**Question 1**

1. $d_1 \sim d_2$ implies the rule
   
   A is open in $(X, d_1) \iff$ is open in $(X, d_2)$

   Now $C$ is closed in $(X, d_1) \iff$
   
   $C^c$ is open in $(X, d_1) \iff$ (bec. $d_1 \sim d_2$)
   
   $C^c$ is open in $(X, d_2) \iff$ (topological definition of closed set)
   
   $C$ is closed in $(X, d_2)$

   as we wanted to show

2. $(\Rightarrow)$ Suppose $d_1 \geq d_2$. We want to show that given $x \in X$ and $r > 0$, we can find $\delta_x, r > 0$ such that $B_1(x, \delta_x, r) \subseteq B_2(x, r)$

   Now, since $A = B_2(x, r)$ is open in $(X, d_2)$ (since open balls are open sets)

   $\Rightarrow$ $d_1 \geq d_2$ implies $A$ is also open in $(X, d_1)$
Then, there exist \( r_x^A > 0 \) such that
\[ B(x, r_x^A) \subseteq A = B_2(x, \epsilon) \]

So, picking \( \delta = \frac{\epsilon}{r_x} \),
\[ \delta_{x, \epsilon} = \frac{\epsilon}{r_x} \]
shows the desired result.

\[ \Rightarrow \]
Suppose the rule is true, and consider

\[ A \text{ an open set in } (x, d_2). \]
We need to show

\[ A \text{ is also open in } (x, d_1) \iff \forall x \in A \text{ we can find } \epsilon > 0 \text{ such that } B_1(x, \epsilon) \subseteq A \]

Now, since \( A \) is open in \( (x, d_2) \),

\[ \exists \, \hat{r}_x > 0 \text{ such that } B_2(x, \hat{r}_x) \subseteq A \]

Using the rule, if we take
\[ r_x = \delta_{x, \hat{r}_x} \quad (\text{use } \epsilon = \hat{r}_x \text{ in the rule}) \]
then the rule tells us that

\[ B_1(x, r_x) = B_1(x, \delta_{x, \hat{r}_x}) \subseteq B_2(x, \hat{r}_x) \subseteq A \]

\[ \Downarrow \text{ DEF. of } \hat{r}_x \]
\[ \Downarrow \text{ RULE HOLDS} \]
\[ \Downarrow \text{ A open in } (x, d_2) \]
So $A$ is open in $(X,d_1)$, like we wanted to show.

3 To prove it, we will show that the condition:

$$\exists \alpha > 0: d_1(x,y) \geq \alpha d_2(x,y)$$

$$\forall x, y \in X$$

guarantees the rule of part 2.

For that, take $x \in X$ and $\epsilon > 0$. We need to find $\sqrt[\alpha]{\epsilon} > 0$ such that

$$B_1(x, \sqrt[\alpha]{\epsilon}) \subseteq B_2(x, \epsilon)$$

if $y: d_1(x, y) < \sqrt[\alpha]{\epsilon} \Rightarrow$
\[ d_2(x, y) < \varepsilon \]

But see that if we define

\[ \forall \varepsilon, x \equiv \alpha \varepsilon \]

then if \( d_1(x, y) < \alpha \varepsilon \) \( \Rightarrow \)

\[ d_2(x, y) \leq \frac{1}{\alpha} d_1(x, y) \]

\[ < \frac{1}{\alpha} \forall \varepsilon = \frac{1}{\alpha} (\alpha \varepsilon) = \varepsilon \]

So \( y \in B_2(x, \varepsilon) \), as we wanted to show.
To show \( d_1 \geq d_2 \), use with \( \alpha = \gamma \). For \( d_2 \geq d_1 \), use \( \alpha = \frac{1}{\beta} \).

We will show that if \( d_2 \geq d_1 \),

then \( x_n \to x \) in \((x, d_2) \implies x_n \to x \) in \((x, d_1) \).

For \( \epsilon > 0 \) we need to find \( N_{\epsilon}^1 \): if \( n \geq N_{\epsilon}^1 \) then

\[ d_1(x_n, x) < \epsilon \]

\[ \uparrow \]

\[ x_n \in B_\epsilon(x, \epsilon) \) for \( n \geq N_{\epsilon}^1 \)

Now, since we assumed \( x_n \to x \) in \((x, d_2) \), we know that for any \( \epsilon > 0 \) \( \exists N_{\epsilon}^2 \):

\[ x_n \in B_\epsilon(x, \epsilon) \) for \( n \geq N_{\epsilon}^2 \)

Then, using 2 with \( r = \epsilon \) and \( \hat{\epsilon} = \delta_{x, \epsilon} \), we know that if \( n \geq N_{\delta_{x, \epsilon}}^2 \).
\[ x_n \in B_2(x, \delta_{x, \epsilon}) \subseteq B_1(x, \epsilon) \]

Therefore, taking \( N_\epsilon^1 \equiv N_{\delta_{x, \epsilon}}^2 \) we show

\[ x_n \to x \text{ in } (X, d_1) \]

