Financial and Monetary Economics
Eco529 Fall 2020
Lecture 05: One Sector Money Model with Idiosyncratic Risk

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Key Takeaways

- Money as a bubble
  - As store of value – link to FTPL
  - Medium of exchange

- Technical Takeaways
  - Recall: Change to total wealth numeraire to derive “money evaluation” equation
  - Idiosyncratic risk
    - Isolating it from value function
Roadmap

- Solve one-sector money model
  - Different ways to derive money evaluation equation
  - Value function with idiosyncratic risk
  - Bubble/Ponzi scheme, $r$ vs. $g$ vs. $\zeta$ and transfersality condition
  - Mining the bubble and MMT

- Fiscal Theory of the Price Level (with a bubble)
  - FTPL equation
  - Price Level Determination
  - Monetary vs. fiscal authority

- Medium of Exchange Role of Money
The 4 Roles of Money

- Unit of account
  - Intratemporal: Numeraire: bounded rationality/price stickiness
  - Intertemporal: Debt contract: incomplete markets

- Store of value
  - “I Theory of Money without I”
    Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- Medium of exchange
  - Overcome double-coincidence of wants problem

- Record keeping device – money is memory
  - Virtual ledger
Safe Assets $\equiv$ (Narrow) Money

- Asset Price = $E[\text{PV(cash flows)}] + E[\text{PV(service flows)}]$
  - dividends/interest
- Service flows/convenience yield
  1. Collateral: relax constraints (Lagrange multiplier)
  2. Safe asset: [good friend analogy]
    - When one needs funds, one can sell at stable price ... since others buy
    - Partial insurance through retrading - market liquidity!
  3. Money (narrow): relax double-coincidence of wants
    - Higher Asset Price = lower expected return

- Problem: safe asset + money status might burst like a bubble
  - Multiple equilibria: [safe asset tautology]
# Models on Money as Store of Value

<table>
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<th>Friction</th>
<th>OLG</th>
<th>Incomplete Markets + idiosyncratic risk</th>
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<tr>
<td>Risk</td>
<td>deterministic</td>
<td>endowment risk borrowing constraint</td>
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<tr>
<td>Only money</td>
<td>Samuelson</td>
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<td>With capital</td>
<td>Diamond</td>
<td>Aiyagari</td>
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One Sector Model with Gov. Bonds/Broad Money as Safe Asset

- Agent $\tilde{i}$’s preferences
  \[ E \left[ \int_0^\infty e^{-\rho t} \left( \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + f(\mathcal{g}_tK_t) \right) dt \right] \]

- Each agent operates one firm
  - Output \[ y_t^i = a k_t^i \]
  - Physical capital $k$
    \[ \frac{dk_t^i}{k_t^i} = (\Phi(\xi_t^i) - \delta) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^i \]

- Financial Friction: Incomplete markets:
  Agents cannot share $d\tilde{Z}_t^i$
Agent $\bar{i}$’s preferences:

$$E \left[ \int_0^\infty e^{-\rho t} \left( \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + f(\varphi_t K_t) \right) dt \right]$$

Each agent operates one firm:

- Output: $y_t^i = a k_t^i$
- Physical capital $k$

$$\frac{dk_t^i}{k_t^i} = (\Phi(i_t^i) - \delta) dt + \sigma dZ_t + \bar{\sigma} d\tilde{Z}_t$$

Financial Friction: Incomplete markets:
Agents cannot share $d\tilde{Z}_t$

Goods market clearing:

$$c_t + \varphi_t K_t = (a - \iota_t) K_t$$
Taxes, Money/Bond Supply, Gov. Budget

- Policy Instruments
  - Government spending $K_t d_g_t$
    - $K_t \mu_t^g dt + K_t \sigma_t^g dZ_t$
  - government debt supply
    - $\frac{dB_t}{B_t} = \mu_t^B dt + \sigma_t^B dZ_t$
  - nominal interest rate $i_t$
  - proportional tax $K_t d \tau_t$ on capital
  - lump-sum tax $\tau_t^{ls}$ ($= 0$ for this talk)

- Government budget constraint (BC)
  - $i_t B_t + \varphi_t K_t d_g_t = \mu_t^B B_t + \varphi_t K_t d \tau_t + \varphi_t \tau_t^{ls} + \text{seigniorage}$

- Assume here:
  - No lump-sum taxes
  - Gov. chooses $\mu^B, i$; while $\tau_t$ adjusts to satisfy (BC)
One Sector Model with Money/Gov. debt

- Seigniorage is distributed
  1. Proportionally to bond/money holdings
     - No real effects, only nominal
  2. Proportionally to capital holdings
     - Bond/Money return decreases with $dB_t$ (change in debt level/money supply)
     - Capital return increases
     - Pushes citizens to hold more capital
  3. Proportionally to net worth
     - Fraction of seigniorage goes to capital - same as 2.
     - Rest of seigniorage goes to money holders - same as 1.
  4. Per capita
     - No real effects:
       people simply borrow against the transfers they expect to receive
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $\frac{C}{N}$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      ▪ Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fnc. of individual investment opportunities $\omega$
      ▪ Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)(\tilde{\eta}^i)^{1-\gamma} u(K)$
   c. Derive $\tilde{\rho} = \frac{C}{N}$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
Postulate Aggregates and Processes

- \( q^K_t K_t \) value of physical capital
- \( q^B_t K_t \) value of nominal capital/outside money/gov. debt
  - \( B_t / \varphi_t = q^B_t K_t \) price level (inverse of “value of money”)
- \( \vartheta_t : = \frac{q^B_t}{q^K_t + q^B_t} \) fraction of nominal wealth
Postulate Aggregates and Processes

- $q_t^K K_t$ value of physical capital
- $q_t^B K_t$ value of nominal capital/outside money/gov. debt
  - $B_t / \varphi_t = q_t^B K_t$ price level (inverse of “value of money”)
- $\vartheta_t := \frac{q_t^B}{q_t^K + q_t^B}$ fraction of nominal wealth

