Course on Continuous-time Macro

1. Introduction: Liquidity, Run-up, Crisis-Amplification, Recovery

**Real Macro-Finance Models with Heterogeneous Agents**

2. A Simple Model

3. General Solution Technique for Real Models

4. International Macro-Finance Model with Sudden Stops/Runs

**Money Models**

5. A Simple Money Model

6. General Solution Technique for Money Models

7. The I Theory of Money

8. Welfare Analysis & Optimal Policy
   - Monetary and Macroprudential Policy

9. International Financial Architecture*

10. Robust Computational Methods – Comparing Nonlinear Models

11. Calibration and Empirical Implications
Why Continuous Time Modeling?

- Discrete time consumption
  - IES/RA within period $= \infty$, but across periods $= 1/\gamma$
- Discrete time: Portfolio choice
  - Linearize: kills $\sigma$-term and all assets are equivalent
  - 2nd order approximation: kills time-varying $\sigma$
  - Log-linearize a la Campbell-Shiller
- As $\Delta t \to 0$ (net)returns converge to normal distribution
  - Constantly adjust the approximation point
  - Continuous compounding

\[ R_t R_{t+1} R_{t+2} \ldots = e^{r_t + r_{t+1} + \ldots} \]
Why Continuous Time Modeling?

- Ito processes... fully characterized by drift and volatility
  - Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$

- Characterization for full volatility dynamics on Prob.-space
  - Discrete time: Probability-loading on states
  - Cts. time: Loading on a Brownian Motion $dZ_t$ (captured by $\sigma$)

- Normal distribution for $dt$, yet with skewness for $\Delta t > 0$
  - If $\sigma_t$ is time-varying

- How restrictive?
Why Continuous Time Modeling with Ito?

- **Continuous path**
  - Information arrives continuously “smoothly” – not in lumps
  - Implicit assumption: can react continuously to continuous info flow
  - Never jumps over a specific point, e.g. insolvency point
  - Simplifies numerical analysis:
    - Only need change from grid-point to grid-point
      (since one never jumps beyond the next grid-points)
- **No default risk**
  - Can continuously delever as wealth declines
    - Might embolden investors ex-ante
- **Collateral constraint**
  - Discrete time: \( b_t R_{t,t+1} \leq \min\{q_{t+1}\}k_t \)
  - Cts. time: \( b_t \leq (p_t + \underbrace{dp_t}_{\to 0})k_t \)
    - For short-term debt – not for long-term debt ... or if there are jumps
- **Levy processes... with jumps**
Why Continuous Time Modeling with Ito?

\[ E[dV(\eta)] = V'(\eta)\mu \eta dt + \frac{1}{2} V''(\eta)(\sigma \eta)^2 \eta^2 dt \]

- More analytical steps
  - Return equations
    - Next instant returns are essentially normal (easy to take expectations)
  - Explicit net worth and state variable dynamics
    - Continuous: only slope of price function determines amplification
    - Discrete: need whole price function (as jump size can vary)
Basics of Ito Calculus

- Geometric Ito Process: \( dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t \)
  - Stock goes up 32% or down 32% over a year.
    - 256 trading days \( \frac{32\%}{\sqrt{256}} = 2\% \)

- Ito’s Lemma:
  \[
df(X_t) = f'(X_t)\mu_t^X X_t dt + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + f'(X_t)\sigma_t^X X_t dZ_t
  \]
  - \( u(c) = \frac{c^{1-\gamma-1}}{1-\gamma} \), \( u'(c) = c^{-\gamma} \) volatility of process \( \frac{dc_t^{-\gamma}}{c_t^{-\gamma}} \) is \(-\gamma \sigma_t^c\)

- Ito product rule: stock price * exchange rate
  \[
  \frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y) dt + (\sigma_t^X + \sigma_t^Y) dZ_t
  \]

- Ito ratio rule:
  \[
  \frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_t^X - \mu_t^Y + \sigma_t^Y (\sigma_t^Y - \sigma_t^X) dt + (\sigma_t^X - \sigma_t^Y) dZ
  \]
Simple Two Sector Model: Basak Cuoco (1998)

