Macro, Money and Finance
Problem Sets 4 – Solutions (selective)
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Problem 2 – Bank Money Creation

- Money model with CIA constraint in production (as Problem 3 last week)

- Now we add money-creating banks:
  - Bankers with wealth share $\eta_t$
  - Can invest in money and loans (return $d\ell_t$)
  - Issue deposits (return $d_d^t$)
  - Deposits relax household CIA constraint

\[
\alpha \cdot a \frac{\theta_t^{k,h,i} n_t^{h,i}}{q_t^K} \leq \left( \theta_t^{m,h,i} + \theta_t^{d,h,i} \right) n_t^{h,i}
\]

instead of

\[
\alpha \cdot a \frac{\theta_t^{k,i} n_t^{i}}{q_t^K} \leq \theta_t^{m,i} n_t^{i}
\]
Part 1 – Formal Model Description

- Return processes
  - Money
    \[ dr_t^m = \left( \Phi(\nu_t) - \delta + \mu^q_tM \right) dt + \sigma dZ_t \]

- Deposits and Loans
  \[ dr_t^d = i_t^d dt + dr_t^m, \quad dr_t^{\ell,i} = i_t^\ell dt + dr_t^m + j_t^\ell d\tilde{J}_t \]

- Net worth evolutions
  - Bankers
    \[ \frac{dn_{t}^{b,i}}{n_{t}^{b,i}} = -\frac{c_{t}^{b,i}}{n_{t}^{b,i}} dt + \theta_{t}^{m,b,i} dr_t^m + \theta_{t}^{\ell,b,i} dr_t^{\ell,i} + \theta_{t}^{d,b,i} dr_t^d \]
  - Households
    \[ \frac{dn_{t}^{h,i}}{n_{t}^{h,i}} = -\frac{c_{t}^{h,i}}{n_{t}^{h,i}} dt + \theta_{t}^{m,h,i} dr_t^m + \theta_{t}^{d,h,i} dr_t^d + \theta_{t}^{k,h,i} dr_t^k + \theta_{t}^{\ell,h,i} dr_t^{\ell,i} \]
Part 1 – Agent Problems

- **Bankers**
  - Maximize utility: \(\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_{t,\theta}^b,\theta \, dt\right]\)
  - Choose: \(\{c_{t,\theta}^b,\theta\}, \{\theta_{t,\theta}^m,b,\theta\}, \{\theta_{t,\theta}^d,b,\theta\}, \{\theta_{t,\theta}^l,b,\theta\}\)
  - s.t.
    - net worth evolution
    - \(\theta_{t,\theta}^m,b,\theta + \theta_{t,\theta}^d,b,\theta + \theta_{t,\theta}^l,b,\theta = 1\)
    - \(\theta_{t,\theta}^m,b,\theta \geq 0\)

- **Households**
  - Maximize utility: \(\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_{t,\theta}^h,\theta \, dt\right]\)
  - Choose: \(\{c_{t,\theta}^h,\theta\}, \{\theta_{t,\theta}^m,h,\theta\}, \{\theta_{t,\theta}^d,h,\theta\}, \{\theta_{t,\theta}^l,h,\theta\}, \{\theta_{t,\theta}^k,h,\theta\}\)
  - s.t.
    - net worth evolution
    - \(\theta_{t,\theta}^m,h,\theta + \theta_{t,\theta}^d,h,\theta + \theta_{t,\theta}^l,h,\theta + \theta_{t,\theta}^k,h,\theta = 1\)
    - \(\theta_{t,\theta}^m,h,\theta, \theta_{t,\theta}^k,h,\theta, \theta_{t,\theta}^d,h,\theta \geq 0\)
    - cash-in-advance constraint
Part 2 – Frictionless Case (no CIA)

- Banker portfolio choice deposits vs loans
  \[ i_t^d = i_t^\ell \]

- Banker and Household choice deposits vs money
  \[ i_t^d \geq 0 \]
  ... and with “\( = \)”, if the agent holds money
  - Someone has to hold money in equilibrium \( \Rightarrow i_t^d = 0 \)

- Conclusions from these preliminary considerations
  - \( dr_t^m = dr_t^\ell = dr_t^d \)
  - The three nominal assets are perfect substitutes for all agents
    \( \Rightarrow \) gross quantities of deposits and loans are not determined by equilibrium conditions (can always add more deposits and back with additional loans)
Part 2 – Frictionless Case – Money Valuation

- Derivation of money valuation equation is standard
- Use loans as benchmark asset
- Portfolio choice of net worth relative to loans

\[ \mu^n + \rho - \mu^\vartheta_t - i_t^\ell = 0 \]

\[ \mu^{1-n} + \rho - \mu^\vartheta_t - i_t^\ell = \zeta_t \sigma_{t,h} \]

