LECTURE 2: ONE PERIOD MODEL STRUCTURE
Overview

1. Securities Structure
   - Arrow-Debreu securities structure
   - Redundant securities
   - Market completeness
   - Completing markets with options
The Economy

- **State space (Evolution of states)**
  - Two dates: $t = 0, 1$
  - $S$ states of the world at time $t = 1$

- **Preferences**
  - $U(c_0, c_1, ..., c_S)$
  - $MRS_{s,0}^A = -\frac{\partial U^A}{\partial U^A} \frac{\partial c_s^A}{\partial c_0^A}$ (slope of indifference curve)

- **Security structure**
  - Arrow-Debreu economy
  - General security structure
Security Structure

- Security $j$ is represented by a payoff vector $(x_1^j, x_2^j, ..., x_S^j)'$

- Security structure is represented by payoff matrix

$$X = \begin{pmatrix}
    x_1^1 & x_1^2 & \cdots & x_1^{J-1} & x_1^J \\
    x_2^1 &       & \cdots &        & x_2^J \\
    \vdots &       & \ddots &        & \vdots \\
    x_{S-1}^1 &       & \cdots & x_{S-1}^J \\
    x_S^1 & x_S^2 & \cdots & x_S^{J-1} & x_S^J
\end{pmatrix}$$
Arrow-Debreu Security Structure in $\mathbb{R}^2$

One A-D asset $e_1 = (1,0)'$

This payoff cannot be replicated!

Payoff Space $\langle X \rangle$

Markets are **incomplete**
Arrow-Debreu Security Structure in $\mathbb{R}^2$

Add second A-D asset $e_2 = (0,1)'$ to $e_1 = (1,0)'$
Arrow-Debreu Security Structure in $\mathbb{R}^2$

Add second A-D asset $e_2 = (0,1)'$ to $e_1 = (1,0)'$

Any payoff can be replicated with two A-D securities
Arrow-Debreu Security Structure in $\mathbb{R}^2$

Add second asset $(1,2)$' to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

*New asset is redundant – it does not enlarge the payoff space*
Arrow-Debreu Security Structure

\[ X = \begin{pmatrix} 1 & 0 & \vdots \\ 0 & \vdots & 0 \\ \vdots & \vdots & 0 & 1 \end{pmatrix} \]

- \( S \) Arrow-Debreu securities
- each state \( s \) can be insured individually
- All payoffs are linearly independent
- Rank of \( X = S \)
- Markets are complete
General Security Structure

Only bond \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\)

Payoff space \(<X>\)
General Security Structure

Only bond \( \left( \frac{1}{1} \right) \) can’t be reached

Payoff space \(<X>\)
General Security Structure

Add security \( \binom{2}{1} \) to bond \( \binom{1}{1} \)
General Security Structure

Add security $\binom{2}{1}$ to bond $\binom{1}{1}$

- Portfolio of
  - buy 3 bonds
  - sell short 1 risky asset
General Security Structure

Payoff space \( <X> \)

Two assets span the payoff space

Market are complete with security structure

\[
\begin{pmatrix}
1 & 1 \\
2 & 1
\end{pmatrix}
\]

Payoff space coincides with payoff space of

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
General Security Structure

- Portfolio: vector $h \in \mathbb{R}^J$ (quantity for each asset)
- Payoff of Portfolio $h$ is $\sum_j h^j x^j = Xh$
- Asset span $\langle X \rangle = \{z \in \mathbb{R}^S: z = Xh \text{ for some } h \in \mathbb{R}^J\}$
  - $\langle X \rangle$ is a linear subspace of $\mathbb{R}^S$
  - Complete markets $\langle X \rangle = \mathbb{R}^S$
  - Complete markets if and only if $\text{rank}(X) = S$
  - Incomplete markets if $\text{rank}(X) < S$
  - Security $j$ is redundant if $x^j = Xh$ with $h^j = 0$
Introducing derivatives

• Securities: property rights/contracts
• Payoffs of derivatives *derive* from payoff of underlying securities
• Examples: forwards, futures, call/put options

• Question:
Are derivatives necessarily redundant assets?
Forward contracts

• Definition: A binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today.

• Futures contracts are same as forwards in principle except for some institutional and pricing differences.

• A forward contract specifies:
  – The features and quantity of the asset to be delivered.
  – The delivery logistics, such as time, date, and place.
  – The price the buyer will pay at the time of delivery.
Payoff diagram for forwards

- Long and short forward positions on the S&R 500 index:
Forward vs. outright purchase

Forward payoff

\[ \text{Forward payoff} = \text{Spot price at expiration} - \$1,020 + \$1,020 = \text{Spot price at expiration} \]

Bond payoff

\[ \text{Bond payoff} = \$1,020 \]
Additional considerations (ignored)

- **Type of settlement**
  - Cash settlement: less costly and more practical
  - Physical delivery: often avoided due to significant costs

