Financial and Monetary Economics
Eco529 Fall 2020
Lecture 04: Jumps and Runs

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Jumps due to multiple equilibria

- Bank runs
  - Diamond Dybvig

- Liquidity spirals
  - Brunnermeier Pedersen

- Sudden stops
  - Calvo, Mendoza, ...

- Currency attacks
  - Obstfeld (2nd generation models), Morris Shin

- Twin crisis models
  - Kaminsky Reinhart (3rd generation models)

- Loss of safe asset status
  - (after introducing safe asset in world with idiosyncratic risk)
Recall: Endogenous Risk due to Amplification

Initial exogenous shock $\sigma dZ_t$ /trigger

$i$’s best response

Run-up

amplification

shock

others’ average actions
Recall: Endogenous Risk due to **Multiple Equilibria Jumps**

- No exogenous shock, but sunspot process
- Higher strategic complementarities
Two Type/Sector Model with Outside Equity

- **Expert sector**
  - Experts must hold fraction $\chi_t^e \geq \alpha \kappa_t^e$ (skin in the game constraint)
  - Return on inside equity $N_t^e$ can differ from outside equity
    - Issue outside equity at required return from HH
    - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return
Two Type Model Setup

Expert sector

- Output: \( y_t^e = a^e k_t^e \), \( a^e \geq a^h \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( i_t^e \)
  \[
  \frac{dk_t^{i,e}}{k_t^{i,e}} = \left( \Phi \left( i_t^{i,e} \right) - \delta \right) dt + \sigma dZ_t + \bar{\sigma} d\tilde{Z}_t^i
  \]
- \( E_0 \left[ \int_0^\infty e^{-\rho^e_t \frac{(c_t^e)^{1-\gamma}}{1-\gamma}} dt \right] \), \( \rho^e \geq \rho^h \)

Friction: Can only issue

- Risk-free debt
- Equity, but most hold \( \chi_t^e \geq \alpha \kappa_t \)

Household sector

- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( i_t^h \)
  \[
  \frac{dk_t^{i,h}}{k_t^{i,h}} = \left( \Phi \left( i_t^{i,h} \right) - \delta \right) dt + \sigma dZ_t + \bar{\sigma} d\tilde{Z}_t^i
  \]
- \( E_0 \left[ \int_0^\infty e^{-\rho^h_t \frac{(c_t^h)^{1-\gamma}}{1-\gamma}} dt \right] \)
Unanticipated Run on Experts

- Can unanticipated withdrawal of all experts’ funding be self-fulfilling?
- Unanticipated crash – jump to $\eta^e = 0$
  - Absent a run: solution as in Lecture 03, since unanticipated
  - When do jump capital losses wipe out experts net worth?

$$\left(q(\eta^e_t) - q(0)\right) \left(\theta_{t,K}^e + \theta_{t,OE}^e\right) \eta^e_t K_t \geq \eta^e_t q(\eta^e_t) K_t$$

$$q(\eta^e_t) \left(1 - \frac{\eta^e_t}{\chi^e(\eta^e_t)}\right) \geq q(0) \quad \text{or} \quad q(\eta^e_t) \left(1 - \frac{1}{\theta_{t,K}^e + \theta_{t,OE}^e}\right) \geq q(0)$$

- Vulnerability region:
  - High price (not very low $\eta^e$)
  - “high risk-leverage” (not very high $\eta^e$)

- After run: $\eta^0 = 0$ forever
2 Types of Runs and Modeling Challenges

- What type of run? What’s the trigger?
  - Funding supply run: Depositor/households run
    - Household withdraw funding to experts
  - Funding demand run: Other experts run
    - Each expert tries to pay back debt and fire-sells assets
    - Drop in \( q \) is driver

- Model advantage: Always jump to the same point \( q(\eta^e = 0) \)!

- Modeling Challenges: (see Mendo (2020))
  1. Experts are whipped out forever.
     - OLG structure:
       - Death: all agents die with Poisson rate \( \lambda^d \),
       - Birth: fraction \( \psi \) of newborns are experts
  2. With anticipated run, expert fear “infinite marginal utility state” \( \eta^e = 0 \).
     - Transfer of \( \tau K \) to bankrupt experts after run
     - Also fixes challenge 1.
     - To keep \( \tau \) small also introduce relative performance penalty
From Ito to Levy and Cox Processes

- Ito process: \( dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t \) (geometric)
  - the Brownian “shocks” \( dZ_t \) are i.i.d. and small s.t. continuous path
  - For non-normal shocks within \( dt \) one needs discontinuities

- Levy process: \( dL_t = a dt + b dZ_t + dJ_t \) – most general class with i.i.d. increments
  \( dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t + j_t^X X_t dJ_t \)

- Restrict attention to Poisson processes:
  - Levy jump process can be written as integral w.r.t. Poisson random measures
  - Poisson process with arrival rate \( \lambda > 0 \):
    - \( J \) takes on values in \( \mathbb{N}_0 = \{0,1,2, \ldots \} \)
    - Increments \( J_{t+\Delta t} - J_t \) are Poisson distributed with Parameter \( \lambda \Delta t \)
    - Stochastic integral w.r.t. Poisson process simply sums up the values of the integrand
      \( \int_0^T a_t J_t = \sum_{n=1}^{J_T} a_{\tau_n} \)
    - Cox process: \( \lambda_t \) can be time-varying
    - Compensated Jump process: \( J_t - \int_0^t \lambda_s ds \) is martingale
    - If \( \int_0^t a_s dJ_s \) and \( a_t \) uses info only up to right before \( t \) then \( J_t - \int_0^t a_s \lambda_s ds \) is martingale
Ito formulas

\[ df(X_t) = f'(X_t)(\mu_t^X X_t dt + \sigma_t^X X_t dZ_t) + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + (f(X_t) - f(X_t-))dJ_t \]

\[ = \left( f'(X_t)\mu_t^X X_t + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 \right) dt + f'(X_t)\sigma_t^X X_t dZ_t + \left( f \left( (1 + j_t^X)X_t- \right) - f(X_t-) \right) dJ_t \]