- Expert sector
  - Capital: $q_tK_t$
  - Debt: $N_t$

- Household sector
  - Loans
  - Net worth: $q_tK_t - N_t$

See Handbook of Macroeconomics 2017, Chapter 18
Two Sector Model Setup

Expert sector

- Output: \( y_t = ak_t \)

Household sector

- Output: \( y_t = ak_t \)

- Consumption rate: \( c_t \)
- Investment rate: \( \dot{d}k_t = \Phi \dot{\delta}_t - \delta d\dot{k}_t + \sigma_d Z_t + \sigma d\sigma_d Z_t \)

- Friction: Can only issue
  - Risk-free debt
  - Equity, but most hold \( \beta_t \geq \alpha \)

- Expected utility: \( E_0 \left[ \int_0^{\infty} e^{-\rho t} c_t^{1-\gamma} \frac{1-\gamma}{1-\gamma} dt \right] \)
Two Sector Model Setup

Expert sector
- Output: $y_t = a k_t$
- Consumption rate: $c_t$
- Investment rate: $\lambda_t$

\[
\frac{d k_t}{k_t} = (\Phi(\lambda_t) - \delta) dt + \sigma dZ_t
\]

Household sector
- Consumption rate: $c_t$

agent $i$ of type $i$ (expert, HH)
Two Sector Model Setup

**Expert sector**

- Output: \( y_t = a k_t \)
- Consumption rate: \( c_t \)
- Investment rate: \( \ell_t \)


\[
\frac{dk_t}{k_t} = (\Phi(\ell_t) - \delta) \, dt + \sigma \, dZ_t
\]

- \( E_0 \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} \, dt \right] \)

**Household sector**

- Consumption rate: \( c_t \)

- \( E_0 \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} \, dt \right] \)

Log-utility in Basak Cuoco 1998
Two Sector Model Setup

Expert sector

- Output: \( y_t = a k_t \)
- Consumption rate: \( c_t \)
- Investment rate: \( \lambda_t \)

\[
\frac{dk_t}{k_t} = \left( \Phi(\lambda_t) - \delta \right) dt + \sigma dZ_t
\]

\[
E_0 \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]
\]

Friction: Can only issue

- Risk-free debt

Household sector

- Consumption rate: \( c_t \)

\[
E_0 \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]
\]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment \( \iota \), (portfolio \( \theta \), & consumption choice of each agent)
      - Toolbox 1: Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities \( \omega \)
      - Special cases
   b. De-scaled value fcn. as function of state variables \( \eta \)
      - Digression: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive \( \zeta \) price of risk, \( C/N \)-ratio from value fcn. envelop condition

3. Evolution of state variable \( \eta \)
   - Toolbox 3: Change in numeraire to total wealth (including SDF)
   - (“Money evaluation equation” \( \mu^\theta \))

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
0. Postulate Aggregates and processes

- Individual capital evolution
  \[
  \frac{d k^i_t}{k^i_t} = (\Phi(l^i_t) - \delta)dt + \sigma dZ_t + d\Delta_{k,i}^t
  \]
  - Where \( \Delta_{k,i}^t \) is the individual cumulative capital purchase process

- Capital aggregation: \( K \equiv \int k^i_t d\tilde{i} \)
  \[
  \frac{dK_t}{K_t} = \int (\Phi(l^i_t) - \delta) d\tilde{i} dt + \sigma dZ_t
  \]
0. Postulate Aggregates and processes

- Individual capital evolution
  \[
  \frac{dk^i_t}{k^i_t} = (\Phi(l^i_t) - \delta)dt + \sigma dz_t + d\Delta^k_t, \hat{\iota}
  \]
  Where $\Delta^k_t$ is the individual cumulative capital purchase process

- Capital aggregation: $K \equiv \int k^i_t d\hat{\iota}$
  \[
  \frac{dK_t}{K_t} = \int (\Phi(l^i_t) - \delta)d\hat{\iota}dt + \sigma dz_t
  \]

