A Macroeconomic Model with a Financial Sector

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Motivation

- Financial instability
  - Persistence of shocks
  - Amplification
  - Non-linear liquidity spirals - adverse feedback loops
    - Go beyond log-linearization
  - Endogenous risk
  - “Volatility paradox”

- Asset pricing implications
  - Fat tails
  - Endogenous correlation structure
Amplification & Instability - Overview

  - Perfect (technological) liquidity, but persistence
  - Bad shocks erode net worth, cut back on investments, leading to low productivity & low net worth of in the next period
Amplification & Instability - Overview

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  - Technological/market illiquidity
  - KM: Leverage bounded by margins; BGG: Verification cost (CSV)
  - Stronger amplification effects through prices (low net worth reduces leveraged institutions’ demand for assets, lowering prices and further depressing net worth)
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- Brunnermeier & Sannikov (2010)
  - only equity constraint
  - Instability and volatility dynamics, volatility paradox
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- Brunnermeier & Pedersen (2009), Geanakoplos
  - Volatility interaction with margins/haircuts (leverage) – debt constraint
Preview of results

- Full equilibrium dynamics + volatility dynamics
  - Near “steady state”
    - (large) payouts balance profit making
    - intermediaries must be unconstrained and amplification is low
  - Below “steady state”
    - intermediaries constrained, try to preserve capital
      leading to high amplification and volatility precaution

- Crises episodes have significant endogenous risk, correlated asset prices, larger spreads and risk premia

- “Volatility paradox”

- SDF is driven by constraint & $c \geq 0$

- Securitization and hedging of idiosyncratic risks can lead to higher leverage, and greater systemic risk
### Model setup

- **Productive**
- **Intermediary**
  - Monitoring
    - Diamond (1984)
    - Holmström-Tirole (1997)
- **Less productive**

![Diagram showing capital sources and risk allocation](image)

- Debt inside equity
- Debt outside equity
- Equity inside
- Equity outside

- \( k_t q_t \)
- \( \alpha^E \)
- \( \alpha^I \)

- Incentive for entrepreneur to exert effort
- Incentive for intermediaries to monitor (have to hold outside equity)
Model details

- **Output** \( y_t = a k_t \) (spend for consumption - investment)
- **Capital** \( dk_t = (\Phi(t_t) - \delta) k_t dt + \sigma k_t dZ_t \)

**Agents**

- **More productive**
  - \( U = E_0 \left[ \int_0^\infty e^{-\rho t} c_t dt \right] \)
  - Production frontier

- **Less productive**
  - \( U = E_0 \left[ \int_0^\infty e^{-rt} c_t dt \right] \)
  - Production frontier
    - \( \bar{\delta} > \delta \)
    - \( t_t = 0 \)

- **Endogenous price process for capital**
  \( dq_t = \mu_t q_t dt + \sigma_t q_t dZ_t \)

**More productive**

\( q_t \geq \underline{q} = \frac{a}{r+\delta} \) if HH limited to buy-hold strategy.
Market value of capital/assets $k_t q_t$

- **Capital**
  - $dk_t = g(\iota)k_t dt + \sigma k_t dZ_t$ “cash flow news” (dividends $a_t$)

- **Price**
  - $dq_t = \mu_t^q q_t dt + \sigma_t^q q_t dZ_t$ “SDF news”

- $k_t q_t$ value dynamics
Market value of capital/assets $k_t q_t$

- **Capital**
  \[ dk_t = g(\lambda)k_t dt + \sigma k_t dZ_t \]  
  exogenous risk

- **Price**
  \[ dq_t = \mu_t q_t dt + \sigma_t q_t dZ_t \]  
  endogenous risk

- **$k_t q_t$ value dynamics**
  \[ d(k_t q_t) = (\Phi(t) - \delta + \mu_t^q + \sigma \sigma_t^q)(k_t q_t)dt + (\sigma + \sigma_t^q)(k_t q_t)dZ_t \]  
  exogenous endogenous risk

- Ito’s Lemma product rule: \[ d(X_t Y_t) = dX_t Y_t + X_t dY_t + \sigma^X \sigma^Y dt \]
Interlinked balance sheets

- **Productive**
- **Intermediary**
  - Monitoring
    - Diamond (1984)
    - Holmström-Tirole (1997)
- **Less productive**

- Capital
- Debt
- Equity

\[ k_t q_t \]

- Incentive for entrepreneur to exert effort
- \( \alpha^E \)
- \( \alpha^I \) of total risk

- Incentive for intermediaries to monitor (have to hold outside equity)
Merging productive HH & Intermediaries

- **Productive**
- **Intermediary**
  - Monitoring
    - Diamond (1984)
    - Holmström-Tirole (1997)
- **Less productive**

\[ \alpha := \alpha^E + \alpha^I \geq b(m) + c(m) \]

