Macro, Money and Finance
Problem Set 1 – Solutions (selective)
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Problem Set 1 – Problem 2
(Basak Cuoco with $\rho \neq \hat{\rho}$)

1. Solve model, closed-form expressions for $\iota, q, \sigma^q, \mu^\eta, \sigma^\eta$

2. Plot $q, r^f, \sigma^\eta \eta, \mu^\eta \eta$

3. Asset prices attenuate risk – why?

4. Stationary distribution
Problem Set 1 – Problem 2 – Model Solution

- Start with goods market clearing

\[
(\rho \eta + \rho (1 - \eta)) q = a - \iota
\]

- Use optimal investment \((q = 1 + \kappa \iota)\), solve for \(\iota\)

\[
\iota (\eta) = \frac{a - \rho \eta - \rho (1 - \eta)}{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}
\]

- Recover \(q\)

\[
q (\eta) = \frac{1 + \kappa a}{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}
\]
Problem Set 1 – Problem 2 – Model Solution

- Return on capital, expert portfolio choice, laws of motion of $N$ and $qK$ all do not depend on $\rho$
  → same as in lecture

$$\frac{d\eta}{\eta} = \left( \frac{a - \iota}{q} - \rho + \theta^2 (\sigma + \sigma^q)^2 \right) dt - \theta (\sigma + \sigma^q) dZ$$

- Capital market clearing

$$1 - \theta = \frac{qK}{\frac{N}{\frac{1}{\eta}}} \Rightarrow \theta = -\frac{1 - \eta}{\eta}$$

- Thus

$$\frac{d\eta}{\eta} = \left( \frac{a - \iota}{q} - \rho + \left( \frac{1 - \eta}{\eta} \right)^2 (\sigma + \sigma^q)^2 \right) dt + \frac{1 - \eta}{\eta} (\sigma + \sigma^q) dZ$$
Problem Set 1 – Problem 2 – Model Solution

- Left to find $\sigma^q$

- Apply Ito’s formula to $q(\eta)$

$$\frac{dq(\eta)}{q(\eta)} = \frac{q'(\eta) \mu^n \eta + \frac{1}{2} q''(\eta) (\sigma^n \eta)^2}{q(\eta)} dt + \frac{q'(\eta)}{q(\eta)} \sigma^n \eta dZ.$$ 

- Recall,
  
  - $q(\eta) = \frac{1 + \kappa \alpha}{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}$
  
  - $\sigma^n = \frac{1 - \eta}{\eta} (\sigma + \sigma^q)$

- So,

$$\sigma^q = \frac{q'(\eta)}{q(\eta)} (1 - \eta) (\sigma + \sigma^q) \Rightarrow \sigma^q = \frac{\frac{q'(\eta)}{q(\eta)} (1 - \eta)}{1 - (1 - \eta) \frac{q'(\eta)}{q(\eta)}} \sigma$$

$$\sigma^q(\eta) = -\frac{(1 - \eta) \kappa (\rho - \rho)}{1 + \kappa \rho} \sigma$$
Conclusion:

\[
\nu(\eta) = \frac{a - \rho \eta - \rho (1 - \eta)}{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}
\]

\[
q(\eta) = \frac{1 + \kappa a}{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}
\]

\[
\sigma^q(\eta) = -\frac{(1 - \eta) \kappa (\rho - \rho)}{1 + \kappa \rho} \sigma
\]

\[
\mu^\eta(\eta) = \left( \frac{1 - \eta}{\eta} \frac{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}{1 + \kappa \rho} \sigma \right)^2 - (\rho - \rho) (1 - \eta)
\]

\[
\sigma^\eta(\eta) = \frac{1 - \eta}{\eta} \frac{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}{1 + \kappa \rho} \sigma.
\]
Problem Set 1 – Problem 2 – Solution Plots

- Risk-free rate (from experts’ portfolio choice)

\[ r_f^f = \frac{a - \nu}{q} + \Phi(\nu) - \delta + \mu^q + \sigma \sigma^q - \zeta (\sigma + \sigma^q) \]

- \( \zeta \), \( \mu^q \)?

\[ \zeta (\eta) = \frac{1}{\eta} (\sigma + \sigma^q (\eta)) \]

\[ \mu^q (\eta) = \frac{q'(\eta) \mu^\eta (\eta) \eta + \frac{1}{2} q''(\eta) (\sigma^\eta (\eta) \eta)^2}{q(\eta)} \]
Problem Set 1 – Problem 2 – Solution Plots
Replication of Lecture ($\rho = \rho$)
Problem Set 1 – Problem 2 – Solution Plots
Heterogeneous Time Preference ($\rho < \bar{\rho}$)
Problem Set 1 – Problem 2 – Amplification?

Is there endogenous amplification in this model?

- No,

\[
\sigma + \sigma^q = \left(1 - \frac{(1 - \eta) \kappa (\rho - \underline{\rho})}{1 + \kappa \rho}\right) \sigma < \sigma
\]

Reason:

1. **Technical:**
   - \( q \) is decreasing in \( \eta \)
   - generates negative amplification term

2. **Economic Answer:**
   - After negative shock, expert wealth is reduced by more than output (due to leverage)
   - If households have lower MPC than experts (due to higher patience), aggregate consumption demand falls
   - Relative price of capital (relative to output good) and physical investment must rise
Problem Set 1 – Problem 2
Stationary Distribution

\[
\mu^n (\eta) = \left( \frac{1 - \eta}{\eta} \frac{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}{1 + \kappa \rho} \sigma \right)^2 - (\rho - \rho_0) (1 - \eta)
\]

\[
\sigma^n (\eta) = \frac{1 - \eta}{\eta} \frac{1 + \kappa \rho \eta + \kappa \rho (1 - \eta)}{1 + \kappa \rho} \sigma.
\]
Problem Set 1 – Problem 4
(Stability of ODEs)

▪ Consider linear test problem (for $\lambda \in \mathbb{C}$, $\text{Re}(\lambda) < 0$)
  \[ y' = \lambda y, \quad y(0) = 1 \]

▪ Solution $y(x) = e^{-\lambda x}$ is bounded, has strictly decreasing absolute value and converges to 0 for $x \to \infty$

▪ When do solutions based on explicit/implicit Euler methods have these properties?

▪ Important for numerical stability: numerical errors in each step get dampened over time
\[ y_i - y_{i-1} = \lambda y_{i-1} \Delta x \Rightarrow y_i = (1 + \lambda \Delta x) y_{i-1}. \]

\[ |y_i| = |1 + \lambda \Delta x|^i |y_0| = |1 + \lambda \Delta x|^i. \]

\[ \left( \frac{1}{\Delta x} + \text{Re}\lambda \right)^2 + (\text{Im}\lambda)^2 < \frac{1}{\Delta x^2}, \]
Numerical Example: $\lambda = -10$

$$y_i = (1 + \lambda \Delta x) y_{i-1}.$$
\[ y_i - y_{i-1} = \lambda y_i \Delta x \Rightarrow y_i = \frac{1}{1 - \lambda \Delta x} y_{i-1}. \]

\[ |y_i| = \frac{1}{|1 - \lambda \Delta x|^i}. \]

- \( \text{Re}(1 - \lambda \Delta x) > 1 \), whenever \( \text{Re}(\lambda) < 0 \)

- hence \( |1 - \lambda \Delta x| > 1 \)

\[ \Rightarrow \text{implicit Euler method unconditionally stable} \]