}_t c_t - k\mathcal{A}_t\right]$$
$$s.t.$$
$$0 \le c_t p_t \le p_t \;\forall t \in V;\;\; c_t, p_t \in \mathbb{R}^+$$
$$v_t\mathcal{A}_t - p_t \ge v_t\mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} \quad \forall t, \hat{t} > t \in V$$

Notice we dropped the constraints for incentives to pretend to be lower types since we know they are not binding and the IR constraints since we know the utilities are increasing and the types in the web have positive utility. Additionally we must have $\mathcal{A}_t \ge \mathcal{A}_{t-k}$ for any k and we know c_t, p_t must be 0 for all the types in the web so we can drop them from the problem as well except the constraints involving $\underline{t}$ pretending to be some higher type. Since $\mathcal{A}$'s are given we can equivalently write the above problem as follows:

$$\max_{\langle c\rangle,\langle p\rangle} \sum_{t=\underline{t}+1}^{n} \left[-p_t + k\mathcal{A}_t c_t\right]$$
$$s.t.$$
$$c_t \le 1 \;\forall t \in V;\;\; c_t, p_t \in \mathbb{R}^+$$
$$v_t\mathcal{A}_t \ge v_t\mathcal{A}_{\hat{t}} c_{\hat{t}} - p_{\hat{t}} + p_t \quad \forall t, \hat{t} \in \{\underline{t}, n\}, \hat{t} > t$$

This is a standard linear programming problem, which can be solved via Simplex method. Now let us consider the dual of this problem, letting $y_{i,j}$ denote the coefficient of constraint regarding type i pretending to be type j. Since our inequalities are all of the form $\ge$, we must have $y_{i,j} \ge 0$ for all i, j. Similarly let $r_j \ge 0$ denote the coefficient for the constraint $c_j \le 1$. The dual of the problem takes the form

$$\min_{\langle y_{i,j}\rangle,\langle r\rangle} \sum_{i\in\{\underline{t},n\}} \sum_{j>i} y_{i,j} v_i(\mathcal{A}_i) + \sum_{j\in \underline{t}+1,n} r_j$$
$$\sum_{i=\underline{t}}^{j-1} v_i \mathcal{A}_j y_{i,j} + r_j \ge k\mathcal{A}_j, \quad \forall j \in \{\underline{t}+1, n\}$$
$$-\sum_{i=\underline{t}}^{j-1} y_{i,j} + \sum_{l=j+1}^{n} y_{j,l} \ge -1 \quad \forall j \in \{\underline{t}+1, n\}$$

Now, towards a contradiction, we are going to assume $p_{\underline{t}+1} = 0$, which means, the constraint $-y_{\underline{t},\underline{t}+1} + \sum_{l=\underline{t}+2}^{n} y_{\underline{t}+1,l} \ge -1$ is slack. But then we must have $v_{\underline{t}}\mathcal{A}_{\underline{t}+1} y_{\underline{t},\underline{t}+1} = k\mathcal{A}_{\underline{t}+1}$ and $r_{\underline{t}+1} = 0$ as otherwise we would have $c_{\underline{t}+1} = 1$, which contradicts $\underline{t}$ being the highest type in the web. But since this holds with equality we also must have, $\sum_{l=\underline{t}+2}^{N} y_{\underline{t}+1,l} > \frac{k}{v_{\underline{t}}} - 1 > 0$. Which in turn implies there is at least one l with $y_{\underline{t}+1,l} > 0$. Since $-y_{\underline{t},\underline{t}+1} + \sum_{l=\underline{t}+2}^{n} y_{\underline{t}+1,l} > -1$ we can reduce $y_{\underline{t}+1,l}$ by $\varepsilon > 0$ that is small enough and increase $y_{\underline{t},l}$ by $\delta = \frac{\varepsilon v_{\underline{t}+1}}{v_{\underline{t}}}$. Then the objective decreases by $\varepsilon v_{\underline{t}+1}(\mathcal{A}_{\underline{t}+1} - \frac{\mathcal{A}_{\underline{t}}}{v_{\underline{t}}}) > 0$. Hence we must have $-y_{\underline{t},\underline{t}+1} + \sum_{l=\underline{t}+2}^{n} y_{\underline{t}+1,l} = -1$, which by complementary slackness implies $p_{\underline{t}+1} > 0$.

To see that $c_{\underline{t}+1} = 1$, for a contradiction assume this is not true, $0 < c_{\underline{t}+1} < 1$ which implies $r_{\underline{t}+1} = 0$ and $v_{\underline{t}}\mathcal{A}_{\underline{t}+1}y_{\underline{t},\underline{t}+1} = k\mathcal{A}_{\underline{t}+1}$. Now consider decreasing by $\varepsilon > 0$ and setting $r_{\underline{t}+1} = \varepsilon v\underline{t}\mathcal{A}_{\underline{t}+1}$. Then the sum $v_{\underline{t}}\mathcal{A}_{\underline{t}+1}y_{\underline{t},\underline{t}+1} + r_{\underline{t}+1}$ remains unchanged. But the constraint regarding $p_{\underline{t}+1}$, $-y_{\underline{t},\underline{t}+1} + \sum_{l=\underline{t}+2}^{n} y_{\underline{t}+1,l} = -1$ is now relaxed. Thus similar to above we can reduce $y_{\underline{t}+1,l}$ by ε and increase $y_{\underline{t},l}$ by $\delta = \frac{\varepsilon v_{\underline{t}+1}}{v_{\underline{t}}}$. Then the objective decreases by $\varepsilon v_{\underline{t}+1}(\mathcal{A}_{\underline{t}+1} - \frac{\mathcal{A}_{\underline{t}}}{v_{\underline{t}}}) - \varepsilon v_{\underline{t}}(\mathcal{A}_{\underline{t}+1} - \frac{\mathcal{A}_{\underline{t}}}{v_{\underline{t}}}) > 0$, hence we must have $c_{\underline{t}+1} = 1$.

□