- Take \( \eta \)-weighted average, rearrange

\[ \mu^\vartheta_t = \rho - \frac{1}{1 - \eta_t} (1 - \vartheta_t)^2 \tilde{\sigma}^2 - i_t^\ell \]

- Combine with first equation for \( \eta \) drift

\[ \mu^n_t = -\frac{1}{1 - \eta_t} (1 - \vartheta_t)^2 \tilde{\sigma}^2 \]

here \( i_t^\ell = 0 \)
Part 2 – Frictionless Case – Steady State

- Recall:
  \[ \mu_t^\eta = -\frac{1}{1 - \eta_t} \left(1 - \vartheta_t\right)^2 \tilde{\sigma}^2 \]

- This is always negative!
  \[ \Rightarrow \text{steady state must be at } \eta = 0 \]

- Then steady-state money valuation equation is standard
  \[ \rho = (1 - \vartheta)^2 \tilde{\sigma}^2 \]
  \[ \Rightarrow \vartheta = \frac{\tilde{\sigma} - \sqrt{\rho}}{\tilde{\sigma}} \]

- Get same equilibrium as in Lecture 5
Part 3 – Risk-free Loans

- Now add CIA constraint, but loans remain risk-free
- Banker portfolio choice still implies \( i_t^d = i_t^\ell \)

For households now

\[
\nu_t^\ell - \nu_t^d = -\lambda_t^h (\nu_t^\ell - \nu_t^d) = \lambda_t^h \frac{1}{\alpha} \geq 0
\]

Where (compare last week)

- \( \lambda^h \): price of transaction services (for households)
- \( \nu^d = \frac{1}{\alpha} \): transaction services provided by deposits
- \( \nu^\ell = 0 \): transaction services provided by loans

- Combining the two conditions: \( \lambda_t^h = 0 \)
  - As long as \( i_t^\ell - i_t^d > 0 \), bankers issue deposits to make more loans and earn a spread
  - In equilibrium, bankers create so much deposit money that CIA is slack
Part 3 – Risk-free Loans – Balance Sheet Size

- Bank balance sheet minimal, if outside money held by households

- Cash-in-advance constraint requires

\[
\left( \frac{\vartheta_t}{1 - \eta_t} + \theta_{t}^{d,h,i} \right) \geq \alpha \cdot a \frac{1 - \vartheta_t}{1 - \eta_t} \quad \iff \quad (1 - \eta_t) \theta_{t}^{d,h,i} \geq \alpha \cdot a \frac{1}{q_t^K + q_t^M} - \vartheta_t
\]

- Multiply by net worth and integrate

\[
(1 - \eta_t) \int \theta_{t}^{d,h,i} n_{t}^h h_{i}^{h,i} d\tilde{i} \geq \alpha \cdot a \frac{N_t^h}{q_t^K + q_t^M} - \vartheta_t N_t^h = \alpha \cdot a (1 - \eta_t) K_t - (1 - \eta_t) q_t^M K_t
\]

- Thus (using known \( q^M \) expression)

\[
N_t^b + D_t \geq \left( \alpha a + (\eta_t - \vartheta_t) \frac{1 + \phi a}{1 - \vartheta_t + \phi \rho} \right) K_t
\]
Part 3 – Risk-free Loans – Allocation Impact of CIA Constraint

- Does the CIA constraint have any impact on the equilibrium allocation?
  - No, multiplier is zero, $\lambda^h_t = 0$
  - Agents would make same choices if constraint was absent

- Conclusions:
  - again same allocation as in Lecture 5 model
  - in presence of money-creating banks, medium-of-exchange role of money not relevant for determining value of outside money
  - from perspective of this model: BruSan are right to emphasize store of value role of money
Parts 4 and 5

- Why may banks not create arbitrary quantities of money in reality?
  1. Regulation, e.g.
     a) leverage constraints
     b) reserve requirements
  2. Limited competition
     (do not compete spread $i_t^e - i_t^d$ all the way to zero)
  3. Bank assets are risky

- Want to explore 1b) and 3 in parts 4 and 5
Part 4 – Reserve Requirements

- Change: bankers must hold $\psi \in [0,1]$ units of outside money (required reserves) for each unit of deposits they issue

- Adds a portfolio constraint

\[-\psi \theta_{t,d,b,i} \leq \theta_{t,m,b,i}\]

- Questions:
  1. When does the requirement matter? (parameter condition)
  2. Steady state values of $\eta$ and $\vartheta$?
  3. Show that this leads to the “money multiplier model”
Part 4 – When Does Requirement Matter?