“merged experts”
Balance sheet dynamics

- Productive
- Intermediary
- Less productive

\[
\text{assets } k_t q_t \quad \text{debt } d_t \quad \text{equity} = \text{net worth } n_t
\]

assume \( \alpha = 1 \) (for today)
\[ dr_t^k = \left( \frac{a - \nu_t}{\alpha_t} + \Phi(\nu_t) - \delta + \mu_t^q + \sigma \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t \]

\[ dn_t = r n_t dt + (dr_t^k - r dt)(k_t q_t) - dc_t = \cdots \]
Intuition – main forces at work

- **Investment**
  - *Scale up*
    - Scalable profitable investment opportunity
    - Higher leverage (borrow at $r$)
  - *Scale back*
    - **Precaution:** don’t exploit full (GE) debt capacity – “dry powder”
      - Ultimately, stay away from fire-sales prices
      - Debt can’t be rolled over if $d > k_tq$ (note, price is depressed)
    - Solvency constraint

- **Consumption**
  - Consume *early* and borrow $r < \rho$
  - Consume *late* to overcome investment frictions

*aggregate leverage!*
An equilibrium consists of functions that for each history of macro shocks $\{Z_s, s \in [0, t]\}$ specify

- $q_t$ the price of capital
- $k^i_t, k^h_t$ capital holdings and
- $dc^i_t, dc^h_t$ consumption of representative expert and households
- $\iota_t$ rate of internal investment of a representative expert, per unit of capital
- $r_t$ the risk-free rate

such that

- intermediaries and households maximize their utility, given prices $q_t$ as given and
- markets for capital and consumption goods clear
1. **Households:** risk free rate of $r_t = \text{households discount rate}$
   - Makes HH indifferent between consuming and saving, s.t. consumption market clears
   - Required return when their capital >0
     \[
     \frac{a}{q_t} - \delta + \mu^q_t + \sigma^q_t = r
     \]
     *expected return from capital*

2. **Experts** choose \{k_t, i_t, c_t\} dynamically to maximize utility
   \[
   \max_{c,t,k} E \left[ \int_0^\infty e^{-\rho t} dc_t \right] \quad \text{s.t.}
   \]
   \[
   dn_t = -dc_t + (\Phi(i_t) - \delta + \mu^q_t + \sigma^q_t)(k_tq_t)dt
   \]
   \[
   + (\sigma + \sigma^q_t)(k_tq_t)dZ_t + [(a - i_t)k_t - rd_t]dt
   \]
   \[
   dn_t \geq 0
   \]

3. Markets clear: total demand for capital is $K_t$
Solving for equilibrium

1. Internal investment (static)

2. External investment $k_t$
   - Given price dynamics $dq_t/q_t = \mu^q_t dt + \sigma^q_t dZ_t$
   - Solvency constraint $n_t \geq 0$

3. When to consume? $dc_t$
   - Bellman equation w/ value function $\theta_t n_t$

Payoff experts generate from a dollar of net worth by trading undervalued capital. Proportional to net worth, atomistic experts have no price impact.
Solving dynamic optimization

- Let value of extra $d\theta_t = \mu_t^\theta \theta_t dt + \sigma_t^\theta \theta_t dZ_t$
  - recall $dn_t = ....$

- Use Ito’s lemma to expand the Bellman equation
  $$\rho \theta_t n_t dt = \max_{k_t, dc_t} E[dc_t + d(\theta_t n_t)]$$

- Risk free:
  $$\overset{\text{risk-free}}{r} + \overset{\text{E[change of investment opportunities]}}{\mu_t^\theta} = \overset{\text{required return}}{\rho}$$

- Capital:
  $$\overset{\text{E[excess return of capital]}}{\frac{a}{q_t} + g_t + \mu_t^q + \sigma \sigma_t^q} - r = \overset{\text{capital risk premium}}{-\sigma_t^\theta (\sigma + \sigma_t^q)}$$

- $\theta_t \geq 1$, and $dc_t^i > 0$ only when $\theta_t = 1$.

- $e^{-\rho t \theta_t / \theta_0}$ is the experts’ stochastic discount factor.
Scale invariance

- Model is scale invariant
  - $K_t$ total physical capital
  - $N_t$ total net worth of all experts
- Solve $q_t$ and $\theta_t$ as a function of the single state variable $\eta_t$
  - $\eta_t = \frac{N_t}{K_t}$

Mechanic application of Ito’s lemma
Pricing equations get transformed into ordinary differential equations for $q(\eta)$ and $\theta(\eta)$
Equilibrium

- Boundary conditions: \( q(0) = q, \, \theta(0) = \infty, \, \theta(\eta^*) = 1, \, q'(\eta^*) = \theta'(\eta^*) = 0 \)
Equilibrium dynamics
Endogenous risk & “Instability”
Endogenous Risk through Amplification

- Amplification through prices
- Volatility due to endogenous risk

\[ \sigma_t^q = \frac{q'(\eta_t)\sigma(q_t - \eta_t)}{1 - q'(\eta_t)} \]

Key to amplification is \( q'(\eta) \)
  - Depends how constrained experts are

amplification
Dynamics near and away from SS

- Intermediaries choose payouts endogenously
  - Exogenous exit rate in BGG/KM
  - Payouts occur when intermediaries are least constrained
    \[ q'(\eta^*) = 0 \]