\textbf{To finish the proof} we show the analogous result for \( d_1 \geq d_2 \), and putting both together

\[ \text{if } d_1 \sim d_2 \implies x_n \to x \text{ in } (X, d_1) \iff x_n \to x \text{ in } (X, d_2) \]

\[ \textbf{6) Just an application of } (5): \]

\[ f \text{ is continuous at } x \text{ in } (X, d_1) \iff \text{sequential definition of continuity} \]

\[ \text{if } x_n \to x \text{ in } (X, d_1) \implies f(x_n) \to f(x) \iff d_1 \sim d_2 \]

\[ \text{if } x_n \to x \text{ in } (X, d_2) \implies f(x_n) \to f(x) \iff \text{again def. of continuity} \]

\[ f(.) \text{ continuous at } x \text{ in } (X, d_2) \]
\[ N_1 \leq n \cdot N_{\infty} \]

\[ N_1(x) = \sum \nolimits_i |x_i| \leq \sum \nolimits_i [m_{\max} \cdot |x_j|] = n \cdot m_{\max} \cdot |x_j| = n \cdot N_{\infty}(x) \]

\[ N_2 \leq N_1 \]

\[ N_1(x) = \sum \nolimits_i |x_i| = \sqrt{\left( \sum \nolimits_i |x_i| \right)^2} = \]

\[ \sqrt{\sum \nolimits_i |x_i|^2 + \sum \nolimits_{i \neq \hat{j}} |x_i||x_j|} \leq \sqrt{\sum \nolimits_i |x_i|^2} = N_2(x) \]

\[ N_{\infty} \leq N_2 \]

\[ N_{\infty}(x) = \max \nolimits_i |x_i| = x_{i^*} \quad \text{where} \quad i^* = \arg \max_i \]

\[ = \sqrt{x_{i^*}^2} \leq \sqrt{x_{i^*}^2 + \sum \nolimits_{i \neq i^*} x_i^2} = N_2(x) \]
Application of \( \textcircled{4} \)

\[
d_1 \sim d_2
\]

\[d_1(x,y) \leq n d_\infty(x,y) \leq n d_2(x,y)\]

\[\Rightarrow\]

\[d_1(x,y) \geq d_2(\alpha,\gamma)\]

So use \( \textcircled{5} \) with \( \gamma = 1 \)

\[\beta = \eta\]

\[
d_2 \sim d_\infty
\]

\[d_2(x,y) \leq d_1(x,y) \leq n d_\infty(x,y)\]

\[\Rightarrow\]

\[d_\infty(x,y) \leq d_2(x,y)\]

Use again \( \textcircled{5} \) with \( \gamma = 1 \)

\[\beta = \eta\]
Already know $d_1 \leq n \leq d_\infty$

and

$d_1 \geq d_2 \geq d_\infty$ so again, use $\gamma \equiv \beta \equiv n$.

**Question 2**

**Part 1** if $f \in C(A, IR)$ and $A$ is compact

$\Rightarrow$ Weierstrass Theorem ensures that

$\exists x, \bar{x} \in X$:

$f \equiv f(x) \leq f(\bar{x}) \leq \bar{f}(x) \equiv \bar{f}$

and hence $f \in \mathcal{B}(A, IR)$.

Therefore

$C(A, IR) = \{ f \text{ real, and bounded} \}$
Take a sequence of concave functions $f_n \in D$ such that $f_n \to f$. We want to show that $f \in D$ as well.

**Proof by contradiction:** Suppose it is not concave. For this to be the case, there must exist $\hat{x} \neq \hat{y} \in A$ and $\lambda \in (0,1)$ such that

$$f[\lambda \hat{x} + (1-\lambda) \hat{y}] > \lambda f(\hat{x}) + (1-\lambda) f(\hat{y})$$

Given $\hat{x}, \hat{y}$, define the sequence of real numbers:

$$a_n = f_n[\lambda \hat{x} + (1-\lambda) \hat{y}] - \lambda f_n(\hat{x}) - (1-\lambda) f_n(\hat{y})$$

Since $f_n$ is concave $\implies a_n \geq 0$ for all $n$. Moreover

$$\lim_{n \to \infty} a_n = f[\lambda \hat{x} + (1-\lambda) \hat{y}] - \lambda f(\hat{x}) - (1-\lambda) f(\hat{y}) = a < 0$$

Which is impossible, since the set $C = [0, +\infty)$ is closed, and $a_n \in C \forall n$ and $a_n \to a \notin C$, a contradiction.
**PART 3**  

**H is not**.

Take $A = \left[ \frac{1}{4}, \frac{1}{2} \right]$ and the sequence

$$f_n(x) = -x^{2n}$$

Notice that:

$$\int f_n(x) = -2n(2n+1)x^{2n-1} < 0$$

$\forall x \in \left[ \frac{1}{4}, \frac{1}{2} \right]$

**Moreover**

$$\lim_{n \to \infty} (f_n,0) = \max_{x \in \left[ \frac{1}{4}, \frac{1}{2} \right]} \left| -x^{2n} - 0 \right| = \max_{x \in \left[ \frac{1}{4}, \frac{1}{2} \right]} x^{2n} = \left( \frac{1}{2} \right)^{2n} \to 0$$

So $f_n \to 0$, and $O(x) = 0$ $\forall x \in \left[ \frac{1}{4}, \frac{1}{2} \right]$ is concave (since it is linear) but **NOT STRICTLY CONCAVE!!!**

**Therefore, the set of strictly concave functions is NOT CLOSED**
**Question 3**

Proof is a direct application of the Kuhn-Tucker theorem for convex programs.