- For $\psi < 1$, it is efficient for banks to hold all outside money.
- Then aggregate version of reserve requirement is:
  \[ \psi D_t \leq q_t^M K_t \]
- In equilibrium from part 3, $D_t$ must be large to satisfy CIA:
  \[ D_t \geq \alpha \cdot aK_t \]
- So requirement does not matter if $\left[ \alpha a, \frac{q_t^M}{\psi} \right] \neq \emptyset$ (for $q_t^M$ as in parts 2-3).
- Conversely: requirement matters if
  \[ \psi \alpha \cdot aK_t > q_t^M K_t \iff \psi \alpha a > \vartheta \frac{1 + \phi a}{1 - \vartheta + \phi \rho} \]
Part 4 – Steady State – Money Valuation

- Can still use loans as benchmark asset (does not enter CIA, so no additional terms relative to part 2)
- Then money valuation and $\eta$ drift are as before

$$\mu_t^g = \rho - \frac{1}{1 - \eta_t} (1 - \vartheta_t)^2 \tilde{\sigma}^2 - i_t$$

$$\mu_t^\eta = -\frac{1}{1 - \eta_t} (1 - \vartheta_t)^2 \tilde{\sigma}^2$$

- Again, $\eta = 0$ in steady state (bankers do not need net worth to create money)
Part 4 – Steady State – Deposit Rate

- Steady-state version of money valuation equation
  \[ \rho = (1 - \vartheta)^2 \tilde{\sigma}^2 + \dot{i}^l \]

- Need to determine loan rate
  - Household portfolio choice (loans vs deposits)
    \[ i^l - i^d = -\lambda^h (v^l - v^d) = \lambda^h \frac{1}{\alpha} \]
  - Banker self-financing strategy:
    - 1 additional unit of deposits
    - invest in \( \psi \) units of money and \( 1 - \psi \) units of loans
      \[ (1 - \psi)i^l - i^d = 0 \]
  - Combine the two:
    \[ i^l = \frac{1}{\psi} (i^l - i^d) = \lambda^h \frac{1}{\psi \alpha} \]
Part 4 – Steady State Solution

- Substitute into money valuation equation

\[ \rho = (1 - \vartheta)^2 \tilde{\sigma}^2 + \lambda^h \frac{1}{\psi \alpha} \]

- This looks like equation from last week
  (with \(\psi \alpha\) instead of \(\alpha\))

- We solve it the same way:
  - Either constraint slack, then valuation equation determines \(\vartheta\)
  - Or constraint binds, then CIA

\[
\alpha \cdot a \frac{1 - \vartheta}{q^K} = \frac{D}{N^h} \iff \alpha \cdot a \frac{1}{q^M} = \frac{D}{q^M K} \iff \alpha \cdot \frac{a}{1 + \phi a} \frac{1 - \vartheta + \phi \rho}{\vartheta} = \frac{D}{q^M K_t} \\
\vartheta = \frac{\alpha a (1 + \phi \rho)}{\frac{D}{q^M K} (1 + \phi a) + \alpha a} \iff \vartheta = \frac{\psi \alpha a (1 + \phi \rho)}{1 + a (\psi \alpha + \phi)}
\]
Part 4 – Money Multiplier Model

- When constraint is binding,

\[ i^d = (1 - \psi) i^e = (1 - \psi) \lambda^h \frac{1}{\psi \alpha} \]

- This is positive, so households strictly prefer to hold deposits \( \Rightarrow \) all outside money most be held by banks

- Reserve requirement must be binding for banks
  (otherwise they are unwilling to pay \( i^d_t > 0 \) while earning 0 on excess reserves)

- Integrating reserve requirement yields

\[ \psi D_t = \int \psi \theta_{t}^{d, b, i} n_{t}^{b, i} d\bar{i} = \int \theta_{t}^{m, b, i} n_{t}^{b, i} d\bar{i} = q_M K_t \]
Part 5 – Risky Loans

- Changes:
  - No reserve requirement
  - But default risk in loans
  - At idiosyncratic jump times, $d\tilde{J}_t$, household $\tilde{i}$ can walk away from fraction $\hat{\beta}$ ($= 1 - \beta$ in problem set) of debt
  - Bankers cannot diversify (lose fraction $\hat{\beta}$ of loan value)

- New loan return
  \[
  dr_t^l = i_t^l dt + dr_t^m - \hat{\beta} d\tilde{J}_t
  \]

- Questions:
  1. Money valuation equation and $\eta$ evolution?
  2. Characterize steady state
  3. $\hat{\beta}$ comparative statics (numerically)
Part 5 – Money Valuation