0. Postulate

- $q^K$-price process
  \[ dq_t^K / q_t^K = \mu_t^q dK_t + \sigma_t^q dZ_t, \]
- $q^B$-price process
  \[ dq_t^B / q_t^B = \mu_t^q dZ_t, \]
- SDF for each $\tilde{i}$ agent
  \[ d\xi_t^\tilde{i} / \xi_t^\tilde{i} = -r_t^f dt - \zeta_t^\tilde{i} dZ_t - \tilde{\zeta}_t^\tilde{i} d\tilde{Z}_t^\tilde{i} \]

Poll 13: Why is risk-free rate and aggregate price of risk the same for all $\tilde{i}$?

a) Because risk-free debt can be traded.

b) Because they identical up to size (scaled versions of each other).
0. Return on Gov. Bond/Money

- Number of Bonds/coins follows:
  \[
  \frac{dB_t}{B_t} = \bar{\mu}_t^B dt + idt + \sigma_t^B dZ_t
  \]
  - Where \( i \) is interest paid on government bonds/outside money (reserves)

- Return on Gov. Bond/Money: in output numeraire

  \[
  d\tilde{r}_t^B = idt + \frac{d(q_t^B K_t/B_t)}{q_t^B K_t/B_t} = \frac{d(q_t^B K_t)}{q_t^B K_t} - \bar{\mu}_t^B dt - \sigma_t^B dZ_t + \sigma_t^B (\sigma_t^B - \sigma) dt
  \]

- Seigniorage (excluding interest paid to money holders)
0. Return on Capital (with seigniorage rebate terms)

\[ dr_t^{K,i} = \frac{a-i_t^i}{q_t^K} dt + \frac{d\left(q_t^K k_t^i\right)}{q_t^K k_t^i} - \frac{k^l d\tau}{q_t^K k_t^i} \]

\[ = \left(\frac{a-i_t^i}{q_t^K} + \Phi\left(i_t^i\right) - \delta + \mu_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t + \tilde{\sigma}_t d\tilde{Z}_t - \frac{k^l d\tau}{q_t^K k_t^i} \]

- Use government budget constraint to substitute out \( d\tau_t \) (and \( B_t/\varphi_t = q_t^B K_t \))

\[ K_t d\tau_t = \left(1 + \frac{1}{\varphi_t}\right)(d B_t - i_t B_t dt) = K_t d\varphi_t \]

\[ K_t d\tau_t = K_t \mu_t^q dt + K_t \sigma_t^q dZ_t - q_t^B K_t \left\{ \mu_t^B - i_t + \left(\sigma + \sigma_t^q - \sigma_t^B\right)\sigma_t^B \right\} dt + \sigma_t^B dZ_t \]

\[ dr_t^{K,i} = \left(\begin{array}{c}
\frac{\bar{a}}{a-\mu_t^q-i_t^i} + \Phi\left(i_t^i\right) - \delta + \mu_t^q + \frac{q_t^B}{q_t^K} \left(\mu_t^B + \left(\sigma + \sigma_t^q - \sigma_t^B\right)\sigma_t^B\right) \end{array}\right) dt
\]

\[ + \left(\sigma + \sigma_t^q + \frac{q_t^B \sigma_t^B - \sigma_t^q}{q_t^K}\right) dZ_t + \tilde{\sigma}_t d\tilde{Z}_t \]
Household Problem

- Wealth evolution (budget constraint)

\[
\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i} dt + dr_t^B + (1 - \theta_t^i) \left(dr_t^K(\nu_t^i) - dr_t^B\right)
\]

- HJB equation of household

\[
\rho V_t(n) = \max_{c, \theta, n} \left\{ \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + V_t'(n) \left[ -c_t^i + n \left( \Phi(\nu_t^i) - \delta + \mu_t^\rho - \bar{\mu}^B + (1 - \theta) \left( \frac{\bar{a} - \bar{\nu}}{q_t} + \Phi(\bar{\nu}) - \Phi(\nu_t) - \frac{\mu_t^\theta - \bar{\mu}^B}{1 - \vartheta_t} \right) \right] + E[dr_t^B]/dt \left[ \frac{1}{2} V_t''(n)n^2(1-\theta)^2 \bar{\sigma}^2 \right] \right\} = E[dr_t^K - dr_t^B]/dt
\]
Optimal Choices

- Guess (and verify) value function $V_t(n) = \alpha_t + \frac{1}{\rho} \log n_t$

- Optimal investment rate
  - $\phi t = q_t - 1$

\[
\frac{1}{q_t} = \Phi'(\tilde{\iota}_t) \quad \text{Tobin's } q
\]
All agents $\tilde{\iota}_t = \iota_t$
Special functional form:
\[
\Phi(\iota_t) = \frac{1}{\phi} \log(\phi \iota_t + 1) \Rightarrow \phi \iota_t = q_t - 1
\]
Optimal Choices

- Guess (and verify) value function \( V_t(n) = \alpha_t + \frac{1}{\rho} \log n_t \)

- Optimal investment rate
  - \( \phi_t = q_t - 1 \)

- Consumption
  - \( \frac{c_t}{n_t} =: \tilde{\rho}_t \Rightarrow C_t = \tilde{\rho}_t (q_t^B + q_t^K) K_t \)

- Looking ahead to Step 3:
  - When is \( \frac{c}{n} \) constant? Recall \( \frac{c}{n} = \rho^{1/\gamma} \omega^{1-1/\gamma} \)
    - Log utility, \( \gamma = 1 \): \( \tilde{\rho} = \rho \)
    - In steady state:
      - \( \omega \) investment opportunity/net worth multiplier is constant
Optimal Choices & Market Clearing

- Optimal investment rate
  - \( \phi_t = q_t - 1 \)

- Consumption
  - Goods market
  - \( \frac{c_t}{n_t} =: \tilde{\rho}_t \Rightarrow C_t = \tilde{\rho}_t (q^B_t + q^K_t)K_t = (\bar{a} - \iota_t)K_t \)

