Macro, Money and Finance
Problem Set 3 – Solutions (selective)
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Problem 1 – Cash-in-advance Constraint

- One sector money model of lecture 5 („I theory without I“)
- No idiosyncratic risk
- No money growth
- CIA constraint: must hold a fraction $1/\ell$ of consumption expenditures in money

Tasks:
1. HJB of individual agent, derive liquidity-adjusted choice conditions
3. How do velocity ($\ell$) changes affect equilibrium allocation? Is there crowding in/out?
Problem 1 – Choice Conditions

- Take generic returns:

\[
\begin{align*}
    dr^k &= \mu^{r,k} dt + \sigma^{r,k} dZ \\
    dr^m &= \mu^{r,m} dt + \sigma^{r,m} dZ
\end{align*}
\]

- HJB equation:

\[
\rho V(n) = \max_{c,\theta} \left( u(c) + V'(n) \left( -c + \theta n \mu^{r,m} + (1 - \theta) n \mu^{r,k} \right) + \frac{1}{2} V''(n) \left( \theta \sigma^{r,m} + (1 - \theta) \sigma^{r,k} \right)^2 n^2 \right)
\]

- Constraint: \( c \leq \ell \theta n \)

- \( \hat{\lambda} \) Lagrange multiplier and \( \lambda := \hat{\lambda} / V'(n) \) „price of liquidity“

- FOCs:

\[
\begin{align*}
    u'(c) &= (1 + \lambda) V'(n) \\
    \mu^{r,k} - \mu^{r,m} &= -\frac{V''(n)}{V'(n)} \sigma^n \left( \sigma^{r,m} - \sigma^{r,k} \right) + \lambda \ell
\end{align*}
\]
Problem 1 – Model Solution

- Log utility, thus \( V'(n) = \frac{1}{\rho n} \) and \(-\frac{V''(n)n}{V'(n)} = 1\)
- In addition:
  \[\mu^{r,k} - \mu^{r,m} = \frac{a - \iota}{q} \quad \text{and} \quad \sigma^{r,k} = \sigma^{r,m} = \sigma\]
- Substitute into consumption and portfolio condition
  \[\frac{1}{c} = \frac{1 + \lambda}{\rho n} \Rightarrow \zeta = \frac{\rho}{1 + \lambda}\]

\[\zeta (p + q) = a - \iota\]
- Goods market clearing

\[\frac{a - \iota}{q} = \sigma (\sigma - \sigma) + \lambda \ell = \lambda \ell\]
- Divide by \(q\), combine with choice conditions

\[\frac{\rho}{1 + \lambda} \frac{1}{1 - \psi} = \lambda \ell \iff 1 - \psi = \frac{\rho}{(1 + \lambda) \lambda \ell} \]
Problem 1 – Model Solution

- From last slide
  \[ 1 - \vartheta = \frac{\rho}{(1 + \lambda) \lambda \ell} \]

- Two possibilities:
  1. Constraint does not bind, then \( \lambda = 0 \Rightarrow 1 - \vartheta = \infty \) → not possible
  2. Constraint binds, then \( \zeta = \ell \vartheta \), thus
    \[ \vartheta = \frac{\rho}{(1 + \lambda) \ell} \]

- Combine two equations to get „price of liquidity“
  \[ 1 = \frac{\rho}{(1 + \lambda) \ell} \left(1 + \frac{1}{\lambda}\right) = \frac{\rho}{\ell \lambda} \Rightarrow \lambda = \frac{\rho}{\ell} \]

- Recover \( \vartheta \) and \( \zeta \)
  \[ \vartheta = \frac{\rho}{\ell + \rho} \quad \zeta = \frac{\rho \ell}{\ell + \rho} \]
Problem 1 – Model Solution

\[ \vartheta = \frac{\rho}{\ell + \rho} \quad \zeta = \frac{\rho \ell}{\ell + \rho} \quad \lambda = \frac{\rho}{\ell} \]

- Remaining steps are standard (see lecture):

\[ \iota = \frac{(1 - \vartheta) a - \zeta}{1 - \vartheta + \kappa \zeta} = \frac{a - \rho}{1 + \kappa \rho} \]

\[ q = (1 - \vartheta) \frac{1 + \kappa a}{1 - \vartheta + \kappa \zeta} = \frac{1 + \kappa a}{1 + \kappa \rho} \]

\[ p = \vartheta \frac{1 + \kappa a}{1 - \vartheta + \kappa \zeta} = \frac{\rho \frac{1 + \kappa a}{\ell}}{1 + \kappa \rho} \]
Problem 1 – How Does $\ell$ Affect Allocations?