- Steady state: experts unconstrained
  - Bad shock leads to lower payout rather than lower capital demand
    \[ q'(\eta^*) = 0, \sigma^q_t(\eta^*) = 0 \]

- Below steady state: experts constrained
  - Negative shock leads to lower demand
  - \( q'(\eta^*) \) is high, strong amplification, \( \sigma^q_t(\eta^*) \) is high
  - ... but when \( \eta \) is close to 0,
    \[ q \approx q(\eta_t), q'(\eta) \text{ and } \sigma^q_t(\eta^*) \text{ is low} \]

Note difference to BGG/KM
"Volatility Paradox" ... $\sigma (.025, .05, .1)$

- As $\sigma$ decreases, $\eta^*$ goes down, $q(\eta^*)$ goes up, $\sigma^\eta(\eta^*)$ may go up, $\max \sigma^\eta$ goes up
**Ext: asset pricing (cross section)**

- **Capital:** Correlation increases with $\sigma^q$
  - Extend model to many types $i$ of capital

  \[ \frac{dk_t^i}{k_t^i} = (\Phi(i_t^i) - \delta)dt + \sigma dZ_t + \sigma' dz_t^i \]

  - Aggregate shock
  - Uncorrelated shock

- Experts hold diversified portfolios
  - Equilibrium looks as before, (all types of capital have same price) but
  - Volatility of $q_t k_t$ is $\sigma + \sigma' + \sigma^q$
  - Endogenous risk is perfectly correlated, exogenous risk not
  - For uncorrelated $z^i$ and $z^j$
    - Correlation \((q_t^i k_t^i, q_t^j k_t^j)\) is \((\sigma + \sigma^q)/(\sigma + \sigma' + \sigma^q)\)
    - Which is increasing in $\sigma^q$
Ext1: asset pricing (cross section)

- **Outside equity:**
  - Negative sknewness
  - Excess volatility
  - Pricing kernel: $e^{-rt}$
    - Needs risk aversion!

- **Derivatives:**
  - Volatility smirk (Bates 2000)
  - More pronounced for index options (Driessen et al. 2009)
Ext2: Idiosyncratic jump losses

\[ dk_t^i = g k_t^i \, dt + \sigma k_t^i \, dZ_t + k_t^i \, dJ_t^i \]

- \( J_t^i \) is an idiosyncratic compensated Poisson loss process, recovery distribution \( F \) and intensity \( \lambda(\sigma_t^q) \)
- \( q_t k_t^i \) drops below debt \( d_t \), costly state verification

- Time-varying interest rate spread
- Allows for direct comparison with BGG
### Ext. 2: Idiosyncratic losses

\[ d k_t^i = g k_t^i dt + \sigma k_t^i dZ_t + k_t^i dJ_t^i \]

- \( J_t^i \) is an idiosyncratic compensated Poisson loss process, recovery distribution \( F \) and intensity \( \lambda(\sigma^q_t) \)
- \( q_t k_t^i \) drops below debt \( d_t \), costly state verification
- Debt holders’ loss rate
  \[ \lambda(\sigma^p) v \int_{0}^{d} \left( \frac{d}{v} - x \right) dF(x) \]
- Verification cost rate
  \[ \lambda(\sigma^p) v \int_{0}^{\frac{d}{v}} c x dF(x) \]
- Leverage bounded not only by precautionary motive, but also by the cost of borrowing

<table>
<thead>
<tr>
<th>Asset</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_t = k_t q_t )</td>
<td>( d_t = k_t q_t - n_t )</td>
</tr>
<tr>
<td></td>
<td>( n_t )</td>
</tr>
</tbody>
</table>
Ext2: Equilibrium

- Experts borrowing rate $> r$
  - Compensates for verification cost
- Rate depends on leverage, price volatility
- $d\eta_t = \text{diffusion process (without jumps) because losses cancel out in aggregate}$
Experts can contract on shocks $Z_t$ and $dJ_t^i$ directly among each other, zero contracting costs

In principle, good thing (avoid verification costs)

Equilibrium

- experts fully hedge idiosyncratic risks
- experts hold their share (do not hedge) aggregate risk $Z_t$, market price of risk depends on $\sigma_t^\theta (\sigma + \sigma_t^q)$
- with securitization experts lever up more (as a function of $\eta_t$) and bonus payments occur “sooner”
- financial system becomes less stable
- risk taking is endogenous (Arrow 1971, Obstfeld 1994)
Conclusion

- Incorporate financial sector in macromodel
  - Higher growth
  - Exhibits instability
    - similar to existing models (BGG, KM) in term of persistence/amplification, but
    - non-linear liquidity spirals (away from steady state) lead to instability

- Risk taking is endogenous
  - “Volatility paradox:” Lower exogenous risk leads to greater leverage and may lead to higher endogenous risk
  - Correlation of assets increases in crisis
  - With idiosyncratic jumps: countercyclical credit spreads
  - Securitization helps share idiosyncratic risk, but leads to more endogenous risk taking and amplifies systemic risk

- Welfare: (Pecuniary) Externalities
  - excessive exposure to crises events
Thank you! 😊