- Portfolio
  - Capital market
  - Solve for \( \theta_t \) later
  - \( 1 - \theta_t = 1 - \vartheta_t \)
  - Debt market
  - clears by Walras law
Equilibrium (before solving for portfolio choice)

<table>
<thead>
<tr>
<th>Equilibrium</th>
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<tbody>
<tr>
<td>$q_t^B = \vartheta_t \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \tilde{\rho}_t}$</td>
</tr>
<tr>
<td>$q_t^K = (1 - \vartheta_t) \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \tilde{\rho}_t}$</td>
</tr>
<tr>
<td>$\iota_t = \frac{(1 - \vartheta_t) a - \tilde{\rho}_t}{(1 - \vartheta_t) + \phi \tilde{\rho}_t}$</td>
</tr>
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- Moneyless equilibrium with $q_t^B = 0 \Rightarrow \vartheta_t = 0$
- Next, determine portfolio choice.
Portfolio choice $\theta$ (consumption numeraire)

- **Asset pricing equation (martingale method) for $\sigma_t^B = \sigma^\tau = 0$**

\[
\frac{E[dr^K_t]}{dt} = \frac{a - \nu_t}{q^K_t} + \Phi(\nu_t) - \delta + \mu^K_t + \sigma \sigma^K_t + \frac{q^B_t}{q^K_t} \hat{\mu}^B_t = r^f_t + \zeta_t \left( \sigma + \sigma^K_t \right) + \hat{\zeta}_t \tilde{\sigma}
\]

\[
\frac{E[dr^B_t]}{dt} = \Phi(\nu_t) - \delta + \mu^B_t + \sigma \sigma^B_t - \hat{\mu}^B_t = r^f_t + \zeta_t \left( \sigma + \sigma^B_t \right)
\]

Price of Risk:

\[
\zeta_t = -\sigma^v + \sigma^B_t + \gamma \sigma, \quad \hat{\zeta}_t = \gamma \tilde{\sigma}_t^n = \gamma (1 - \theta_t) \tilde{\sigma}
\]

For stationary equilibria

\[
\frac{a - \nu_t}{q^K_t} + \mu^K_t - \mu^B_t + \sigma \left( \sigma^K_t - \sigma^B_t \right) + \frac{1}{1 - \vartheta_t} \hat{\mu}^B_t = \zeta_t \left( \sigma^K_t - \sigma^B_t \right) + (1 - \theta_t) \gamma \tilde{\sigma}
\]

\[
\frac{(a - \nu) q^K_t + \hat{\mu}^B_t}{\gamma \tilde{\sigma}^2} = 1 - \theta = 1 - \vartheta \quad \text{capital market clearing}
\]

\[
\hat{\rho} \frac{(q^B_t + q^K_t) / q^K_t}{1 - \vartheta} = (a - \nu) / q^K_t \quad \text{goods market clearing}
\]

\[
(1 - \vartheta) = \sqrt{\hat{\rho} + \hat{\mu}^B_t / (\gamma \tilde{\sigma})}
\]


## Two Stationary Equilibria

<table>
<thead>
<tr>
<th>Non-Monetary</th>
<th>Monetary</th>
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<tbody>
<tr>
<td>$q_0^B = 0$</td>
<td>$q^B = \frac{(\sqrt{\gamma\delta} - \sqrt{\rho + \mu^B})(1 + \phi a)}{\sqrt{\rho + \mu^B + \kappa\sqrt{\gamma\delta}\rho}}$</td>
</tr>
<tr>
<td>$q_0^K = \frac{1 + \phi a}{1 + \phi \rho_0}$</td>
<td>$q^K = \frac{\sqrt{\rho + \mu^B} (1 + \phi a)}{\sqrt{\rho + \mu^B + \phi \sqrt{\gamma\delta}\rho}}$</td>
</tr>
<tr>
<td>$\iota = \frac{a - \rho_0}{1 + \phi \rho_0}$</td>
<td>$\iota = \frac{a \sqrt{\rho + \mu^B} - \sqrt{\gamma\delta}\rho}{\sqrt{\rho + \mu^B + \phi \sqrt{\gamma\delta}\rho}}$</td>
</tr>
</tbody>
</table>

- For log utility
  - $\rho = \rho_0 = \rho$
  - $\gamma = 1$
Remark

- Money is a bubble
  - But provides store of value/insurance role
- Comparative static: As $\tilde{\sigma}$ increases
  - Flight to safety to bubbly money
    - $q^B$ rises (disinflation)
    - $q^K$ falls and so does
      - $i$ and
      - growth rate of economy

- Can be extended to a model with stochastic idiosyncratic volatility ($\tilde{\sigma}_t$ becomes state variable)

- how is investment rate affected?
Remark: Pecuniary externalities

1. Agents’ portfolio choice takes $r^B$ as given, but ... it is affected by agents’ portfolio choice $\theta$, which affects $q^K$, which in turn affects $\iota$ (esp. for low $\kappa$), which affects the real return on money

2. Agents’ portfolio choice takes $q^K$ as given, but ... tilting the portfolio towards money, lower $q^K$ (esp. for high $\phi$), which in turn reduces risk per unit of output/capital

- Government acts like a “diversifier”
  - Individual tax liability is idiosyncratically risky, but “dividend” is not
  - Tax = co-ownership with dividends paid in form of $r^B$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
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   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)(\vec{n}^i)^{1-\gamma} u(K)$
   c. Derive $\bar{\rho} = C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
Remark: Portfolio choice Money vs. $N_t$