- **Power rule:**
  \[ \frac{dX_t^\gamma}{X_t^\gamma} = \left( \gamma \mu_t^X + \gamma (\gamma - 1)(\sigma_t^X)^2 \right) dt + \gamma \sigma_t^X dZ_t + \left( (1 + j_t^X)^\gamma - 1 \right) dJ_t \]

- **Product rule:**
  \[ \frac{d(X_t Y_t)}{X_t - Y_t -} = \left( \mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y \right) dt + \left( \sigma_t^X + \sigma_t^Y \right) dZ_t + \left( j_t^X + j_t^Y + j_t^X j_t^Y \right) dJ_t \]

- **Quotient rule:**
  \[ \frac{d(X_t / Y_t)}{X_t - Y_t -} = \left( \mu_t^X - \mu_t^Y + (\sigma_t^Y)^2 - \sigma_t^X \sigma_t^Y \right) dt + \left( \sigma_t^X - \sigma_t^Y \right) dZ_t + \frac{j_t^X - j_t^Y}{1 + j_t^Y} dJ_t \]

- **Memorize simple rules:**
  - \[ 1 + j_t^X = (1 + j_t^X)^Y \]
  - \[ 1 + j_t^{XY} = (1 + j_t^X)(1 + j_t^Y) \]
  - \[ 1 + j_t^{X/Y} = \frac{1 + j_t^X}{1 + j_t^Y} \]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $\nu^i(\eta)u(K)$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $\nu^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
0. Postulate Aggregates and Processes

- **Individual capital evolution:**

\[
\frac{dk_{t}^{i,i}}{k_{t}^{i,i}} = (\Phi(i^{i,i}) - \delta)dt + \sigma dZ_t + d\Delta_{t}^{k,i,i}
\]

- Where \(\Delta_{t}^{k,i,i}\) is the individual cumulative capital purchase process

- **Capital aggregation:**
  - Within sector \(i\):
    \[K_{t}^{i} \equiv \int k_{t}^{i,i} d\bar{i}\]
  - Across sectors:
    \[K_{t} \equiv \sum_{i} K_{t}^{i}\]
  - Capital share:
    \[\kappa_{t}^{i} \equiv \frac{K_{t}^{i}}{K_{t}}\]

\[
\frac{dK_{t}}{K_{t}} = (\Phi(i_{t}) - \delta)dt + \sigma dZ_t
\]

- **Net worth aggregation:**
  - Within sector \(i\):
    \[N_{t}^{i} \equiv \int n_{t}^{i,i} d\bar{i}\]
  - Across sectors:
    \[N_{t} \equiv \sum_{i} N_{t}^{i}\]
  - Wealth share:
    \[\eta_{t}^{i} \equiv \frac{N_{t}^{i}}{N_{t}}\]

- **Value of capital stock:**

\[dq_{t}/q_{t} = \mu_{t}^{q}dt + \sigma_{t}^{q}dZ_{t} + j_{t}^{q}dJ_{t}\]

(c is numeraire)
0. Postulate Aggregates and Processes

- **Individual capital evolution:**
  \[
  \frac{dk_t^{i,i}}{k_t^{i,i}} = (\Phi\left(i_t^{i,i}\right) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,i,i}
  \]
  - Where \(\Delta_t^{k,i,i}\) is the individual cumulative capital purchase process

- **Capital aggregation:**
  - Within sector \(i\):
    \[
    K_t^i \equiv \int k_t^{i,i} d\bar{i}
    \]
  - Across sectors:
    \[
    K_t \equiv \sum_i K_t^i
    \]
  - Capital share:
    \[
    \frac{dK_t}{K_t} = (\Phi\left(i_t\right) - \delta)dt + \sigma dZ_t
    \]

- **Net worth aggregation:**
  - Within sector \(i\):
    \[
    N_t^i \equiv \int n_t^{i,i} d\bar{i}
    \]
  - Across sectors:
    \[
    N_t \equiv \sum_i N_t^i
    \]
  - Wealth share:
    \[
    \eta_t^i \equiv N_t^i / N_t
    \]

- **Value of capital stock:**
  \[
  dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t + j_t^q dJ_t
  \]
  - Postulate
    \[
    \frac{d\xi_t^i}{\xi_t^i} = \mu_t^{\xi^i} dt + \sigma_t^{\xi^i} dZ_t + j_t^{\xi^i} (dJ_t - \lambda_t dt)
    \]
    \(c\) is numeraire

- **Postulated SDF-process:**
  \[
  \frac{d\bar{\xi}_t^i}{\bar{\xi}_t^i} = \mu_t^{\bar{\xi}_t^i} dt + \sigma_t^{\bar{\xi}_t^i} dZ_t + j_t^{\bar{\xi}_t^i} (dJ_t - \lambda_t dt)
  \]
  \(c\) is numeraire

