Macro, Money and Finance
Problem Sets 2 – Solutions
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Problem Set 2 – Problem 2 (KFE OU Process)

- General Kolmogorov Forward Equation (KFE)
  \[
  \frac{\partial p}{\partial t}(x, t) = -\frac{\partial}{\partial x} (\mu(x, t)p(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(x, t)p(x, t))
  \]
- Describes density evolution of process \( X \) with
  \[
  dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dZ_t
  \]
- For this problem: \( X = \) Ornstein-Uhlenbeck process (continuous-time AR(1))
  \[
  dX_t = \theta(\bar{x} - X_t)dt + \sigma dZ_t
  \]
- Get then special KFE
  \[
  \frac{\partial p}{\partial t}(x, t) = \theta(x - \bar{x}) \frac{\partial}{\partial x} p(x, t) + \theta p(x, t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} p(x, t)
  \]
Problem Set 2 – Problem 2 (KFE OU Process)

- For this problem: $X = \text{Ornstein-Uhlenbeck process}$ (continuous-time AR(1))

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dZ_t$$

- Get then special KFE

$$\frac{\partial p}{\partial t}(x, t) = \theta(x - \bar{x}) \frac{\partial}{\partial x} p(x, t) + \theta p(x, t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} p(x, t).$$

- Tasks:
  - Solve equation numerically using different schemes and parameters
  - Compare with known closed-form solution
  - Identify problems with some schemes
Digression: a More General PDE

- OU-KFE PDE is

\[
\frac{\partial p}{\partial t}(x, t) = \theta(x - \bar{x}) \frac{\partial}{\partial x} p(x, t) + \theta p(x, t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} p(x, t).
\]

- Instead, let’s solve the generic linear PDE

\[
\frac{\partial p}{\partial t}(x, t) = a(x, t) \frac{\partial^2}{\partial x^2} p(x, t) + b(x, t) \frac{\partial}{\partial x} p(x, t) + c(x, t)p(x, t) + d(x, t)
\]
Digression: a More General PDE

- Let’s solve the generic linear PDE
  \[
  \frac{\partial p}{\partial t}(x, t) = a(x, t) \frac{\partial^2}{\partial x^2} p(x, t) + b(x, t) \frac{\partial}{\partial x} p(x, t) + c(x, t) p(x, t) + d(x, t)
  \]
- Let \( x_0 < x_1 < \cdots < x_{N-1} < x_N \) be a grid in \( x \)-dimension
- A finite difference approximation transforms the PDE into a vector ODE
  \[
  \frac{d\hat{p}}{dt}(t) = A(t) D^2 \hat{p}(t) + B(t) D^1 \hat{p}(t) + C(t) \hat{p}(t) + \hat{d}(t)
  \]
  where
  \[
  \hat{p}(t) := \begin{pmatrix}
  p(x_0, t) \\
p(x_1, t) \\
  \vdots \\
p(x_N, t)
\end{pmatrix},
  A(t) := \begin{pmatrix}
a(x_0, t) & 0 & \cdots & 0 \\
0 & a(x_1, t) & \cdots & \vdots \\
\vdots & \vdots & & \vdots \\
0 & \cdots & \cdots & a(x_N, t)
\end{pmatrix},
  B(t) := \cdots,
  C(t) := \cdots,
  \hat{d}(t) := \begin{pmatrix}
d(x_0, t) \\
d(x_1, t) \\
\vdots \\
d(x_N, t)
\end{pmatrix}
  \]
  and the matrices \( D^2, D^1 \) represent the finite difference approximation.
Solving the Vector ODE – time step

- We can write this vector ODE more concisely as
  \[ \hat{\mathbf{p}}'(t) = M(t)\hat{\mathbf{p}}(t) + \hat{\mathbf{d}}(t) \]

  where
  \[ M(t) = A(t)D^2 + B(t)D^1 + C(t) \]

- We use two methods for the ODE (time step)
  1. Explicit Euler:
     - Evaluate right-hand side at the previous time grid point \( t \)
     - Equation for new vector \( \hat{\mathbf{p}}(t + \Delta t) \)
       \[ \hat{\mathbf{p}}(t + \Delta t) = \hat{\mathbf{p}}(t) + \Delta t \cdot (M(t)\hat{\mathbf{p}}(t) + \hat{\mathbf{d}}(t)) \]
  2. Implicit Euler:
     - Evaluate right-hand side at the new time grid point \( t + \Delta t \)
     - Equation for new vector \( \hat{\mathbf{p}}(t + \Delta t) \)
       \[ (I - \Delta t \cdot M(t + \Delta t))\hat{\mathbf{p}}(t + \Delta t) = \hat{\mathbf{p}}(t) + \Delta t \cdot \hat{\mathbf{d}}(t + \Delta t) \]

- Remark: on the two boundaries, we ignore the ODE and impose the boundary conditions instead
Solving the Vector ODE – the matrix $M(t)$

- Left to do: construct matrix $M(t)$
  - recall $M(t) = A(t)D^2 + B(t)D^1 + C(t)$
- The matrices $A(t)$, $B(t)$ and $C(t)$ are diagonal
- $D^2$ and $D^1$ are tridiagonal

$$\begin{align*}
D^k &= \begin{pmatrix}
* & * & * \\
& d_{1,-}^k & d_{1,0}^k & d_{1,+}^k \\
& d_{2,-}^k & d_{2,0}^k & d_{2,+}^k \\
& & \ddots & \ddots & \ddots \\
& & & d_{N-1,-}^k & d_{N-1,0}^k & d_{N-1,+}^k \\
& & & & * & * 
\end{pmatrix}
\end{align*}$$

- Conclusion: $M(t)$ is again tridiagonal, construct as sparse matrix
Example: $D^2$ for equally spaced grid

- Suppose we have an equally spaced grid,
  \[ x_{i+1} = x_i + \Delta x \]
- Then $D^2$ is given by

\[
D^2 = \begin{pmatrix}
\ast & \ast & & & & \\
\frac{1}{\Delta x^2} & \frac{1}{\Delta x^2} & \frac{1}{\Delta x^2} & & & \\
-\frac{2}{\Delta x^2} & -\frac{2}{\Delta x^2} & -\frac{2}{\Delta x^2} & \ddots & & \\
& \ddots & \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots & \frac{1}{\Delta x^2} \\
& & & \ddots & \frac{1}{\Delta x^2} & \frac{1}{\Delta x^2}
\end{pmatrix}
\]
Solution of KFE for $\theta = 0$
What Happens for Finer Space Grid?

- Implicit Method: errors slightly smaller

- Explicit Method:
What Happens for Finer Space Grid?

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- Explicit Method:
What Goes Wrong with the Explicit Method?

- Recall, Stability of ODEs:
  - Stable, if eigenvalue in stability region
  - Smallest eigenvalue of equation’s space discretization $\lambda \approx -\frac{2\sigma^2}{\Delta x^2}$
- If too small, explicit method becomes unstable:
Solution for $\theta = 3$