- Networth aggregation:
  - Within sector: $N_t \equiv \int n^i_t d\hat{\iota}$, \(\overline{N}_t \equiv \int \overline{n}^i_t d\hat{\iota}\)
  - Wealth share: $\eta_t \equiv N_t/(N_t + \overline{N}_t)$
0. Postulate Aggregates and processes

- Individual capital evolution
  \[
  \frac{d k_t^i}{k_t^i} = (\Phi(l_t^i) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,i}
  \]
  - Where \(\Delta_t^{k,i}\) is the individual cumulative capital purchase process

- Capital aggregation:
  \[
  K \equiv \int k_t^i d\tilde{i}
  \]
  \[
  \frac{dK_t}{K_t} = \int (\Phi(l_t^i) - \delta)d\tilde{i}dt + \sigma dZ_t
  \]

- Networth aggregation:
  - Within sector:
    \[N_t \equiv \int n_t^i d\tilde{i}, \quad \overline{N_t} \equiv \int \overline{n_t^i} d\tilde{i}\]
  - Wealth share:
    \[\eta_t \equiv N_t / (N_t + \overline{N_t})\]
  - Value of capital stock:
    \[q_t K_t\]
    Postulate
    \[dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t\]
0. Postulate Aggregates and processes

- Individual capital evolution
  \[
  \frac{d k_t^i}{k_t^i} = \left( \Phi(l_t^i) - \delta \right) dt + \sigma dZ_t + d\Delta_{t}^{k,i}
  \]
  Where \(\Delta_{t}^{k,i}\) is the individual cumulative capital purchase process

- Capital aggregation: \(K \equiv \int k_t^i \, d\tilde{i}\)
  \[
  \frac{dK_t}{K_t} = \int \left( \Phi(l_t^i) - \delta \right) d\tilde{i}dt + \sigma dZ_t
  \]

- Networth aggregation:
  - Within sector: \(N_t \equiv \int n_t^i \, d\tilde{i}\), \(\hat{N}_t \equiv \int \hat{n}_t^i \, d\tilde{i}\)
  - Wealth share:
    \(\eta_t \equiv N_t / (N_t + \hat{N}_t)\)

- Value of capital stock:
  \(q_t K_t\)
  Postulate
  \[
  dq_t/q_t = \mu_t^q \, dt + \sigma_t^q \, dZ_t
  \]

- Postulate SDF-process:
  \[
  \frac{d\xi_t}{\xi_t} = -r_t dt - \zeta_t dZ_t, \quad \frac{d\xi_t}{\xi_t} = -r_t dt - \zeta_t dZ_t
  \]

Price of risk

\(\Delta_{t}^{k,i}\) add up to zero

Same Brownian

\(\xi_t\) add up to zero
Aside: Basics of Ito Calculus

- Geometric Ito Process \( dX_t = \mu_t X_t dt + \sigma_t X_t dZ_t \)

- Ito’s Lemma:
  \[
  df(X_t) = f'(X_t)\mu_t X_t dt + \frac{1}{2} f''(X_t)(\sigma_t X_t)^2 dt + f'(X_t)\sigma_t X_t dZ_t
  \]

- \( u(c) = \frac{c^{1-\gamma-1}}{1-\gamma}, \quad u'(c) = c^{-\gamma} \quad \) volatility of process \( \frac{dc_t^{-\gamma}}{c_t^{-\gamma}} \) is \( -\gamma \sigma_t c \)

- Ito product rule: stock price * exchange rate
  \[
  \frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t X_t + \mu_t Y_t + \sigma_t X_t \sigma_t Y_t) dt + (\sigma_t X_t + \sigma_t Y_t) dZ_t
  \]
0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define \( \tilde{k}_t \): \[
    \frac{d\tilde{k}_t}{\tilde{k}_t} = \left( \Phi(\tilde{i}_t) - \delta \right) dt + \sigma dZ_t + d\Delta^\kappa_t \]
    without purchases/sales

    \[
    dr^K_t(i_t) = \left( a - \frac{i^q_t}{q} \right) + \Phi(i^q_t) - \delta + \mu^q_t + \sigma \sigma^q_t \right) dt \]
    \[
    + \left( \sigma + \sigma^q_t \right) dZ_t
    \]