Since only risky debt and not risk-free debt is traded.
0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Ito)
- Use Ito product rule to obtain capital gain rate \( (\text{in absence of purchases/sales}) \)
  - Define \( \tilde{k}^i_t \): \( \frac{d\tilde{k}^i_t}{\tilde{k}^i_t} = \left( \Phi \left( l^i_t \right) - \delta \right) dt + \sigma dZ_t + d\Delta \tilde{k}^i_t \)

\[
dr^k_t \left( l^i_t \right) = \left( \frac{a^i - l^i_t}{q} + \Phi \left( l^i_t \right) - \delta + \mu^q_t + \sigma^q \right) dt \\
+ \left( \sigma + \sigma^q \right) dZ_t + j^q_t dJ_t
\]

- Return on defaultable debt

\[
dr^D_t = r^i_t dt + j^D,^i_t dJ_t
\]

- Postulate SDF-process: (Example: \( \xi^i_t = e^{-\rho t}V'(n^i_t) \))

\[
\frac{d\xi^i_t}{\xi_t} = -r^F,^i_t dt - \xi^i_t dZ_t - \gamma^i_t \left( dJ_t - \lambda_t dt \right)
\]
The Big Picture

- allocation of physical assets
- output $A(\kappa)$
- consumption + investment
- net worth distribution $\eta$
- capital growth $\Phi(\iota) - \delta$
- debt
- outside equity
- value function

Precautionary

Backward equation

Forward equation with expectations

Kappa

Drift

Volatility
Solving MacroModels Step-by-Step

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2. Evolution of state variable \( \eta \) (and \( K \)) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities \( \omega \)
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. \( V^i(n^i; \eta, K) \) into \( v^i(\eta)u(K) \)
   c. Derive \( C/N \)-ratio and \( \zeta \) price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. \( v^i(\eta) \) into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1a. Individual Agent Choice of $l, \theta, c/n$

- Choice of $l$ is static problem (and separable) for each $t$

- \[
\max_{l_t^i} dr^k_t(l_t^i)
= \max_{l_t^i} \left( \frac{a^i - l_t^i}{q_t} + \Phi(l_t^i) - \delta + \mu^q + \sigma \sigma^q \right) + (\sigma + \sigma_t^q) dZ_t + j_t^q dJ_t
\]

- FOC: $\frac{1}{q_t} = \Phi'(l_t^i)$ Tobin’s $q$
  - All agents $l_t^i = l_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(l_t) - \delta) dt + \sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota) = \frac{1}{\phi} \log(\phi \iota + 1) \Rightarrow \phi \iota = q - 1$

- Goods market clearing: $(A(\kappa) - l_t)K_t = \sum_i C_i$
  \[
  \kappa_t a^e K_t + (1 - \kappa_t) a^h K_t - l(q_t)K_t = \eta^e_t \frac{c_t^e}{N_t^e} q_t K_t + (1 - \eta^e_t) \frac{c_t^h}{N_t^h} q_t K_t
  \]
  For aggregate capital return, Replace $a^i$ with $A(\kappa)$
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5. KFE: Stationary distribution, Fan charts
1a. Individual Agent Choice of $i, \theta, c/n$

$$\max_{\{i_t, \theta_t, c_t\}_{t=0}^\infty} E \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

s.t. \( \frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + \sum_j \theta_t^j dr_t^j + \text{labor income/endow/taxes} \)

- Portfolio Choice: Martingale Approach
  - Let $x_t^A$ be the value of a “self-financing trading strategy” (reinvest dividends)

- Theorem: $\xi_t x_t^A$ follows a Martingale, i.e. drift = 0.
  - Let
    $$\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t + j_t^A dJ_t,$$
  - Recall SDF
    $$\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \xi_t^i dZ_t - \nu_t^i (dJ_t - \lambda_t dt)$$
  - By Itô product rule
    $$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = (-r_t + \mu_t^A - \xi_t^i \sigma_t^A - (1 - \nu_t^i) \lambda_t) dt + (\sigma_t^A - \xi_t^i) dZ_t + \left( j_t^A - (1 - \nu_t^i)(1 + j_t^A) \right) dJ_t$$

- Expected return: $\mu_t^A + \lambda j_t^A = r_t + \xi_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$
1a. Individual Agent Choice of $i$, $\theta$, $c/n$

- Expected return: $\mu_t^A + \lambda j_t^A = r_t^i + \zeta_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$
  - $r_t^i$ is the shadow risk-free rate (need not to be same across groups)
  - $\zeta_t^i$ is the price of Brownian risk of agents $i$, $\zeta_t^i \sigma_t^A$ is the required Brownian risk premium of agents $i$
  - $\nu_t^i \lambda_t$ is the price of Poisson upside risk if $j_t^A > 0$
    For risk-neutral agents $\nu_t^i = 0$

- Remark:
  - $dr^{e,K}$ experts return on capital
  - $dr^{h,OE}$ households return on outside equity
  - $dr^{h,D}$ households’ return on debt is risky (due to bankruptcy)
1a. Individual Agent Choice of $i$, $\theta$, $c/n$

- Expected return: $\mu_t^A + \lambda j_t^A = r_t^i + \zeta_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$
  - $r_t^i$ is the shadow risk-free rate (need not to be same across groups)
  - $\zeta_t^i$ is the price of Brownian risk of agents $i$,
    $\zeta_t^i \sigma_t^A$ is the required Brownian risk premium of agents $i$
  - $\nu_t^i \lambda_t$ is the price of Poisson upside risk if $j_t^A > 0$
    For risk-neutral agents $\nu_t^i = 0$