- Use now deposits as benchmark asset
  \[ \mu_t^{\eta} + \rho - \mu_t^{\vartheta} - i_t^d + \tilde{\lambda} j_t^{n,b} = \tilde{\lambda} \tilde{\nu}_t^b \tilde{\gamma}_t^{n,b}, \]
  \[ \mu_t^{1-\eta} + \rho - \mu_t^{\vartheta} - i_t^d + \tilde{\lambda} j_t^{n,h} = \tilde{\zeta}_t^h \tilde{\sigma}_t^{n,h} + \tilde{\lambda} \tilde{\nu}_t^h \tilde{\gamma}_t^{n,h} + \lambda t \frac{1}{\alpha} \]

- \( \eta \)-weighted average
  \[ \mu_t^{\vartheta} = \rho - \left( (1 - \eta_t) \tilde{\zeta}_t^h \tilde{\sigma}_t^{n,h} + \eta_t \tilde{\lambda} \left( \tilde{\nu}_t^b - 1 \right) \tilde{j}_t^{n,b} + (1 - \eta_t) \tilde{\lambda} \left( \tilde{\nu}_t^h - 1 \right) \tilde{j}_t^{n,h} + (1 - \eta_t) \lambda t \frac{1}{\alpha} \right) \]

- Substitue back into first equation
  \[ \mu_t^{\eta} = (1 - \eta_t) \left( \tilde{\lambda} \left( \tilde{\nu}_t^b - 1 \right) \tilde{j}_t^{n,b} - \tilde{\zeta}_t^h \tilde{\sigma}_t^{n,h} - \tilde{\lambda} \left( \tilde{\nu}_t^h - 1 \right) \tilde{j}_t^{n,h} - \lambda t \frac{1}{\alpha} \right) \]
Part 5 – Money Premium

- Need to determine risk premia and money premium
- Start with money premium

- Banker portfolio choice loans vs deposits
  \[ i_t^l - \tilde{\lambda} \hat{\beta} - i_t^d = -\tilde{\lambda} \tilde{\nu}_t^b \hat{\beta} \]

- Household portfolio choice loans vs deposits
  \[ i_t^l - \tilde{\lambda} \hat{\beta} - i_t^d = -\tilde{\lambda} \tilde{\nu}_t^h \hat{\beta} + \lambda_t \frac{1}{\alpha} \]

- Combine
  \[ \lambda_t \frac{1}{\alpha} = \tilde{\lambda} \hat{\beta} (\tilde{\nu}_t^h - \tilde{\nu}_t^b) \]

- So money premium related to jump risk premium differential
Part 5 – Brownian risk premium

- Brownian risk premium is standard

- Log utility, so \( \tilde{\zeta}_t^h = \tilde{\sigma}_t^{n,h} = \theta_t^{k,h} \tilde{\sigma} \)

- Capital market clearing: \( \theta_t^{k,h} = \frac{1-\vartheta_t}{1-\eta_t} \)

- Thus: \( \tilde{\zeta}_t^h \tilde{\sigma}_t^{n,h} = \left( \frac{1-\vartheta_t}{1-\eta_t} \tilde{\sigma} \right)^2 \)
Part 5 – Jump risk premia

- Price of jump risk (generic formula)
  \[ 1 - \tilde{\nu}_t^i = 1 + \tilde{j}_t^{\xi, i} = 1 + \tilde{j}_t^{1/n, i} = \frac{1}{1 + \tilde{j}_t^{n, i}} \Rightarrow \tilde{\nu}_t^i = \frac{\tilde{j}_t^{n, i}}{1 + \tilde{j}_t^{n, i}} \]

- Net worth jumps
  \[ \tilde{j}_t^{n, b} = -\theta_t^{\ell, b} \hat{\beta} \]
  \[ \tilde{j}_t^{n, h} = \frac{\eta_t \theta_t^{\ell, b}}{1 - \eta_t} \hat{\beta} \]

- Prices of jump risk (here)
  \[ \tilde{\nu}_t^b = \frac{-\theta_t^{\ell, b} \hat{\beta}}{1 - \theta_t^{\ell, b} \hat{\beta}} \]
  \[ \tilde{\nu}_t^h = \frac{\eta_t \theta_t^{\ell, b} \hat{\beta}}{1 - \eta_t + \eta_t \theta_t^{\ell, b} \hat{\beta}} \]
Part 5 – Money Valuation and $\eta$ Drift