Notice that

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} f_0(x) & \equiv -\max_{x \in \mathbb{R}^n} (-f_0)(x) \\
\leq & \sum_{i=1}^k f_i(x) \leq 0 \quad \forall i \in [k] \\
\text{subject to} & \quad a_j^T x = b_j \quad \forall j \in [m] \\
\text{and} & \quad -a_j^T x + b_j = 0 \quad \forall j
\end{align*}
\]

Now

- \( f_0(x) \) is convex \( \iff \) \(-f_0(x)\) is concave
- \( f_i(x) \) is convex \( \iff \) \(-f_i(x)\) is concave

Therefore, the RHS max problem satisfies the Kuhn-Tucker theorem.

\[ \chi^* \in \arg \max_{x \in \mathbb{F}} -f_0(x) \iff \exists (\chi^*, \lambda^*) \text{ such that} \]
1. \nabla \mathcal{L} = 0

Here \( \mathcal{L}(x, \lambda, \mu) = -f_0(x) + \sum \lambda_i [-f_i(x)] + \sum \mu_j h_j(x) \)

\[ \nabla \mathcal{L} = 0 \iff 0 = -\nabla f_0(x^*) - \sum_i \lambda_i^* f_i(x^*) - \sum_j \mu_j^* h_j(x) \]

\[ \iff \nabla f_0(x^*) + \sum \lambda_i^* f_i(x^*) + \sum \mu_j^* a_j = 0 \]

2. \( \lambda_i^* [-f_i(x^*)] = 0 \iff \lambda_i^* f_i(x^*) = 0 \)

3. \( -f_i(x) \geq 0 \iff f(x^*) \leq 0 \)

4. \( \lambda_i \geq 0 \)

Hence, we finish our proof.
Given \( \varphi(\lambda) = \sup_{x \in \mathbb{R}^n} (c^T - \sum \lambda_i a_i^T) x + \sum \lambda_i b_i \)

\[ = \sup_{x \in \mathbb{R}^n} (c^T - a_i^T) x + b_i^T \lambda \]

First, we show that unless \( c = A^T \lambda \), then \( \varphi(\lambda) = +\infty \) (i.e., there is no supremum).

Suppose \( c \neq A^T \lambda \) \( \Rightarrow \exists i : c_i - (A^T \lambda)_i \neq 0 \)

Suppose first that \( c_j > (A^T \lambda)_j \)

Then, for any \( n \in \mathbb{N} \), the vector \( x_n = (0, 0, \ldots, 0, 0, \ldots, 0) \)

is such that

\( \varphi(x) \geq (c - A^T \lambda)^T x_n + b^T \lambda = (c_j - (A^T \lambda)_j)_j \cdot n + b^T \lambda \)

but the \( \varphi(x) \geq \lim_{n \to \infty} (c_j - (A^T \lambda)_j)_j \cdot n + b^T \lambda = +\infty \)
If \( c_j < (A^T \lambda)_j \), we do a similar construction, with
\[
X_n = (0, 0, \ldots, \underbrace{-n}_{j\text{-th place}}, 10 \ldots 0)
\]
AND GET SOME RESULT.

Suppose now that \( C = A^T \lambda \). In that case
\[
\mathcal{S}(\lambda) = \sup_{X \in \mathbb{R}^n} \; 0 \cdot X + b^T \lambda = b^T \lambda
\]
I.e. the obj. function is constant in \( X \)

\( \square \) We can apply Kuhn-Tucker theorem

To note that if
\[
\lambda^* \in \arg\min \; \mathcal{S}(\lambda) \\
\lambda \geq 0 \\
A^T \lambda = C
\]

\( \Rightarrow \exists \; X^* \in \arg\min_{X} \; C^T X \\
X: A \cdot X \leq b \)
AND MOREOVER \((x^*, \lambda^*)\) SATISFY THE KUHN-TUCKER CONDITIONS

1. \(\nabla_x \mathcal{L}(x; \lambda^*) = 0 \iff c - A^T \lambda^* = 0\)  
   (which we already know is satisfied by \(\lambda^*\))

2. \(\lambda_i^* (a_i^T x^* - b_i) = 0 \ \forall i\) 

   Now, for all \(i \in I^*\) we know that \(\lambda_i^* > 0\).
   **Therefore**  
   \(a_i^T x^* = b_i \ \forall i \in I^*\)  
   \(\iff A^* x^* = b^*\)

   where \(A^*\) and \(b^*\) pile up the equations that we know \(x^*\) satisfies
   
   if \(A^*\) is invertible \(\Rightarrow\) 
   
   \(x^* = (A^*)^{-1} \cdot b^*\)  
   
   is the solution