- Remark:
  - For CRRA utility: SDF is $\xi_t = e^{-\rho} \omega_t^{1-\gamma} n_t^{-\gamma}$
    $1 - \nu_t = (1 + j_t^\omega)^{1-\gamma} (1 + j_t^n)^{-\gamma}$
  - For log utility: $\nu_t = 1 - \frac{1}{1 + j_t^n} = \frac{j_t^n}{1 + j_t^n}$
  - For Epstein-Zin: part of $\omega_t$-process
1a. Individual Agent Choice of $i$, $\theta$, $c/n$

- Of experts with outside equity issuance (after plugging in households’ outside equity choice)
  \[
  \frac{a^e - \nu_t}{q_t} + \Phi(\nu_t) - \delta + \mu^q_t + \sigma \sigma^q_t - \left[ \frac{\chi^e_t}{\kappa^e_t} r^F,e_t + \left( 1 - \frac{\chi^e_t}{\kappa^e_t} \right) r^F,h_t \right] + \lambda_t \cdot j^q_t = \\
  \left[ \frac{\epsilon^e_t}{\kappa^e_t} \gamma^e_t + \xi^h_t \left( 1 - \frac{\chi^e_t}{\kappa^e_t} \right) \right] (\sigma + \sigma^q_t) + \left[ \nu^e_t \frac{\chi^e_t}{\kappa^e_t} + \nu^h_t \left( 1 - \frac{\chi^e_t}{\kappa^e_t} \right) \right] \lambda_t \cdot j^q_t
  \]

- Of households’ capital choice
  \[
  \frac{a^h - \nu_t}{q_t} + \Phi(\nu_t) - \delta + \mu^q_t + \sigma \sigma^q_t - r^F,h_t + \lambda_t \left( j^q_t - j^D_t \right) \\
  \leq \xi^h_t (\sigma + \sigma^q_t) + \nu^h_t \lambda_t (j^q_t - j^D_t)
  \]
  with equality if $\kappa^e_t < 1$

- Note: Later approach replaces this step with Fisher Separation Social Planners’ choice (see below)
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$  
   a. Real investment $\iota_i$ + Goods market clearing (static)  
      ▪ Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach  
   b. Portfolio choice $\theta_i$ + Asset market clearing or Asset allocation $\kappa_i$ & risk allocation $\chi_i$  
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem  

   ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)  
   ▪ forward equation

3. Value functions  
   ▪ backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega_i$  
      ▪ Special cases: log-utility, constant investment opportunities  
   b. Separating value fcn. $V^i(n^i, \eta, K)$ into $v^i(\eta)u(K)$  
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution  
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE  
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1b. Asset/Risk Allocation across I Types

- **Price-Taking Planner’s Theorem:**
  A social planner that takes prices as given chooses an physical asset allocation, \( \kappa_t \), and Brownian risk allocation, \( \chi_t \), and a Jump risk allocation, \( \zeta_t \), that coincides with the choices implied by all individuals’ portfolio choices.

\[
\begin{align*}
\mathbb{E}_{t} \left[ \frac{d r_{t}^{N}(\kappa_{t})}{dt} \right] - \zeta_t \sigma(\chi_t) - \lambda v j(\zeta_t) &= dr^F / dt \\
\text{subject to friction: } F(\kappa_t, \chi_t, \zeta_t) &\leq 0
\end{align*}
\]

- **Example:**
  1. \( \chi_t = \zeta_t = \kappa_t \) (can’t issue outside equity to offload Brownian or risky debt to offload Jump risk)
  2. \( \chi_t \geq \alpha \kappa_t \) (skin in the game constraint, outside equity up to a limit)

Let \( d N_t / N_t = \mu_t^N dt + \sigma_t^N d Z_t + j_t^N d J_t \)
1b. Asset/Risk Allocation across \textit{I} Types

- Sketch of Proof of Theorem

1. Fisher Separation Thm: (delegated portfolio choice by firm)
   - FOC yield the martingale approach solution
   - Each individual agent \((i, \bar{i})\) portfolio maximization is equivalent to the maximization problem of a firm
     \[
     \max_{\{\theta^j,i\}} \frac{E_t \left[dr^{n(i,i)}\right]}{dt} - \zeta \sigma^n - \lambda v^i j n^i(\zeta_t)
     \]
     \[
     dr^{n(i,i)} = \sum_j \theta^j,i dr^j = \sum_j \theta^j,i E[dr^j] + \sum_j \theta^j,i (\sigma^j dZ_t + v^i j (dJ_t - \lambda_t dt))
     \]
     is linear in \(\theta\)s
     - Either bang-bang solution for \(\theta\)s s.t. portfolio constraints bind
     - Or prices/returns/risk premia are s.t. that firm is indifferent

2. Aggregate
   - Taking \(\eta\)-weighted sum to obtain return on aggregate wealth

3. Use market clearing to relate \(\theta\)s to \(\kappa\)s & \(\chi\)s & \(\zeta\)s (incl. \(\theta\)-constraint)
1b. Allocation of Capital/Risk: 2 Types

- **Expert:** \( \theta^e = (\theta^{e,K}, \theta^{e,OE}, \theta^{e,D}) \) for capital, outside equity, debt