- Postulate SDF-process: (Example: \( \xi_t = e^{-\rho^t u'(c_t)} \))

  \[
  \frac{d\xi_t}{\xi_t} = -r_t dt - \zeta_t dZ_t \]
  \[
  \frac{d\xi_t}{\xi_t} = -r_t dt - \zeta_t dZ_t
  \]

Recall discrete time \( e^{-r_F} = E[SDF] \)
0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\tilde{k}_t^i$: $\frac{d\tilde{k}_t^i}{\tilde{k}_t^i} = (\Phi(l_t^i) - \delta) dt + \sigma dZ_t + d\Delta_t$ without purchases/sales

\[
dr^K_t(l_t^i) = \left( \frac{a - l_t^i}{q} + \Phi(l_t^i) - \delta + \mu_t^q + \sigma \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t
\]

- Postulate SDF-process: (Example: $\xi_t = e^{-\rho_t u'(c_t)}$)

\[
\frac{d\xi_t}{\xi_t} = -r_t dt - \zeta_t dZ_t \quad \frac{d\xi_t}{\xi_t} = -r_t dt - \zeta_t dZ_t
\]

Recall discrete time $e^{-r^F} = E[SDF]$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment $\iota$, (portfolio $\theta$, & consumption choice of each agent)
      - *Toolbox 1*: Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      - *Toolbox 2*: “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - *Special cases*
   b. De-scaled value fcn. as function of state variables $\eta$
      - *Digression*: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive $\zeta$ price of risk, $C/N$-ratio from value fcn. envelop condition

3. Evolution of state variable $\eta$
   - *Toolbox 3*: Change in numeraire to total wealth (including SDF)
   - (“Money evaluation equation” $\mu^{\theta}$)

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Replace endogenous $\mu s, \sigma s$ with $f, f', f''(\eta)$
   c. Value function iteration by solving PDE
1. Individual Agent Choice of $l$, $\theta$, $c$

- Choice of $l$ is static problem (and separable) for each $t$

- $\max_i dr^K(i) = \max_i \left( \frac{a - i}{q} + \Phi'(i) - \delta + \mu^q + \sigma^q \right)$

- FOC: $\frac{1}{q} = \Phi'(i)$ Tobin’s $q$
  - All agents $i = i$ $\Rightarrow \frac{dK_t}{K_t} = (\Phi(i) - \delta) dt + \sigma dZ_t$

- Special functional form:
  - $\Phi(i) = \frac{1}{\kappa} \log(\kappa i + 1) \Rightarrow \kappa i = q - 1$
1a. Martingale Approach – Discrete Time

\[
\max_{\{c, \theta\}} \mathbb{E}_t \left[ \sum_{\tau=t}^{T} \frac{1}{(1+\rho)^{\tau-t}} u(c_{\tau}) \right]
\]

s.t. \( \theta_t p_t = \theta_{t-1} (p_t + d_t) - c_t \) for all \( t \)

- FOC w.r.t. \( \theta_t \): (deviate from optimal at \( t \) and \( t+1 \))
  \[
  \xi_t p_t = E_t [\xi_{t+1} (p_{t+1} + d_{t+1})]
  \]
  - where \( \xi_t = \frac{1}{(1+\rho)^t} \frac{u'(c_t)}{u'(c_0)} \) is the (multi-period) stochastic discount factor (SDF)
  - If projected on asset span, then pricing kernel \( \xi^* \)
  - Note: MRS_{t, \tau} = \frac{\xi_{t+\tau}}{\xi_t}

- Consider portfolio, where one reinvests dividend \( d \)
  - Portfolio is a self-financing trading strategy, \( A \), with price, \( p_t^A \)
    \[
    \xi_t p_t^A = E_t [\xi_{t+1} p_{t+1}^A]
    \]
  - Stochastic process, \( \xi_t p_t^A \), is a martingale
1a. Martingale Approach – Cts. Time

\[
\max_{\{\mu_t, \theta_t, c_t\}_{t=0}^\infty} E \left[ \int_0^\infty e^{-\rho u(c_t)} dt \right]
\]

s.t. \[
\frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + \sum_j \theta^j_t dr^j_t + \text{labor income/endow/taxes}
\]

Portfolio Choice: Martingale Approach

- Let \( x_t^A \) be the value of a “self-financing trading strategy” (reinvest dividends)

Theorem: \( \xi_t x_t^A \) follows a Martingale, i.e. drift = 0.