- **Credit risk of the counter party**
  - Major issue for over-the-counter contracts
    - Credit check, collateral, bank letter of credit
  - Less severe for exchange-traded contracts
    - Exchange guarantees transactions, requires collateral
Call options

• A non-binding agreement (right but not an obligation) to buy an asset in the future, at a price set today
• Preserves the upside potential (😊), while at the same time eliminating the unpleasant (👎) downside (for the buyer)
• The seller of a call option is obligated to deliver if asked
Definition and Terminology

- A call option gives the owner the right but not the obligation to buy the underlying asset at a predetermined price during a predetermined time period.
- Strike (or exercise) price: The amount paid by the option buyer for the asset if he/she decides to exercise.
- Exercise: The act of paying the strike price to buy the asset.
- Expiration: The date by which the option must be exercised or become worthless.
- Exercise style: Specifies when the option can be exercised.
  - European-style: can be exercised only at expiration date.
  - American-style: can be exercised at any time before expiration.
  - Bermudan-style: can be exercised during specified periods.
Payoff/profit of a purchased call

• Payoff = \( \max [0, \text{spot price at expiration} - \text{strike price}] \)
• Profit = Payoff – future value of option premium
• Examples 2.5 & 2.6:
  – S&R Index 6-month Call Option
    • Strike price = $1,000, Premium = $93.81, 6-month risk-free rate = 2%
    – If index value in six months = $1,100
      • Payoff = \( \max [0, \$1,100 - \$1,000] \) = $100
      • Profit = $100 – ($93.81 \times 1.02) = $4.32
    – If index value in six months = $900
      • Payoff = \( \max [0, \$900 - \$1,000] \) = $0
      • Profit = $0 – ($93.81 \times 1.02) = - $95.68
Diagrams for purchased call

- **Payoff at expiration**

- **Profit at expiration**

![Payoff Diagram](image1.png)

- Payoff at expiration

- Profit at expiration

![Profit Diagram](image2.png)

- Index price = $1020
- Profit = $-95.68

Index price = $1000

- Purchased call
- Long forward
Put options

- A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period.
- The seller of a put option is obligated to buy if asked.
- Payoff/profit of a purchased (i.e., long) put:
  - Payoff = \( \max(0, \text{strike price} - \text{spot price at expiration}) \)
  - Profit = Payoff – future value of option premium
- Payoff/profit of a written (i.e., short) put:
  - Payoff = - \( \max(0, \text{strike price} - \text{spot price at expiration}) \)
  - Profit = Payoff + future value of option premium
A few items to note

• A call option becomes more profitable when the underlying asset appreciates in value
• A put option becomes more profitable when the underlying asset depreciates in value
• Moneyness:
  – In-the-money option: positive payoff if exercised immediately
  – At-the-money option: zero payoff if exercised immediately
  – Out-of-the-money option: negative payoff if exercised immediately
Options and insurance

- Homeowner’s insurance as a put option:
Equity linked CDs

- The 5.5-year CD promises to repay initial invested amount and 70% of the gain in S&P 500 index:
  
  - Assume $10,000 invested when S&P 500 = 1300
  - Final payoff =

    $10000 \times \left(1 + 0.7 \times \max\left\{0, \frac{S_{\text{final}}}{1300} - 1\right\}\right)$

  - where $S_{\text{final}} = $ value of the S&P 500 after 5.5 years

Fig. 2.14
Option and forward positions

A summary

Profit
Long forward
Stock Price

Profit
Short forward
Stock Price

Profit
Long call
Stock Price

Profit
Short call
Stock Price

Profit
Long put
Stock Price

Profit
Short put
Stock Price
Options to Complete the Market

- Stock’s payoff: \( x^j = (1, 2, \ldots, S)' (= \text{state space}) \)
- Introduce call options with final payoff at \( T \):
  \[ C_T = \max\{S_T - K, 0\} = [S_T - K]^+ \]
- Thus
  \[
  C_{K=1} = (0, 1, 2, \ldots, S - 2, S - 1)'
  
  C_{K=2} = (0, 0, 1, \ldots, S - 3, S - 2)'
  
  \vdots
  
  C_{K=S-1} = (0, 0, 0, \ldots, 0, 1)'
  
  \]
Options to Complete the Market

• Together with the primitive asset:

\[
X = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
2 & 1 & 0 & \ldots & 0 & 0 & 0 \\
3 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
S - 2 & S - 3 & S - 4 & \ldots & 1 & 0 & 0 \\
S - 1 & S - 2 & S - 3 & \ldots & 2 & 1 & 0 \\
S & S - 1 & S - 2 & \ldots & 3 & 2 & 1
\end{pmatrix}
\]

Homework: check whether markets are complete
General Security Structure

• Price vector $p \in \mathbb{R}^J$ of asset prices
• Cost of portfolio $h$,
  \[
  p \cdot h := \sum_j p^j h^j
  \]
• If $p^j \neq 0$ the (gross) return vector of asset $j$ is the vector
  \[
  R^j = \frac{x^j}{p^j}
  \]