- **Restrictions:**
  \[
  \begin{align*}
  \theta^{e,K} & \geq 0, \\
  \theta^{e,OE} & \leq 0, \\
  \theta^{e,OE} & \geq -(1 - \alpha) \theta^{e,K} \quad \text{skin in the game}
  \end{align*}
  \]

maximize
\[
\begin{align*}
\theta^{e,K}_t E[dr^{e,K}_t]dt + \theta^{e,OE}_t (E[dr^{OE}_t]dt) + \theta^{e,D}_t E[dr^{D,e}_t] - \zeta^e_t (\theta^{e,K}_t + \theta^{e,OE}_t) \sigma^{r,eK}_t \\
- \lambda_t \nu^e_t ((\theta^{e,K}_t + \theta^{e,OE}_t) j^{eK}_t + \theta^{e,D}_t j^{D}_t)
\end{align*}
\]

- **Household:** \( \theta^h = (\theta^{h,K}, \theta^{h,OE}, \theta^{h,D}) \)

maximize
\[
\begin{align*}
\theta^{h,K}_t E[dr^{h,K}_t]/dt + \theta^{h,OE}_t E[dr^{OE}_t]/dt + \theta^{h,D}_t E[D_r^{D,h}_t] - \zeta^h_t (\theta^{h,K}_t + \theta^{h,OE}_t) \sigma^{r,hK}_t \\
- \lambda_t \nu^h_t ((\theta^{h,K}_t + \theta^{h,OE}_t) j^{hK}_t + \theta^{h,D}_t j^{D}_t)
\end{align*}
\]
1b. Allocation of Capital/Risk: 2 Types

- **Example 2**: 2 Type + with outside equity

\[
\max_{\{\kappa_t^e, \lambda_t^e\}} \left[ \frac{\kappa_t^e a^e + (1 - \kappa_t^e)a^h - \nu_t}{q_t} + \Phi(\nu_t) - \delta + \right] - (\chi_t^e \zeta_t^e + (1 - \chi_t^e)\zeta_t^h)(\sigma + \sigma_t^q) + \ldots
\]

- **FOC** $\chi$: Case 1: \( \zeta_t^e (\sigma + \sigma_t^q) + \ldots > \zeta_t^h (\sigma + \sigma_t^q) + \ldots \Rightarrow \chi_t^e = \alpha \kappa_t^e \\
Case 2: \Rightarrow \chi_t^e > \alpha \kappa_t^e$

- Case 1: plug $\chi_t^e = \alpha \kappa_t^e$ in objective
  - a. $FOC_{\kappa}: \frac{a^e - a^h}{q_t} > \alpha (\zeta_t^e - \zeta_t^h)(\sigma + \sigma_t^q) + \ldots \Rightarrow \kappa_t^e = 1$
  - b. $= \Rightarrow \kappa_t^e < 1$

- Case 2:
  - a. $FOC_{\kappa}: \frac{a^e - a^h}{q_t} > 0 \Rightarrow \kappa_t^e = 1$
  - b. $= 0 \Rightarrow \kappa_t^e < 1$ impossible

\[
\chi_t^e = \alpha \kappa_t^e
\]
1b. Allocation of Capital, $\kappa$, and Risk, $\chi$

- Summarizing previous slide (2 types with outside equity)

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\chi^e_t \geq \alpha \kappa^e_t$</th>
<th>$\kappa^e_t \leq 1$</th>
<th>$(\alpha^e - \alpha^h)q_t \geq \alpha (\zeta^e_t - \zeta^h_t)(\sigma + \sigma^q_t) + \cdots$</th>
<th>$(\zeta^e_t - \zeta^h_t)(\sigma + \sigma^q_t) + \cdots \geq 0$</th>
</tr>
</thead>
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<tr>
<td>1a</td>
<td>=</td>
<td>&lt;</td>
<td>=</td>
<td>&gt;</td>
</tr>
<tr>
<td>1b</td>
<td>=</td>
<td>=</td>
<td>&gt;</td>
<td>&gt;</td>
</tr>
<tr>
<td>2a</td>
<td>&gt;</td>
<td>=</td>
<td>&gt;</td>
<td>=</td>
</tr>
<tr>
<td>impossible</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

HHs’ short-sale constraint of capital binds, $\kappa^e_t = 1$

Experts’ skin in the game constraint binds, $\chi^e_t = \alpha \kappa^e_t$
Invariance of Relative Capital Demand

- One of the insights of Mendo (2020) is that self-fulfilling jumps do not influence the relative demand for capital of experts relative to households. I.e. the excess market return that experts demand to hold capital is not affected.

- Subtract experts pricing condition from households

\[ \mu_t^{r,k,e} - \mu_t^{r,k,h} \geq \frac{\chi_t^e}{\kappa_t^e} (\zeta_t^e - \zeta_t^h)(\sigma + \sigma_{t}^q) - \frac{\chi_t^e}{\kappa_t^e} \lambda_t (1 - \nu_t^h) \left( \frac{\partial j_t^D}{\partial \theta_t^{e,K}} (\theta_t^{e,K} - 1) + j_t^q - j_t^D \right) \]

\[ = 0 \]

- Losses are split between experts and households (via defaultable debt)

- Since experts’ losses are capped by their net worth due to limited liability, all additional losses from increasing capital holding, \( \theta_t^{e,K} \), are born by households
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      ▪ Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      ▪ Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
Toolbox 3: Change of Numeraire

- $x_t^A$ is a value of a self-financing strategy/asset in $\$
- $Y_t$ price of € in $\$$(exchange rate)
  \[
  \frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t + j_t^Y dJ_t
  \]
- $x_t^A/Y_t$ value of the self-financing strategy/asset in €

\[
e^{-\rho_t u'(c_t)} Y_t \frac{x_t^A}{Y_t} \]

follows a martingale (+ SDF in new numeraire $\hat{\xi}_t = \xi_t Y_t$)