- Let \( \frac{dx_t^A}{x_t^A} = \mu^A_t dt + \sigma^A_t dZ_t \),

- Recall \( \frac{d\xi_t^i}{\xi_t} = -r_t dt - \zeta_t^i dZ_t \)

- By Ito product rule

\[
\frac{d(\xi_t^i x_t^A)}{\xi_t x_t^A} = \left( -r_t + \mu^A_t - \zeta_t^i \sigma_t^A \right) dt + \text{volatility terms} = 0
\]

Expected return: \( \mu^A_t = r_t + \zeta_t^i \sigma_t^A \)

- For risk-free asset, i.e. \( \sigma_t^A = 0 \):
- Excess expected return to risky asset B: \( r_t^F = r_t \), \( \mu_t^A - \mu_t^B = \zeta_t^i (\sigma_t^A - \sigma_t^B) \)
1a. Martingale Approach – Cts. Time

- **Proof 1**: Stochastic Maximum Principle (see Handbook chapter)

- **Proof 2**: Intuition (calculus of variation)

  remove from optimum $\Delta$ at $t_1$ and add back at $t_2$

$$V(n, \omega, t) = \max_{\{\ell_s, \theta_s, c_s\}_{s=t}} \mathbb{E}_t \left[ \int_0^\infty e^{-\rho(s-t)} u(c_s) d\ell | \omega_t = \omega \right]$$

  s.t. $n_t = n$

$$e^{-\rho t_1} \frac{\partial V}{\partial n} (n^*_1, x_{t_1}, t_1) x^A_{t_1} = \mathbb{E}_{t_1} \left[ e^{-\rho t_2} \frac{\partial V}{\partial n} (n^*_2, x_{t_2}, t_2) x^A_{t_2} \right]$$

- See Merkel Handout
1a. Optimal Portfolio Choice

- Using $\mu^A_t - r_t = \zeta_t^i \sigma^A_t$ for capital return (instead of generic asset A)

\[
\frac{a - \mu_t}{q_t} + \Phi(\mu_t) - \delta + \mu^q_t + \sigma \sigma^q_t - r_t = \zeta_t (\sigma + \sigma^q_t)
\]

- Recall
  - $\theta_t$ portfolio share in risk-free debt (short position)
  - $(1 - \theta_t)$ portfolio share in (physical) capital $k_t$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment $\iota$, (portfolio $\Theta$, & consumption choice of each agent)
      • Toolbox 1: Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      • Toolbox 2: “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      • Special cases
   b. De-scaled value fcn. as function of state variables $\eta$
      • Digression: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive $\zeta$ price of risk, $C/N$-ratio from value fcn. envelop condition

3. Evolution of state variable $\eta$
   • Toolbox 3: Change in numeraire to total wealth (including SDF)
   • (“Money evaluation equation” $\mu^\Theta$)

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
The Big Picture

- Allocation of physical assets
- Output $A(\psi)$
- Capital growth $\Phi(t) - \delta$
- Wealth distribution $\eta$
- Value function

Risk amplification

Drift $\lambda = 1$
Volatility $\chi = 1$

Consumption + investment $\iota$

Debt accumulation
Outside equity

Backward equation
Forward equation

Precautionary

Drift
Volatility
2a. CRRA Value Function: relate to $\omega$

- $\omega_t$ Investment opportunity/ “networth multiplier”

- CRRA/power utility $u(c) = \frac{c^{1-\gamma-1}}{1-\gamma}$
  \[ \Rightarrow \text{increase networth by factor, optimal consumption for all future states increases by same factor} \]
  \[ \Rightarrow \left( \frac{c}{n} \right) \text{-ratio is invariant in } n \]