- Substitute everything in $\mu^\vartheta$ and $\mu^\eta$ equations

\[
\mu^\vartheta_t = \rho - \left( \frac{1}{1 - \eta_t} \right) \left[ (1 - \vartheta_t)^2 \tilde{\sigma}^2 + \left( \frac{\eta_t + (1 - \eta_t) \hat{\beta}}{1 - \theta_t^{\ell,b} \hat{\beta}} - \frac{(1 - \eta_t) \eta_t \left( 1 - \hat{\beta} \right)}{1 - \eta_t + \eta_t \theta_t^{\ell,b} \hat{\beta}} \right) \tilde{\lambda} \theta_t^{\ell,b} \hat{\beta} \right]
\]

\[
\mu^\eta_t = (1 - \eta_t) \left( \left[ \frac{1}{1 - \theta_t^{\ell,b} \hat{\beta}} + \frac{\eta_t}{1 - \eta_t + \eta_t \theta_t^{\ell,b} \hat{\beta}} \right] \tilde{\lambda} \left( 1 - \hat{\beta} \right) \theta_t^{\ell,b} \hat{\beta} - \frac{(1 - \vartheta_t)^2}{(1 - \eta_t)^2} \tilde{\sigma}^2 \right)
\]
Part 5 – Steady State

- Recall banker net worth pricing condition

\[ \mu_t^\eta + \rho - \mu_t^\vartheta - i_t^d + \tilde{\lambda} j_t^{n,b} = \tilde{\lambda} \tilde{\nu}_t^b \tilde{j}_t^{n,b} \]

- In steady state

\[ \rho = \tilde{\lambda} (\tilde{\nu}^b - 1) \tilde{j}_t^{n,b} = -\tilde{\lambda} \tilde{\nu}^b \]

- Thus:

\[ \tilde{\nu}^b = -\frac{\rho}{\tilde{\lambda}} \Rightarrow \tilde{j}_t^{n,b} = -\frac{\rho}{\tilde{\lambda} + \rho} \]

- For this jump exposure, need \( \theta_t^{\ell,b} = \frac{\rho}{\tilde{\lambda} + \rho} \cdot \frac{1}{\tilde{\beta}} \)
Part 5 – Steady State

- Next: binding CIA

\[
\alpha \cdot a \frac{\theta^k, h}{q^K} = \theta^m, h + \theta^d, h \iff \alpha \cdot a \frac{\frac{1-\vartheta}{1-\eta}}{q^K} = 1 - \frac{1 - \vartheta}{1 - \eta} + \frac{\eta \theta^\ell, b}{1 - \eta}
\]

\[
\iff \frac{\alpha a}{1 + \phi a} (1 - \vartheta + \phi \rho) = \vartheta + \eta \frac{\rho}{\lambda + \rho \beta} + \frac{1}{1 + \phi a}
\]

- Solve for \(1 - \vartheta\)

\[
1 - \vartheta = \frac{1 + \phi a}{1 + (\phi + \alpha) a} \left(1 + \eta \frac{\rho}{\lambda + \rho \beta}\right) - \frac{\alpha \phi a}{1 + (\phi + \alpha) a} \rho
\]
Part 5 – Steady State

- Final step: plug everything into steady-state money valuation equation

\[ \rho = \frac{1}{(1 - \eta)} \left( \frac{1 + \phi a}{1 + (\phi + \alpha) a} \left( 1 + \eta \frac{\rho}{\bar{\lambda} + \rho \beta} \right) - \frac{\alpha \phi a}{1 + (\phi + \alpha) a \rho} \right)^2 \bar{\sigma}^2 \]

\[ + \left( \frac{\eta + (1 - \eta) \beta}{1 - \theta^{e,b} \beta} - \frac{(1 - \eta) \eta \left( 1 - \beta \right)}{1 - \eta + \eta \theta^{e,b} \beta} \right) \bar{\lambda} \theta^{e,b} \hat{\beta} \]

- Can solve this numerically for \( \eta \ldots \)
- Note: if there is no positive solution, then \( \eta = 0 \) (happens for large \( \hat{\beta} \))
Part 5 – Numerical Results

\begin{align*}
\eta & \quad 0 & 0.1 & 0.2 & 0.3 & 0.4 \\
\hat{\beta} & \quad 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\end{align*}

\begin{align*}
q^K & \quad 1.178 & 1.18 & 1.182 & 1.184 \\
\hat{\beta} & \quad 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\end{align*}

\begin{align*}
\text{money multiplier} & \quad 1 & 1.5 & 2 & 2.5 & 3 \\
\hat{\beta} & \quad 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\end{align*}

\begin{align*}
q^M & \quad 0.4 & 0.6 & 0.8 & 1 \\
\hat{\beta} & \quad 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\end{align*}