Recall $\mu^A_t - \mu^B_t + \lambda_t (j^A_t - j^B_t) = \left(-\sigma_t^\xi \right) \left(\sigma^A - \sigma^B_t\right) + \nu_t \lambda_t (j^A_t - j^B_t)$

\[
\frac{A}{B} \mu^A_t - \mu^Y_t + \lambda_t \left(\frac{A}{B} j^A_t - \frac{B}{B} j^Y_t\right) = \left(-\sigma_t^\xi \right) \left(\sigma^A - \sigma^B_t\right) + (\nu_t - j^Y_t - \nu_t j^Y_t) \lambda_t \frac{j^A_t - j^B_t}{1 + j^Y_t}
\]

- Price of Brownian risk $\xi^\epsilon = \xi^\$ - $\sigma^Y$
- Price of Jump risk $\nu_t^\epsilon = \nu_t^\$ - $j_t^Y - \nu_t j_t^Y$
Change of Numeraire: SDF

- SDF in good numeraire is
  \[ d \xi_t^i / \xi_{t-}^i = -r_t^F dt - \zeta_t^i dZ_t - \nu_t^i (dJ_t - \lambda_t dt) \]

- SDF in total net worth numeraire is
  \[ d \hat{\xi}_t^i / \hat{\xi}_{t-}^i = \mu_t^i dt - (\zeta_t^i - \sigma_t^N) dZ_t - (\nu_t^i - j_t^N - \nu_t j_t^N) dJ_t \]
  \[ = \hat{r}_t^F dt - (\zeta_t^i - \sigma_t^N) dZ_t - (\nu_t^i - j_t^N - \nu_t j_t^N)(dJ_t - \lambda_t dt) \]
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5. KFE: Stationary distribution, Fan charts
2. GE: Markov States and Equilibria

- Equilibrium is a map
  - Histories of shocks $\{Z_s, s \in [0, t]\}$
  - Prices $q_t, \zeta_t^i, \iota_t^i, \theta_t^e$
  - Net worth distribution
    $$\eta_t^e = \frac{N_t^e}{q_tK_t} \in (0,1)$$
    Net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology

- All markets clear
  - Consumption, capital, money, outside equity
2. Law of Motion of Wealth Share $\eta_t$

**Method 1:** Using Ito’s quotation rule $\eta^i_t = N^i_t / (q^i_t K^i_t)$

- Recall
  
  $$\frac{dN^i_t}{N^i_t} = -\frac{C^i_t}{N^i_t} dt + \frac{r^{bm}_t}{N^i_t} dt + \sum_{t} \left( \frac{\chi^i_{t} \kappa^i_{t} (\sigma + \sigma^q_t)}{\eta^i_t} - \sigma^{bm} \right) dt + + \nu \left( j^N_{t} - j^{bm}_t \right) dt$$

- $$\frac{d\eta^i_t}{\eta^i_t} = \ldots \text{(lots of algebra)}$$

**Method 2:** Change of numeraire + Martingale Approach

- New numeraire: Total wealth in the economy, $N_t$
- Apply Martingale Approach for value of $i$’s portfolio

- Simple algebra to obtain drift of $\eta^i_t$: $\mu^i_t$  
  Note that change of numeraire does not affect ratio $\eta^i_t$!
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- **New Numeraire**
  - "Total net worth" in the economy, $N_t$ (without superscript)
  - Type $i$’s portfolio net worth = net worth share

- **Martingale Approach with new numeraire**
  - Asset $A = i$’s portfolio return in terms of total wealth,
    $$\left( \frac{C_t^i}{N_t^i} + \mu^\eta_t^i + \lambda_t j^i \right) dt + \sigma^\eta_t^i dZ_t$$
  - Dividend yield $\frac{C_t^i}{N_t^i}$
  - $E[\text{capital gains}]$ rate $\lambda_t j^i$

  - Asset $B$ (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))
    $$r_t^{bm} dt + \sigma_t^{bm} dZ_t$$

- Apply our martingale asset pricing formula
  $$\mu_t^A - \mu_t^B + \lambda_t (j_t^A - j_t^B) = \hat{\xi}_t^i (\sigma_t^A - \sigma_t^B) + \hat{\nu}_t (j_t^A - j_t^B)$$

Hat notation $\hat{\cdot}$ indicates total net worth numeraire.
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)
\[
\mu_t^\eta + \frac{C_t^i}{N_t^i} - r_t^{bm} + \lambda_t \left( j_t^\eta - j_t^{bm} \right) = \left( \zeta_t^i - \sigma_t^N \right) \left( \sigma_t^{\eta} - \sigma_t^{bm} \right) + \hat{\nu}_t^i \left( j_t^\eta - j_t^{bm} \right)
\]

- Add up across types (weighted),
  (capital letters without superscripts are aggregates for total economy)
\[
\sum_{i'} \eta_t^{i'} \mu_t^{i'} + \frac{C_t}{N_t} - r_t^{bm} + \sum_{i'} \eta_t^{i'} j_t^{i'} - \lambda_t d j_t^{bm} = 0
\]
\[
\sum_{i'} \eta_t^{i'} \hat{\zeta}_t^{i'} \left( \sigma_t^{\eta} - \sigma_t^{bm} \right) + \sum_{i'} \eta_t^{i'} \hat{\nu}_t^{i'} \left( j_t^{i'} - j_t^{bm} \right) = 0
\]