- \[ \Rightarrow \text{value function can be written as } \frac{u(\omega_t n_t)}{\rho}, \text{ that is} \]
  \[ = \frac{1}{\rho} \frac{(\omega_t n_t)^{1-\gamma-1}}{1-\gamma} = \frac{1}{\rho} \frac{\omega_t^{1-\gamma} n_t^{1-\gamma-1}}{1-\gamma} \]

- \[ \frac{\partial V}{\partial n} = u'(c) \text{ by optimal consumption (if no corner solution)} \]
  \[ = \frac{\omega_t^{1-\gamma} n_t^{-\gamma}}{\rho} = c_t^{-\gamma} \iff \frac{c_t}{n_t} = \rho^{1/\gamma} \omega_t^{1-1/\gamma} \]
2a. CRRA Value Function: Special Cases

\[ \frac{c_t}{n_t} = \rho^{1/\gamma} \omega_t^{1-1/\gamma} \]

- For log utility $\gamma = 1$:
  \[ \frac{c_t}{n_t} = \rho \]

\[ \xi_t = e^{-\rho t} / c_t = e^{-\rho t} / (\rho n_t) \text{ for any } \omega_t \Rightarrow \sigma_t^n = \sigma_t^c = \xi_t \]

- Expected excess return: $\mu_t^A - r_t^F = \sigma_t^n \sigma_t^A$
- Recall \[ \frac{d n_t}{n_t} = -\frac{c_t}{n_t} dt + (1 - \theta)dr_t^K + \theta dr_t \]

- For both types: experts and HHs,
  \[ \frac{c_t}{n_t} = \rho \text{ and } \sigma_t^n = \sigma_t^c = \xi_t \]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment $\iota$, (portfolio $\Theta$, & consumption choice of each agent)
      ▪ Toolbox 1: Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      ▪ Special cases
   b. De-scaled value fcn. as function of state variables $\eta$
      ▪ Digression: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive $\varsigma$ price of risk, $C/N$-ratio from value fcn. envelop condition

3. Evolution of state variable $\eta$
   ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)
   ▪ (“Money evaluation equation” $\mu^9$)

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
3. GE: Markov States and Equilibria

- Equilibrium is a map
  Histories of shocks $\{z_{\tau}, 0 \leq \tau \leq t\}$ $\rightarrow$ prices $q_t, \zeta_t, \xi_t, \iota_t, \theta_t, \theta_t \begin{cases} < 0 \\ = 1 \end{cases}$

wealth distribution

$$\eta_t = \frac{N_t}{q_tK_t} \in (0, 1)$$

wealth share
Aside: Basics of Ito Calculus

▪ Geometric Ito Process: \(dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t\)

▪ Ito’s Lemma:

\[
df(X_t) = f'(X_t)\mu_t^X X_t dt + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + f'(X_t)\sigma_t^X X_t dZ_t
\]

▪ Ito product rule:

\[
\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y) dt + (\sigma_t^X + \sigma_t^Y) dZ_t
\]

▪ Ito ratio rule:

\[
\frac{d(X_t / Y_t)}{X_t / Y_t} = \mu_t^X - \mu_t^Y + \sigma_t^Y (\sigma_t^Y - \sigma_t^X) dt + (\sigma_t^X - \sigma_t^Y) dZ_t
\]
3. Law of Motion of Wealth Share $\eta_t$

- **Method 1:** Using Ito’s quotation rule $\eta_t = N_t/(q_t K_t)$

  \[
  \frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + r_t dt + (1 - \theta_t)[d r_t^K - r_t dt]
  \]

  \[
  \frac{dN_t}{N_t} = -\rho dt + r_t dt + (1 - \theta_t) \left[ \left( \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu^q_t + \sigma_t^q - r_t \right) dt + (\sigma + \sigma_t^q) dZ_t \right]
  \]

  \[
  \frac{d q_t K_t}{q_t K_t} = \left( \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t
  \]

  Note that $\zeta_t = \sigma_t^n = (1 - \theta_t)(\sigma + \sigma_t^q)$

- Ito ratio rule:

  \[
  \frac{d(X_t/Y_t)}{X_t/Y_t} = \mu^X_t - \mu^Y_t + \sigma_t^Y (\sigma_t^Y - \sigma_t^X) dt + (\sigma_t^X - \sigma_t^Y) dZ_t
  \]