- Model Solution

$$\ell = \frac{(1 - \vartheta)a - \zeta}{1 - \vartheta + \kappa \zeta} = \frac{a - \rho}{1 + \kappa \rho}$$

$$q = (1 - \vartheta) \frac{1 + \kappa a}{1 - \vartheta + \kappa \zeta} = \frac{1 + \kappa a}{1 + \kappa \rho}$$

$$p = \vartheta \frac{1 + \kappa a}{1 - \vartheta + \kappa \zeta} = \frac{\rho}{\ell} \frac{1 + \kappa a}{1 + \kappa \rho}$$

- Only $p$ depends on $\ell$, investment and consumption are unaffected
- If velocity doubles, price level doubles
- No crowding out/in of investment

Poll 7: When would this result change?

a) With sticky prices

b) With idiosyncratic risk

c) Never, with CIA money is always neutral
Problem 2
(Multiplicity in Money Model)

▪ Simple I Theory without I (+ simplifying assumption)
\[ \frac{d k_t^i}{k_t^i} = \tilde{\sigma} d \tilde{Z}_t^i, \quad A = 1, \quad \tilde{\sigma}^2 > \rho, \quad \kappa \to \infty \]

▪ Tasks

1. Steady states
2. All deterministic equilibria
3. Tax Backing of Money and Uniqueness
Problem 2
General Equilibrium Conditions

- **Goods Market Clearing**
\[ p + q = \frac{1}{\rho} \]

- **Capital Market Clearing**
\[ 1 - \theta = \frac{q}{p+q} = \rho q \]

- **Price of (idiosyncratic) Risk**
\[ \tilde{\zeta} = \rho q \tilde{\sigma} \]
Problem 2 – Steady States

▪ One Steady State without Money

\[ q = \frac{1}{\rho}, \quad p = 0 \]

▪ Steady States with Money
  ▪ From portfolio choice and market clearing

\[ \frac{1}{q} = \rho q\tilde{\sigma}^2 \Leftrightarrow q^2 = \frac{1}{\rho \tilde{\sigma}^2} \Leftrightarrow q = \pm \frac{1}{\sqrt{\rho \tilde{\sigma}}}. \]

▪ Only positive solution is valid equilibrium

\[ q = \frac{1}{\sqrt{\rho \tilde{\sigma}}}, \quad p = \frac{\tilde{\sigma} - \sqrt{\rho}}{\rho} \]
Problem 2 – Deterministic Equilibria

- Postulate

\[
\frac{dq_t}{q_t} = \mu_q^t dt, \quad \frac{dp_t}{p_t} = \mu_p^t dt
\]

- Return processes and portfolio choice

\[
dr_t^k = \left( \frac{1}{q_t} + \mu_P^t \right) dt + \tilde{\sigma} d\tilde{Z}_t, \quad dr_t^m = \mu_P^t dt
\]

\[
\frac{1}{q_t} + \mu_q^t - \mu_P^t = \rho q_t \tilde{\sigma}^2
\]

- Differentiate goods market clearing \((p + q = \frac{1}{\rho})\)

\[
0 = \dot{p}_t + \dot{q}_t \Rightarrow \frac{\dot{q}_t}{q_t} - \frac{\dot{p}_t}{p_t} = \left( 1 + \frac{q_t}{p_t} \right) \frac{\dot{q}_t}{q_t} = \frac{1}{1 - \rho q_t} \frac{\dot{q}_t}{q_t} = \mu^q_t
\]
Problem 2 - Deterministic Equilibria

- Substitute into portfolio choice condition

\[
\frac{1}{q_t} + \frac{\mu_t^q}{1 - \rho q_t} = \rho q_t \tilde{\sigma}^2
\]