- Subtract from first equation
\[
\mu_t^\eta + \lambda_t j_t^{\eta'} = \frac{C_t}{N_t} - \frac{C_t^i}{N_t^i} - \hat{\zeta}_t \left( \sigma_t^{\eta} - \sigma_t^{bm} \right) - \sum_{i'} \eta_t^{i'} \hat{\zeta}_t^{i'} \left( \sigma_t^{\eta'} - \sigma_t^{m} \right)
\]
\[
+ \hat{\nu}_t^i \left( j_t^\eta - j_t^{bm} \right) - \sum_{i'} \eta_t^{i'} \hat{\nu}_t^{i'} \left( j_t^{i'} - j_t^{bm} \right)
\]
2. $\mu^\eta$ Drift of Wealth Share: Two Types $i \in \{e, h\}$

- Subtract from each other yield net worth share dynamics

$$\mu^\eta_t + \lambda_t j^\eta_t = \frac{C^e_t}{N_t^e} - \frac{C^e_t}{N_t^e} - (1 - \eta^e_t)\hat{s}^e_t \left( \sigma^\eta_t + \sigma_{t bm}^N \right) - (1 - \eta^e_t)\hat{s}^h_t \left( \sigma^\eta_t + \sigma_{t bm}^N \right)$$

$$+ (1 - \eta^e_t)\hat{v}^e_t \left( j^\eta_t - j_{t bm}^N \right) - (1 - \eta^e_t)\hat{v}^h_t \left( j^\eta_t - j_{t bm}^N \right)$$

- In our model, benchmark asset is risky debt,
  - $\sigma_{t bm}^N = -\sigma_t^N$,
  - $j_{t bm}^N = \frac{j_D^N - j_{t N}^D}{1 + j_{t N}^N}$ (since $j_D^N$ risky debt jump in c-numeraire, $j_{t N}^N$ wealth jump)
    - Apply quotient rule for jumps

$$\mu^\eta_t + \lambda_t j^\eta_t = \frac{C^e_t}{N_t^e} - \frac{C^e_t}{N_t^e} - (1 - \eta^e_t)\hat{s}^e_t \left( \sigma^\eta_t + \sigma_{t N}^N \right) - (1 - \eta^e_t)\hat{s}^h_t \left( \sigma^\eta_t + \sigma_{t bm}^N \right)$$

$$+ (1 - \eta^e_t)\hat{v}^e_t \left( j^\eta_t - \frac{j_D^N - j_{t N}^D}{1 + j_{t N}^N} \right) - (1 - \eta^e_t)\hat{v}^h_t \left( j^\eta_t - \frac{j_D^N - j_{t N}^D}{1 + j_{t N}^N} \right)$$
2. $\sigma^\eta$ Volatility of Wealth Share

- Since $\eta_t^i = N_t^i / N_t$,

$$
\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} \\
= (1 - \eta_t^i) \sigma_t^{N^i} - \sum_{i' \neq i} \eta_t^{i-} \sigma_t^{N^{i-}}
$$

$$
\eta_t^{j_i} = \frac{j_t^{N^i} - j_t^N}{1 + j_t^N} = \frac{j_t^{N^i} - \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}}{1 + \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}} = \frac{(1 - \eta_t^i) j_t^{N^i} - \sum_{i' \neq i} \eta_t^{i-} j_t^{N^{i-}}}{1 + \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}}
$$

- Note for 2 types example

$$
\eta_t^{j^e} = \frac{(1 - \eta_t^e)(j_t^{N^e} - j_t^{N^h})}{1 + \eta_t^e j_t^{N^e} + (1 - \eta_t^e) j_t^{N^h}}
$$
Note:
- OLG structure and
- transfers $\tau K_t$

also affects net worth evolution and still has to be incorporated!
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   \[ \text{backward equation} \]
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5. KFE: Stationary distribution, Fan charts
The Big Picture

allocation of physical assets

output $A(\kappa)$

capital growth $\Phi(i) - \delta$

net worth distribution $\eta$

value function

precautionary

drift

Backward equation

Debt

Outside equity

risk amplification

volatility

Forward equation

with expectations

consumption + investment

net worth
distribution

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3a. CRRA Value Function
Applies separately for each type of agent

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

\[ V^i(n^i_t; \eta_t, K_t) \] for individuals \( i \)

- For CRRA/power utility
  \[ u(c^i_t) = \frac{(c^i_t)^{1-\gamma-1}}{1-\gamma} \]
  ⇒ increase net worth by factor, optimal \( c^i \) for all future states increases by this factor ⇒ \( \left( \frac{c^i_t}{n^i_t} \right) \)-ratio is invariant in \( n^i_t \)

- ⇒ value function can be written as
  \[ V^i(n^i_t; \eta_t, K_t) = \frac{u(\omega^i(\eta_t, K_t)n^i_t)}{\rho^i} \]

- \( \omega^i_t \) Investment opportunity/ “net worth multiplier”
  - \( \omega^i(\eta_t, K_t) \)-function turns out to be independent of \( K_t \)
  - Change notation from \( \omega^i(\eta_t, K_t) \)-function to \( \omega^i_t \)-process
3a. Special case: log utility

- Result: \( q(\eta^e) \)-function is invariant to run risk, i.e. same as in Lecture 03.
  - ... but expected returns are different.