- \[
  \frac{d\eta_t}{\eta_t} = \left( \frac{a - \iota_t}{q_t} - \rho + \theta_t^2 (\sigma + \sigma_t^q)^2 \right) dt - \theta_t \left( \sigma + \sigma_t^q \right) dZ_t < 0
  \]

- **Method 2:** Change of numeraire + Martingale (Lecture 03)
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment \( \iota \), (portfolio \( \theta \), & consumption choice of each agent)
      - **Toolbox 1:** Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      - **Toolbox 2:** “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities \( \omega \)
      - **Special cases**
   b. De-scaled value fcn. as function of state variables \( \eta \)
      - **Digression:** HJB-approach (instead of martingale approach & envelop condition)
   c. Derive \( \zeta \) price of risk, \( C/N \)-ratio from value fcn. envelop condition

3. Evolution of state variable \( \eta \)
   - **Toolbox 3:** Change in numeraire to total wealth (including SDF)
   - (“Money evaluation equation” \( \mu^\theta \))

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
4a. Market Clearing

- Output good market
  \[ C_t = (a - \iota)K_t \]
  \[ \rho q_t K_t = \left( a - \iota(q_t) \right) K_t \]
  \[ \rho q_t = \left( a - \iota(q_t) \right) \quad \Rightarrow \quad q_t = q \quad \forall t \]

- Capital market
  \[ 1 - \theta_t = \frac{q_t K_t}{N_t} = \frac{1}{\eta_t} \]

- Money market (by Walras Law)
4b. Model Solution

- Using $\rho q_t = (a - \iota(q_t))$, $\kappa \iota_t = q_t - 1$, $\Phi(\iota) = \frac{1}{\kappa} \log(\kappa \iota + 1)$
  \[ q = \frac{1 + \kappa a}{1 + \kappa \rho} \]

- Using portfolio choice, goods & capital market clearing
  \[ r_t = \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t q_t + \sigma \sigma_t q_t - \zeta_t (\sigma + \sigma_t^2) \]
  \[ = \rho + \Phi(\iota_t) - \delta - (1 - \theta_t) \sigma^2 \]
  \[ = \rho + \Phi(\iota_t) - \delta - \frac{\sigma^2}{\eta_t} \quad \text{from capital market clearing} \]
  \[ r_t = \rho + \frac{1}{\kappa} \log\left(\frac{1 + \kappa a}{1 + \kappa \rho}\right) - \delta - \frac{\sigma^2}{\eta_t} \]

- Goods & capital market clearing and $\eta$-evolution
  \[ \frac{d\eta_t}{\eta_t} = \frac{(1 - \eta_t)^2}{\eta_t^2} \sigma^2 dt + \frac{1 - \eta_t}{\eta_t} \sigma dZ_t \]
Numerical Example

\[ a = .11, \rho = 5\%, \sigma = .1, \Phi(t) = \frac{\log(\kappa t + 1)}{\kappa}, \kappa = 10 \]
Observation of Basak-Cuoco Model

- $\eta_t$ fluctuates with macro shocks, since experts are levered
- Price of risk, i.e. Sharpe ratio, is
  $$\frac{\sigma}{\eta_t} = \frac{\rho + \Phi(l) - \delta - r_t}{\sigma}$$
  - Goes to $\infty$ as $\eta_t$ goes to zero
  - Achieved via risk-free rate
    $$r_t = \rho + \Phi(l) - \delta - \frac{\sigma^2}{\eta_t} \to -\infty$$
  - Rather than depressing price of risky asset, $q_t = q \ \forall t$
- No endogenous risk $\sigma^q = 0$
  - No amplification
  - No volatility effects
- $\mu_t^\eta = \frac{(1-\eta_t)^2}{\eta_t^2} \sigma^2 > 0 \Rightarrow$ in the long run HH vanish
  - Way out:
    - Different discount rates $\rho \quad \text{(KM)}$
    - Switching types $\quad \text{(BGG)}$
Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock

- Endogenous risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news

- Volatility paradox

- Financial innovation less stable economy

- (“Net worth trap” double-humped stationary distribution)