- ... and rearrange

\[
\dot{q}_t = (\rho q_t \tilde{\sigma}^2 - 1) (1 - \rho q_t) = \rho^2 \tilde{\sigma}^2 \left( q_t + \frac{1}{\sqrt{\rho \tilde{\sigma}}} \right) \left( q_t - \frac{1}{\sqrt{\rho \tilde{\sigma}}} \right) \left( \frac{1}{\rho} - q_t \right)
\]
Problem 2 – Deterministic Equilibria
Problem 2 – Deterministic Equilibria

Proposition

- Set of possible initial conditions \((q^0, p^0)\) is

\[
\{(p, q) \mid q \in [\underline{q}, \bar{q}], p = \frac{1}{\rho} - q\}
\]

- For each \((q^0, p^0)\) there is exactly one equilibrium path with \(q_{t_0} = q^0, p_{t_0} = p^0\)

Asymptotic behavior

\[
\lim_{t \to \infty} p_t = \begin{cases} p^*, & p_{t_0} = p^* \\ 0, & \text{otherwise} \end{cases}, \quad \lim_{t \to \infty} q_t = \begin{cases} \frac{q}{\bar{q}}, & q_{t_0} = \bar{q} \\ \frac{q}{\underline{q}}, & \text{otherwise} \end{cases}
\]
Problem 2 – Deterministic Equilibria

- Proof of Proposition (idea)
Problem 2 – Tax Backing

- Government imposes output tax, tax rate $\tau$

- Subsidizes money by
  a) Real dividends to money holders
  b) Shrinking of money supply

- After-tax return on capital
  $$dr_t^k = \left( \frac{1 - \tau}{q_t} + \mu_t^q \right) dt$$

- Return on money
  - Policy a)
    - dividend yield
    $$\frac{\tau K}{p_t K} dt = \frac{\tau}{p_t} dt$$
    - Capital gains
    $$\frac{d(p_t K_t)}{p_t K_t} = \mu_t^p dt$$
  - Policy b)
    - dividend yield
    0
    - capital gains
    $$\frac{dp_t^m}{p_t^m} = \frac{dp_t}{p_t} - \frac{dM_t}{M_t} = \left( \frac{\tau}{p_t} + \mu_t^p \right) dt$$

\[ dM_t = -\frac{\tau K}{p_t^m} dt = -\frac{\tau}{p_t} M_t dt \]
Problem 2 – Tax Backing

- **Asset Pricing Condition**

\[
\frac{1 - \tau}{q_t} - \frac{\tau}{p_t} + \mu_t^q - \mu_t^p = \rho q_t \tilde{\sigma}
\]

- **Equilibrium ODE**

\[
\dot{q}_t = \rho^2 \tilde{\sigma}^2 \left( q_t + \frac{1}{\sqrt{\rho \tilde{\sigma}}} \right) \left( q_t - \frac{1}{\sqrt{\rho \tilde{\sigma}}} \right) \left( \frac{1}{\rho} - q_t \right) + \tau (1 - \rho q_t) + \frac{\tau}{p_t} q_t (1 - \rho q_t)
\]

\[
\tau (1 - \rho q_t) + \frac{\tau}{p_t} q_t (1 - \rho q_t) = \tau \frac{p_t}{p_t + q_t} + \frac{\tau}{p_t} q_t \frac{p_t}{p_t + q_t} = \tau
\]
Problem 2 – Tax Backing
Uniqueness of Money Steady State

\[ \dot{q}_t = \rho^2 \tilde{\sigma}^2 \left( q_t + \frac{1}{\sqrt{\rho \tilde{\sigma}}} \right) \left( q_t - \frac{1}{\sqrt{\rho \tilde{\sigma}}} \right) \left( \frac{1}{\rho} - q_t \right) + \tau \]
Problem 2 – Can Tax Backing Achieve Uniqueness without ever Taxing?

- Capital taxation to back the money stock may be undesirable from a welfare perspective (in this model for high idiosyncratic risk want to inflate/subsidize capital, compare Brunnermeier, Sannikov 2016)

- If government can credibly commit to future taxation:
  - Start taxing as soon as value of money falls below some threshold $\hat{p} \in (0, p^*)$:
    \[
    \tau(p, q) = \begin{cases} 
    0, & p \geq \hat{p} \\
    \bar{\tau}, & p < \hat{p}
    \end{cases}
    \]
  - Eliminates all equilibrium paths with $p < \hat{p}$ (same argument as before)
  - Sufficient to eliminate all equilibria other than $(q, p^*)$