- Proof (sketch)
  - Log utility implies, prices of risk:
    - \( \sigma_t^i = \sigma^n_i \)
    - \( \lambda_t \nu_t^i = \lambda_t / (1 + j^n_i) \)
  - Goods market clearing
  - Brownian amplification equation
    \[
    \sigma + \sigma^q_t = \frac{\sigma}{1 - \frac{q'}{q}(\kappa - \eta)}
    \]
  - Relative asset pricing equation
    \[
    \frac{a^e - a^h}{q_t} \geq \left( \frac{\kappa_t}{\eta_t} - \frac{1 - \kappa_t}{1 - \eta_t} \right) \left( \sigma + \sigma^q_t \right)^2
    \]
3a. Value function in OLG setting

- Note: with OLG structure we have to take care that individual value function differs from sector wide.

\[ V_t^i = \frac{1}{\rho^i} \left( \omega_t^{i_n} \right)^{1-\gamma} = \frac{1}{\rho^i} \left( \omega_t^{i_n} \eta_t^{i_i} N^i \right)^{1-\gamma} \]

- Where \( \eta_t^{i_i} \) is the net worth share of individual \((i, i)\) within sector \(i\)
- It is time-varying deterministically, and hence does not affect asset pricing.
3a. CRRA Value Function: relate to $\omega$

- Value function can be written as $\frac{u(\omega_t n_t^i)}{\rho}$, that is
  
  $$
  \frac{1}{\rho^i} \frac{\left(\omega_t n_t^i\right)^{1-\gamma}}{1-\gamma} - 1 = \frac{1}{\rho^i} \frac{\left(\omega_t^i\right)^{1-\gamma} \left(n_t^i\right)^{1-\gamma}}{1-\gamma} - 1
  $$

- $\frac{\partial v}{\partial n^i} = u'(c^i)$ by optimal consumption (if no corner solution)
  
  $$
  \frac{\left(\omega_t^i\right)^{1-\gamma} \left(n_t^i\right)^{-\gamma}}{\rho^i} = (c_t^i)^{-\gamma} \iff \frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma} \left(\omega_t^i\right)^{1-1/\gamma}
  $$

**Optimal consumption is different:**

$$
\frac{\omega^{1-\gamma} n^{-\gamma}}{n} = \frac{\partial v}{\partial n} = \frac{\partial f}{\partial c} = \rho (\omega n)^{1-\gamma} \frac{1}{c}
$$

$$
\frac{\omega^{1-\gamma} n^{-\gamma}}{\rho} = c_t^{-\gamma} \iff \frac{c_t}{n_t} = \rho
$$
3a. CRRA Value Function: relate to $\omega$

- value function can be written as
  \[ \frac{u(\omega_t^n_t)}{\rho} \]

\[ \frac{1}{\rho^i \frac{1}{1-\gamma} - 1} = \frac{1}{\rho^i \frac{(\omega_t^n_t)^{1-\gamma} (n_t)^{1-\gamma}}{1-\gamma}} - 1 \]

- SDF now
  \[ \xi_t = e^{\int_0^t \frac{\partial f}{\partial V}(c_s, V_s) ds} V_t \frac{\partial V}{\partial n} \]

\[ \xi_t n_t = (1 - \gamma) V_t \]

- Get new discounting term
  \[ e^{-\int_0^t \frac{\partial f}{\partial V}(c_s, V_s) ds} \xi_t n_t = (1 - \gamma) V_t \]

\[ E_t [dV_t] / V_t = (-\partial f / \partial V_t - c_t / n_t) dt \]
3a. CRRA Value Function: Special Cases

\[
\frac{c_t^i}{n_t^i} = (\rho_i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}
\]

- For log utility $\gamma = 1$:
  \[
  \xi_t^i = e^{-\rho t} / c_t^i = e^{-\rho t} / (\rho n_t^i) \text{ for any } \omega_t^i \Rightarrow \sigma_t^n = \sigma_t^c = \sigma_t
  \]
- Expected excess return: $\mu_t^A - r_t^F = \sigma_t^n \sigma_t^A$
- Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i} dt + (1 - \theta_t^i)dr_t^K + \theta_t dr_t$

\[
\]
3a. CRRA Value Function: Special Cases

\[
\frac{c_t^i}{n_t^i} = (\rho_i)^{1/\gamma} (\omega_t^i)^{1-1/\gamma}
\]

- For log utility \( \gamma = 1 \):
  \[
  \xi_t^i = e^{-\rho_i t} / c_t^i = e^{-\rho_i t} / (\rho n_t^i)
  \]
  for any \( \omega_t^i \)
  \( \sigma_t^{n_i} = \sigma_t^{c_i} = \xi_t^i \)

- Expected excess return:
  \( \mu_t^A - r_t^F = \sigma_t^{n_i} \sigma_t^A \)

- Recall
  \[
  \frac{d n_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i} dt + (1 - \theta^i)dr_t^K + \theta^i dr_t
  \]

- For constant investment opportunities \( \omega_t^i = \omega^i \),
  \[ c^i / n^i \] is constant and hence \( \sigma_t^{c_i} = \sigma_t^{n_i} \)

- Expected excess return:
  \( \mu_t^A - r_t^F = \gamma \sigma_t^{n_i} \sigma_t^A \)

Poll 49: Which term refers to (dynamic/Mertonian) hedging demand?

- a) \( \gamma \)
- b) \( \sigma_t^{n_i} \)
- c) hidden in risk-free rate
- d) none of the above
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      • Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      • Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   • Toolbox 3: Change in numeraire to total wealth (including SDF)
2. Evolution of state variable $\eta$ (and $K$) forward equation
3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      • Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)$
   c. Derive $C/N$-ratio and $\zeta$ price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts