Question 1: Topologically equivalent metrics

Let \((X, d_1)\) and \((X, d_2)\) be metric spaces defined on the same set \(X\). We say that \(d_1\) is finer than \(d_2\) (and we write \(d_1 \succeq d_2\)) if the following rule holds:

If \(A \subseteq X\) is an open set in \((X, d_2)\) \(\implies\) \(A\) is also open in \((X, d_1)\)

We say \(d_1\) and \(d_2\) are topologically equivalent (and we write \(d_1 \sim d_2\)) if \(d_1 \succeq d_2\) and \(d_2 \succeq d_1\); i.e. the open sets in \((X, d_2)\) are the same as the open sets in \((X, d_1)\).

1. Prove that if \(d_1 \sim d_2\) then \(C \subseteq X\) is a closed set in \((X, d_1)\) \(\iff\) \(C\) is closed in \((X, d_2)\).

2. (*) Let \(B_i(x, r) = \{y \in X : d_i(x, y) < r\}\) be the open ball with center \(x\) and radius \(r\), on metric space \((X, d_i)\) with \(i \in \{1, 2\}\). Prove the following rule:

\[ d_1 \succeq d_2 \iff \text{for all } x \in X \text{ and all } r > 0, \exists \delta_{x,r} > 0 \text{ such that } B_1(x, \delta_{x,r}) \subseteq B_2(x, r) \]

3. Show that if a number \(\alpha > 0\) exists such that for all \(x, y \in X\):

\[ d_1(x, y) \geq \alpha d_2(x, y) \]

Then \(d_1 \succeq d_2\) (Tip: Use part 2, even if you did not show it)

4. Using (3), show that if you can find constants \(\beta, \gamma > 0\) such that for all \(x, y \in X\):

\[ \gamma d_2(x, y) \leq d_1(x, y) \leq \beta d_2(x, y) \]

Then \(d_1 \sim d_2\)

5. Show that if \(d_1 \sim d_2\) then the following rule holds: for all sequences \(\{x_n\}_{n \in \mathbb{N}}\) and \(x \in X\):

\[ x_n \to x \text{ in } (X, d_1) \iff x_n \to x \text{ in } (X, d_2) \]

6. Show that if \(d_1 \sim d_2\), \((Y, d_Y)\) is a metric space and \(f : X \to Y\) is a function, then

\[ f \text{ is continuous for all } x \in X \text{ in } (X, d_1) \iff f \text{ is continuous for all } x \in X \text{ in } (X, d_2) \]
7. Let $X = \mathbb{R}^n$ and define the following norms:

$$N_\infty (x) = \max_{i \in \{1, \ldots, n\}} |x_i|$$

$$N_1 (x) = \sum_{i=1}^n |x_i|$$

$$N_2 (x) = \sqrt{\sum_{i=1}^n x_i^2}$$

show that

$$N_\infty (x) \leq N_2 (x) \leq N_1 (x) \leq nN_\infty (x)$$

8. Using (7), show that the distances

$$d_\infty (x, y) \equiv N_\infty (x - y)$$

$$d_2 (x, y) \equiv N_2 (x - y)$$

$$d_1 (x, y) \equiv N_1 (x - y)$$

are all topological equivalent (i.e. $d_1 \sim d_2$, $d_2 \sim d_3$ and $d_3 \sim d_1$)

**Remark:** This question shows that when studying $\mathbb{R}^n$ with the typical topology, you can use either of this distances to prove all your results about topology, convergence or continuity.

**QUESTION 2**

Given a convex set $A \subseteq \mathbb{R}^n$ let $B (A, \mathbb{R}) \equiv \{ f : A \to \mathbb{R} \text{ where } f \text{ is bounded} \}$ and

$$d_\infty (f, g) = \sup_{x \in A} |f(x) - g(x)|$$

1. Let $C (A, \mathbb{R}) = \{ f : A \to \mathbb{R} \text{ with } f \text{ continuous for all } x \in A \}$. Show that if $A$ is a compact subset of $\mathbb{R}^n$, then $C (A, \mathbb{R})$ is a closed subset of $B (A, \mathbb{R})$

2. Let $D = \{ f \in B (A, \mathbb{R}) : f \text{ is concave} \}$. Show $D$ is a closed subset of $B (A, \mathbb{R})$

3. Let $D' \subset D$ be the set of strictly concave functions. Is it also closed?

**QUESTION 3 - KUHN TUCKER IN MINIMIZATION PROBLEMS**

Consider the following minimization problem:

$$\min_{x \in \mathbb{R}^n} f_0 (x)$$  \hspace{1cm} (1)
Show that if \( f_0 (\cdot) \) and \( f_i (\cdot) \) are convex functions for all \( i = 1, \ldots, k \) and \( h_j (x) = a_j^T x - b_j \) for all \( j = 1, 2, \ldots, m \) for some \( a_j \in \mathbb{R}^n \) and \( b_j \in \mathbb{R} \), then the following statement is true:

If \( \exists \hat{x} \in \mathbb{R}^n : f_i (\hat{x}) < 0 \) for all \( i \) and \( h_j (\hat{x}) = 0 \) for all \( j \), then \( x^* \) is a solution to the minimization problem \( 1 \iff \exists \lambda^* \in \mathbb{R}^k, \mu^* \in \mathbb{R}^n \) such that:

1. \( \nabla f_0 (x^*) + \sum_i \lambda_i^* \nabla f_i (x^*) + \sum_j \mu_j^* a_j = 0 \)
2. \( \lambda_i^* f_i (x^*) = 0 \) for all \( i = 1, 2, \ldots, k \)
3. \( f_i (x^*) \geq 0 \) for all \( i \) and \( a_j^T x^* = b_j \) for all \( j \)
4. \( \lambda_i^* \geq 0 \)

**QUESTION 4 - LINEAR PROGRAMMING**

Consider the linear programming problem

\[
\max c^T x \\
\text{s.t. } a_i^T x \leq b
\]

where the matrix \( A = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \end{pmatrix} \) is of full rank, with \( k \leq n \) and \( b \in \mathbb{R}^k \).

1. Show that the Lagrange dual function \( g (\lambda) = \sup_{x \in \mathbb{R}^n} c^T x + \sum_i \lambda_i (b_i - a_i^T x) \) is

\[
g (\lambda) = \begin{cases} 
  b^T \lambda & \text{if } A^T \lambda = c \\
  \infty & \text{otherwise}
\end{cases}
\]

2. Suppose the dual problem

\[
\min_{\lambda \in \mathbb{R}^n} b^T \lambda \\
\text{s.t. } \begin{cases} 
  A^T \lambda = c \\
  \lambda_i \geq 0
\end{cases}
\]

has a solution \( \lambda^* \in \mathbb{R}^n \), such that \( \lambda_i^* > 0 \) for \( k \) constraints; i.e. the set

\[
I^* = \{ i \in \{1, 2, \ldots, k\} : \lambda_i^* > 0 \}
\]

has \( k \) elements (\( \#(I^*) = k \)). Show that if \( \exists \hat{x} : A \hat{x} < 0 \) then the solution of the program 2 is

\[
x^* = (A^*)^{-1} b
\]
where $A^*$ is the matrix that piles up the $k$ binding constraints: i.e.

$$A^* = \begin{pmatrix}
a^T_{i_1}
a^T_{i_2}
\vdots
a^T_{i_h}
\end{pmatrix} \text{ where } i_z \in I^*$$

3. Show that if $n = k$ and $\det(A) \neq 0$, then the dual problem 3 has a solution if and only if $(A^{-1})^T c \geq 0$