- Portfolio choice problem is simplified if money vs. total net worth (instead of money vs. capital)
- Seigniorage is part of $N_t$ and doesn’t have to be broken out

\[
E \left[ \frac{dr_t^{K^1}}{dt} \right] = \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t q^K + \sigma \sigma_t q^K + \frac{q_t^B \tilde{\mu}_t}{q^K} = r_t^f + \zeta_t \left( \sigma + \sigma_t q^K \right) + \bar{\zeta} \tilde{\sigma}
\]

\[
E \left[ \frac{dr_t^{N^1}}{dt} \right] = \tilde{\rho}_t + \Phi(\iota_t) - \delta + \mu_t q^K + q^B + \sigma \sigma_t q^K + q^B + 0 = r_t^f + \zeta_t \left( \sigma + \sigma_t q^K + q^B \right) + \bar{\zeta} (1 - \theta_t) \tilde{\sigma}
\]

\[
E \left[ \frac{dr_t^B}{dt} \right] = \Phi(\iota_t) - \delta + \mu_t^p + \sigma \sigma_t^p - \mu_t^M = r_t^f + \zeta_t \left( \sigma + \sigma_t^p \right)
\]
Change to total net worth numeraire $N_t$

- SDF in consumption numeraire
  \[
  \frac{d\xi_t^i}{\xi_t^i} = -r_t^f dt - \zeta_t dZ_t - \tilde{\zeta}_t^i d\tilde{Z}_t^i
  \]

- SDF in $N_t$-numeraire
  \[
  \frac{d\xi_t^i}{\xi_t^i} = \frac{d(\xi_t^i N_t)}{(\xi_t^i N_t)} = -(r_t^f - \mu_t^N + \zeta_t \sigma_t^N) dt - (\zeta_t - \sigma_t^N) dZ_t - \tilde{\zeta}_t^i d\tilde{Z}_t^i
  \]

- Return in consumption numeraire:
  \[
  dr_t^j = \mu_t^r^j dt + \sigma_t^r^j dZ_t - \tilde{\sigma}_t^r^j d\tilde{Z}_t^i
  \]

- Return in $N_t$-numeraire
  \[
  dr_{t,N}^j = \left(\mu_t^r^j - \mu_t^N - \sigma_t^N \left(\sigma_t^r^j - \sigma_t^N\right)\right) dt + \left(\sigma_t^r^j - \sigma_t^N\right) dZ_t - \tilde{\sigma}_t^r^j d\tilde{Z}_t^i
  \]

- Value of self-financing strategy investing in asset in the consumption numeraire, e.g. $x_t^j$ satisfies $dx_t^j/x_t^j = dr_t^j$. The same holds in the $N_t$-numeraire, but now the value is $x_t^j/N_t$.  

Portfolio choice $\theta$ (consumption numeraire)

Total net worth $N_t$ return relative to bond/money return

- Asset pricing equation (martingale method)

$$
E\left[\frac{dr_t^N}{dt}\right] = \ddot{p}_t + \Phi(\mu_t) - \delta + \mu_t^{q+p} + \sigma_t^{q+p} + 0 = r_t^f + \zeta_t (\sigma + \sigma_t^{q+p}) + \bar{\zeta}_t (1 - \theta_t) \bar{\sigma}
$$

$$
E\left[\frac{dr_t^B}{dt}\right] = \Phi(\mu_t) - \delta + \mu_t^p + \sigma_t^p - \bar{\mu}_t^B = r_t^f + \zeta_t (\sigma + \sigma_t^p)
$$
Portfolio choice $\theta$ ($N_t$-numeraire)

Total net worth $N_t$ relative to single bond/coin

- Asset pricing equation (martingale method)

$$E \left[ \frac{d r_t}{d t} \right] = \tilde{\rho}_t + 0 = \left( r_t^f - (\Phi(t) - \delta) - \mu_t^{q^K+q^B} - \sigma_t^{q^K+q^B} + \zeta_t (\sigma + \sigma_t^{q^K+q^B}) \right) + (\zeta_t - \sigma_t^N) 0 + \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma}$$

$$E \left[ \frac{d r_t}{d t} \right] = \mu_t^{\theta/B} = \left( r_t^f - (\Phi(t) - \delta) - \mu_t^{q^K+q^B} - \sigma_t^{q^K+q^B} + \zeta_t (\sigma + \sigma_t^{q^K+q^B}) \right) + \frac{(\zeta_t - \sigma_t^N) \sigma_t^{\theta/B}}{price of risk in N_t-numeraire}$$
Portfolio choice $\theta$ ($N_t$-numeraire)

Total net worth $N_t$ relative to a single bond/coin of money

- **Asset pricing equation (martingale method)**

$$
E \left[ d\tilde{r}_t \right] = \frac{\tilde{\rho}_t + 0}{dt} = \left( r_t^f - (\Phi(t)) - \delta - \mu_t^{q^K+q^B} - \sigma_t^{q^K+q^B} + \zeta_t (\sigma + \sigma_t^{q^K+q^B}) \right) + (\zeta_t - \sigma_t^N)0 + \tilde{\zeta}_t (1 - \theta_t)\tilde{\sigma}
$$

$$
E \left[ d\tilde{r}_{t/B} \right] = \mu_t^{\theta/B} = \left( r_t^f - (\Phi(t)) - \delta - \mu_t^{q^K+q^B} - \sigma_t^{q^K+q^B} + \zeta_t (\sigma + \sigma_t^{q^K+q^B}) \right) + (\zeta_t - \sigma_t^N)\sigma_t^{\theta/B}
$$

- **Remark:**
  - **Value of a single bond/coin in $N_t$-numeraire**

$$
d(\theta_t/B_t) = \mu_t^\theta + \sigma_t^\theta dZ_t - \mu_t^B dt - \sigma_t^B dZ_t + \sigma_t^B (\sigma_t^B - \sigma_t^\theta) dt
$$

$$
= \mu_t^{\theta/B} dt + \sigma_t^{\theta/B} dZ_t \quad \text{(defining return-drift and volatility)}
$$

- Terms are shifted into risk-free rate in $N_t$-numeraire, which drop out when differencing
Portfolio choice $\theta$ ($N_t$-numeraire)

Total net worth $N_t$ relative to single bond/coin of money

- Asset pricing equation (martingale method)

$$E\left[ \frac{d\tilde{r}_t}{\mu_t} \right] = \tilde{\rho}_t + 0 = \left( r^f_t - (\Phi(t) - \delta) - \mu^K + q^B - \sigma^K + q^B + \xi_t (\sigma + \sigma^K + q^B) \right) + \left( \zeta_t - \sigma^N_t \right) 0 + \tilde{\xi}_t (1 - \theta_t) \tilde{\sigma}$$

$$E\left[ \frac{d\tilde{r}_t^{\theta/B}}{\mu_t^{\delta/B}} \right] = \mu_t^{\delta/B} = \left( r^f_t - (\Phi(t) - \delta) - \mu^K + q^B - \sigma^K + q^B + \xi_t (\sigma + \sigma^K + q^B) \right) + \left( \zeta_t - \sigma^N_t \right) \sigma^{\delta/B}_t$$

- Price of Risk: $\zeta_t = -\sigma^p + \sigma^p + q + \gamma \sigma$, $\tilde{\zeta}_t = \gamma \tilde{\sigma}^n_t = \gamma (1 - \theta_t) \tilde{\sigma}$

$$\tilde{\rho}_t - \mu_t^{\delta/B} = -(\zeta_t - \sigma^N_t) \sigma^{\delta/B}_t + \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma}$$

- Capital market clearing: $1 - \theta = 1 - \vartheta$

- For stationary equilibria and $\sigma^M_t = 0$:

$$1 - \vartheta = \sqrt{\tilde{\rho} + \tilde{\mu}_t^B} / (\gamma \tilde{\sigma})$$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ (finance block)
   a. Real investment $\iota$ + Goods market clearing (static)
      ▪ Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) (forward equation)

3. Value functions (backward equation)
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      ▪ Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)(\tilde{\eta}^i)^{1-\gamma} u(K)(n^i / n^i)^{1-\gamma}$
   c. Derive $\bar{\rho} = \frac{C}{N}$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
3a.+b. + Isolating Idio. Risk

- Rephrase the conjecture value function as
  \[ V_t^i = \left( \frac{\omega_t n_t^i}{1 - \gamma} \right)^{1 - \gamma} = \left( \frac{\omega_t}{K_t} \right)^{1 - \gamma} \left( \frac{n_t^i}{N_t^i} \right)^{1 - \gamma} \frac{K_t^{1 - \gamma}}{(1 - \gamma)} \]
  \[ =: v_t^i \]
  \[ =: (\tilde{\eta}_t^i)^{1 - \gamma} \]

- \( v_t^i \) depend only on aggregate state \( \eta_t \)

- Ito’s quotation rule
  \[ \frac{d\tilde{n}_t^i}{\tilde{\eta}_t^i} = \frac{d(n_t^i/N_t)}{n_t^i/N_t} = \left( \mu_t^i - \mu_t^{N_t} + \left( \sigma_t^{N_t} \right)^2 - \sigma_t^{N_t} \sigma_t^{ni} \right) dt + \left( \sigma_t^{n_t^i} - \sigma_t^{N_t^i} \right) dZ_t + \tilde{\sigma}_t^{n_t^i} d\tilde{Z}_t \]

- Ito’s Lemma
  \[ \frac{d(\tilde{n}_t^i)^{1 - \gamma}}{(\tilde{\eta}_t^i)^{1 - \gamma}} = -\frac{1}{2} \gamma(1 - \gamma) \left( \tilde{\sigma}_t^{n_t^i} \right)^2 dt + (1 - \gamma)\tilde{\sigma}_t^{n_t^i} d\tilde{Z}_t \]
3b. BSDE for $\nu_t^i$

\[
\frac{dV_t^i}{V_t^i} = \frac{d \left( \nu_t^i (\tilde{\eta}_t^i)^{1-\gamma} (K_t)^{1-\gamma} \right)}{\nu_t^i (\tilde{\eta}_t^i)^{1-\gamma} (K_t)^{1-\gamma}}
\]

- By Ito’s product rule
  \[
  = \left( \mu_t^\nu^i + (1-\gamma)(\Phi(\eta_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\tilde{\sigma}^n_t \right)^2 + (1-\gamma)\sigma_t^\nu^i \right) dt + \text{volatility terms}
  \]

- Recall by consumption optimality
  \[
  \frac{dV_t^i}{V_t^i} - \rho dt + \frac{c_t^i}{n_t} dt \text{ follows a martingale}
  \]
  - Hence, drift above = $\rho - \frac{c_t^i}{n_t}$

- BSDE:
  \[
  \mu_t^\nu^i + (1-\gamma)(\Phi(\eta_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\tilde{\sigma}^n_t \right)^2 + (1-\gamma)\sigma_t^\nu^i = \rho - \frac{c_t^i}{n_t}
  \]
3. Deriving $C/N$-ratio $\bar{\rho}$ in stationary setting

- In stationary equilibrium
  \[
  \mu_t^v + (1 - \gamma)(\Phi(u_t) - \delta) - \frac{1}{2}\gamma(1 - \gamma)\left(\sigma^2 + (\tilde{n}^i)^2\right) + (1 - \gamma)\sigma_t^v = \rho - \frac{c_t^i}{n_t^i} = \bar{\rho}
  \]

- Recall and plug in
  \[
  \tilde{n}^i = (1 - \vartheta)\tilde{\sigma} = \sqrt{\bar{\rho} + \mu^M / \gamma}
  \]

  \[\lambda = \frac{a\sqrt{\bar{\rho} + \mu^M} - \sqrt{\gamma}\sqrt{\sigma} \bar{\rho}}{\sqrt{\bar{\rho} + \mu^M} + \kappa\sqrt{\gamma}\sqrt{\sigma} \bar{\rho}}\]

  yields an equation for $\bar{\rho}$
  \[
  (1 - \gamma)\left(\frac{1}{\kappa} \log \frac{\sqrt{\bar{\rho} + \mu^M}}{\sqrt{\bar{\rho} + \mu^M} + \phi\sqrt{\gamma}\sqrt{\sigma} \bar{\rho}} - \delta\right) - \frac{1}{2}\gamma(1 - \gamma)\left(\sigma^2 + \frac{\tilde{\rho} + \mu^M}{\gamma^2}\right) = \rho - \bar{\rho}
  \]

- For $\gamma = 1$: $\bar{\rho} = \rho$
Roadmap

- Solve one-sector money model
  - Different ways to derive money evaluation equation
  - Value function with idiosyncratic risk
  - Bubble/Ponzi scheme, $r$ vs. $g$ vs. $\zeta$ and transfersality condition
  - Mining the bubble and MMT

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  - FTPL equation
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- Medium of Exchange Role of Money
Remark: Bubble/Ponzi Scheme and Transversality

- Gov. Debt/Money is a Ponzi scheme/bubble
  - But provides store of value/insurance role

- Why does the transversality condition not rule out the bubble?
  - High individual discount rate (low SDF) since net worth (and also optimal money holdings) is idiosyncratically risky
    \[ \lim_{T \to \infty} E[\xi_T n_T] = 0 \]
  - Low “social” discount rate (high SDF)
    \[ \lim_{T \to \infty} E[e^{-r^F_T n_T}] > 0 \]

- Similarity to OLG/perpetual youth models
Remark: $r_t^f$

- From $r^f = E[dr^B]/dt$

\[
r^f = \Phi(t) - \delta - \bar{\mu}^B
\]
\[
r^f = \frac{1}{\kappa} \log \frac{\sqrt{\bar{\rho}+\bar{\mu}^B} (1+\phi a)}{\sqrt{\bar{\rho}+\bar{\mu}^B} + \phi \bar{\sigma} \bar{\rho}} - \delta - \bar{\mu}^B
\]

- Remark: bond supply growth
  - Increases $\iota$ as portfolio choice is tilted towards capital
  - Depresses real $r^f$ one-to-one because ...

\[
r^f = \rho + \mu^c - \left( (\sigma^c)^2 + (\bar{\sigma}^c)^2 \right)
\]
\[
r^f = \rho + \left( \Phi \left( \iota(\bar{\mu}^B) \right) - \delta \right) - (\sigma^2 + \left( 1 - \vartheta(\bar{\mu}^B) \right)^2 \bar{\sigma}^2)
\]

- ... agents hold more idiosyncratic risk
Remark: “sticky price of risk $\tilde{\zeta}$”

- The price of risk $\tilde{\zeta}$ depends on $\tilde{\sigma}$ only via $\rho$

$$\tilde{\zeta} = \tilde{\sigma}^n = (1 - \vartheta)\tilde{\sigma} = \sqrt{\rho + \bar{\mu}^B} \text{ using } (1 - \vartheta) = \sqrt{\rho + \bar{\mu}^B / \tilde{\sigma}}$$

- $\tilde{\zeta}$ is independent of $\tilde{\sigma}$

- Intuition:
  - Increasing $\tilde{\sigma}$ the value of money adjusts in such a way that $\tilde{\sigma}^c$ is not affected
    - For large $\kappa$, higher risk translate into smaller $q$ that reduces idiosyncratic risk driven by capital shocks.
    - For small $\kappa$, $\iota$ is lower and capital in the long-run is reduced
$r^f$ vs. $g$ vs. $E[dr^K]/dt$ for different $\tilde{\sigma}$

\begin{itemize}
  \item $E[dr^K]/dt > r^f > g$ for small $\tilde{\sigma}/\sqrt{\rho}$
  \item $E[dr^K]/dt > g > r^f$ for large $\tilde{\sigma}/\sqrt{\rho}$
  \item $E[dr^K]/dt < g$ can never happen
\end{itemize}

$a = .27, g = \frac{a}{3}, \delta = .1, \rho = .02, \kappa = 3, \bar{\mu}_B = .005$
$r^f$ versus $g$ for different $\bar{\mu}^B$

\[
g = \frac{1}{\phi} \log \frac{\sqrt{\rho + \bar{\mu}^B} (1 + \phi a)}{\sqrt{\rho + \bar{\mu}^B + \phi \bar{\sigma} \rho}} - \delta
\]

\[
r^f = \Phi(t) - \delta - \bar{\mu}^B = g
\]

$a = .27, g = \frac{a}{3}, \delta = .1, \\
\rho = .02, \bar{\sigma} = .25, \phi = 3$,
Bubble and Transversality

- Government debt is a bubble: provides risk-free store of value

- Bonds allow for self-insurance through trading
  - \( d \tilde{Z}_t^i < 0 \Rightarrow \) buy capital, sell bonds
  - \( d \tilde{Z}_t^i > 0 \Rightarrow \) sell capital, buy bonds
  \( \Rightarrow \) lowers volatility of total wealth \( n_t^i \), but increases volatility of bond wealth \( n_t^{b,i} := \theta_t^i n_t^i \)

- Why does the transversality condition (TVC) not rule out the bubble?
  - TVC for bond wealth: \( \lim_{T \to \infty} \mathbb{E}[\xi_T n_T^{b,i}] = 0 \)
  - effective discount rate in TVC = discount rate for stochastic bond portfolio \( n^{b,i} \)
    \( = \) risk-free rate \( r^f \) + (risk premium for idiosyncratic \( n^{b,i} \)-fluctuations)
  - discount rate for individual bond = discount rate for aggregate bond stock \( \int n^{b,i} di \)
    \( = \) risk-free rate \( r^f \)
  - risk premium: (self-insurance) service flow from retrading bonds (like a convenience yield)

- More general point: beneficial equilibrium trades are essential feature of (rational) bubbles
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In all three cases, the bubble – or its mere possibility – grants government some leeway:

- $s < 0$: perpetual deficits are funded out of the bubble, never have to raise taxes ("bubble mining")
- $s = 0$: government debt enjoys positive value despite zero surpluses (debt "backed" by the bubble)
- $s > 0$: no equilibrium bubble, yet possibility of bubble makes debt more sustainable unexpected (persistent) drop in surpluses below zero
  \[ \Rightarrow \text{bubble emerges instead of collapse of the value of debt} \]
Bubble Mining Laffer Curve

see Brunnermeier, Merkel, Sannikov (2020): “The Limits of Modern Monetary Theory”
MMT connection?

- MMT: “as long as inflation is not rising, budget constraints don’t matter”

- Can we lower primary surplus (more negative), $s/a$, without causing inflation?
  
  \[ \pi = \mu^B - \left[ \Phi \left( \frac{i(\mu^B - i)}{g} \right) - \delta \right] \]

  - Increase $\mu^B$
    - Direct $\pi$-effect: higher
    - Indirect $\pi$-effect: lower since growth rate $g$ increases ($q, \ell$ rises)
  - Lower $i$ to increase $g$ further.
  - Steady state inflation will be the same, but jump in price level

- No clear MMT connection (full employment/utilization in our setting)
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The 4 Roles of Money

- **Unit of account**
  - Intratemporal: Numeraire bounded rationality/price stickiness
  - Intertemporal: Debt contract incomplete markets

- **Store of value**
  - “I Theory of Money without I”
    - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Record keeping device – money is memory**
  - Virtual ledger
Motivation

- Can a country permanently run primary fiscal deficits without destabilizing its currency? (MMT?)
- The Japan FTPL critique: E.g. Noah Smith (2017)
  \[
  \frac{B_t + M_t}{\varphi_t} = \mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} (T_s - G_s) ds + \mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\varphi_s} ds, \ (\xi_t = \text{SDF})
  \]
- How to rescue the FTPL?
Motivation

- Can a country permanently run primary fiscal deficits without destabilizing its currency? (MMT?)
- The Japan FTPL critique: E.g. Noah Smith (2017)

\[ \frac{B_t + M_t}{\varphi_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t} (T_s - G_s) ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\varphi_s} ds, (\xi_t = \text{SDF}) \]

- Can a country permanently run primary fiscal deficits without destabilizing its currency? (MMT?)
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\[ \frac{B_t + M_t}{\varphi_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t} (T_s - G_s) ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\varphi_s} ds, (\xi_t = \text{SDF}) \]

- \( \Delta i_s = i_s - i^m_s \) goes towards zero
- Low interest rate environment
- Digital money + Narrow banking/FinTech

- How to rescue the FTPL?
Deriving FTPL equation (in cts time)

- Nominal government budget constraint
  \[
  \left( \mu_t^B B_t + \mu_t^M M_t + \varphi_t T_t \right) dt = \left( i_t B_t + i_t^m M_t + \varphi_t G_t \right) dt
  \]

- Multiply by nominal SDF \( \xi_t / \varphi_t \), rearrange
  \[
  \left[ (\mu_t^B - i_t) \frac{\xi_t}{\varphi_t} B_t + (\mu_t^M - i_t) \frac{\xi_t}{\varphi_t} M_t \right] dt = -\xi_t (T_t - G_t - \frac{(i_t - i_t^m) M_t}{\varphi_t}) dt
  \]

- Suppose \( \xi_t / \varphi_t \) prices the nominal bond
  - Then \( E_t \left[ \frac{d(\xi_t/\varphi_t)}{(\xi_t/\varphi_t)} \right] = i_t dt \)
  - Substitute into above, use product rule, take expectations
    \[
    E_t \left[ d \left( \frac{\xi_t}{\varphi_t} (B_t + M_t) \right) \right] = -E_t \left[ \xi_t \left( T_t - G_t - \Delta i_t \frac{M_t}{\varphi_t} \right) \right] dt
    \]

- In integral form
  \[
  \frac{B_t + M_t}{\varphi_t} = E_t \int_t^T \frac{\xi_s}{\xi_t} (T_s - G_s) ds + E_t \int_t^T \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\varphi_s} ds + \frac{\xi_T B_T + M_T}{\varphi_T}
  \]
Deriving FTPL equation (in cts time)

- Take limit $T \to \infty$

$$\frac{B_t + M_t}{\phi_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t} (T_s - G_s) ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\phi_s} ds + \lim_{T \to \infty} E_t \frac{\xi_T}{\xi_t} \frac{B_T + M_T}{\phi_T}$$

- Remark 1:
  - Literature focuses on settings, in which private-sector transversality eliminates the bubble term
  - Here: fiscal theory in setting, in which where transversality does not rule out bubbles

- Remark 2:
  - The sum of the three limits on the right may not be well-defined mathematically, because they can be infinite with opposite signs
  - The limit of the sum may nevertheless exist and be finite
    - This is what matters economically (cannot separately trade the bubble and fundamental components)
3 Forms of Seigniorage

\[
\frac{B_t + M_t}{\varphi_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t} (T_s - G_s) ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\varphi_s} ds + \lim_{T \to \infty} E_t \frac{\xi_T}{\xi_t} \frac{B_T + M_T}{\varphi_T}
\]

1. *Surprise devaluation*
   - Irrational expectations
   - Small (Hilscher, Raviv, Reis 2014)
     - Inflation options imply likelihood of exceeding 5% of GDP is less than 1%

2. *Exploiting liquidity benefits of “narrow” cash*
   - Only for “narrow” cash that provides medium-of-exchange services
   - \( \Delta i = i - i^M \)
   - 0.36 % of GDP, NPV = 20% (at most 30%) of GDP, (Reis 2019)

3. *“Money bubble mining”*
FTPL Equation with a Bubble in BruSan notation (& $\Delta i = 0$)

- Primary surplus: $sK_t = \tau aK_t - gK_t = -\hat{\mu}^B pK_t$

- FTPL equation:
  \[
  \frac{B_0}{\delta_0} = q^B K_0 = \lim_{T \to \infty} \int_0^T e^{-(r^f - g)t} sK_0 dt + \lim_{T \to \infty} e^{-(r^f - g)T} pK_0
  \]

- $r^f = (\Phi(t) - \delta) - \mu^B$

- Bubble:
  
  - $s > 0$
  - $s = 0$
  - $s < 0$

  - $r^f > g$
  - $r^f = g$
  - $r^f < g$

  - $PV(s) > 0$
  - $PV(s) = 0$
  - $PV(s) < 0$

  - No Bubble
  - Bubble > 0

Government spending $g \neq g$
Mining the “FTPL-Bubble”

- $\tilde{\mu}^B > 0$: perpetual deficits are funded out of the bubble
- PV of surpluses is $-\infty$, bubble is $\infty$, so consider finite-horizon version of FTPL equation

\[
q^B = \int_0^T e^{-(r^f-g)t} dt \cdot s + e^{-(r^f-g)T} q^B = (1-e^{\tilde{\mu}BT})q^B = e^{\tilde{\mu}BT} q^B
\]

- For $\tilde{\mu}^B > 0$: as $\tilde{\mu}^B$ increases
  - PV of surpluses over $[0,T]$ decreases
  - For $T$ large, the continuation value $e^{\tilde{\mu}^B T} p$ increases
- In this sense, mining the bubble increases its value
1. In a particular equilibrium:
   FTPL equation with bubble alone doesn’t determine price level (because size of bubble is not determined),
   **Goods market clearing** determines price level
   - ... and FTPL equation determines size of the bubble, because
   - Bubble generates a consumption demand from wealth effects

2. Multiple equilibria:
   - Off-equilibrium fiscal backing is sufficient
FTPL: Resolving Equilibrium Multiplicity

- In any equilibrium (not necessarily with constant $\tilde{\mu}_t^B$), the path of $\vartheta$ must satisfy
  \[ \dot{\vartheta}_t = (\rho - (1 - \vartheta_t)^2 \tilde{\sigma}^2 + \tilde{\mu}_t^B)\vartheta_t \]

- With constant $\tilde{\mu}_t^B$, there is continuum of solution paths in $[0,1]$ (even for positive surpluses, $\tilde{\mu}_t^B < 0$)
  - Fiscal policy adjusts surpluses $s_t = -(q_t^B + q_t^K)\tilde{\mu}_t^B\vartheta_t$ proportionally with $\vartheta_t$

- All but one solution asymptotically approach $\vartheta_t = 0$

- If surpluses $s_t = -(q_t^B + q_t^K)\tilde{\mu}_t^B\vartheta_t > \varepsilon > 0$ whenever $\vartheta_t$ is close to 0, these solutions are all eliminated
FTPL: Resolving Equilibrium Multiplicity

- Thus: uniqueness requires fiscal commitment to tax, if inflation breaks out
- If credible, off-equilibrium commitment is sufficient

\[
\dot{\vartheta}_t = \begin{cases} 
(\rho - (1 - \vartheta_t)^2 \bar{\vartheta}^2 + \bar{\mu}^B) \vartheta_t, & \text{if } \vartheta_t > \vartheta \\
(\rho - (1 - \vartheta_t)^2 \bar{\vartheta}^2) \vartheta_t - \frac{\tau a}{p_t + q_t}, & \text{if } \vartheta_t \leq \vartheta
\end{cases}
\]

With $\tau, \vartheta > 0$
FTPL: Resolving Equilibrium Multiplicity

- **Equilibria**
  - Moneyless steady state with $p^0 = 0$
  - Price $p_t$ converges over time to zero (hyperinflation)

- With $\varepsilon > 0$ fiscal backing $p_t > \varepsilon$, these equilibria are eliminated
  $\Rightarrow$ only steady state money equilibrium remains

- Off equilibrium fiscal backing suffices to rule out moneyless and hyperinflation equilibria
  - If after a hypothetical jump into the moneyless equilibrium, one can pay (a small amount) of taxes with money.
    Hence, money is not worthless and the moneyless equilibrium does not exist.
FTPL: Who controls inflation?

- Monetary dominance
  - Fiscal authority is forced to adjust budget deficits

- Fiscal dominance
  - Inability or unwillingness of fiscal authorities to control long-run expenditure/GDP ratio
  - Limits monetary authority to raise interest rates

- 0/1 Dominance vs. battle: “dynamic game of chicken”

See YouTube video 4, minute 4:15
Solve one-sector money model
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- Mining the bubble and MMT

Fiscal Theory of the Price Level (with a bubble)
- FTPL equation
- Price Level Determination
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Medium of Exchange Role of Money
The 4 Roles of Money

- Unit of account
  - Intratemporal: Numeraire
  - Intertemporal: Debt contract
- Store of value
  - “I Theory of Money without I”
  - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level
- Medium of exchange
  - Overcome double-coincidence of wants problem
- Record keeping device – money is memory
  - Virtual ledger

bounded rationality/price stickiness
incomplete markets

Money, today

Good 1, today

A

B

Good 2, day after

Money, day after

C

Good 3, tomorrow

Money, tomorrow
Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants
- Quantity equation: \( \phi_t T_t = \nu M_t \)
  - \( \nu \) (nu) is velocity (Monetarism: \( \nu \) exogenous, constant)
  - \( T \) transactions
    - Consumption
    - New investment production
    - Transaction of physical capital
    - Transaction of financial claims

\( C \)
\( tK \)
\( d\Delta^k \)
\( d\theta^{i\in M} \)

produce own machines
infinite velocity
infinite velocity
Models of Medium of Exchange

- Reduced form models
  - Cash in advance
  
  \[ T_t = \nu \frac{M_t}{\phi_t} \]
  \[ c_t \leq \sum_{j \in M} \nu^j \theta^j n_t \]
  \[ c = (c^c, l) \]

- Shopping time models
- Money in the utility function
  - New Keynesian Models
  - No satiation point
- New Monetary Economics

For generic setting encompassing all models:
see Brunnermeier-Niepelt 2018

Only assets \( j \in M \) with money-like features
Cash in Advance

- Liquidity/cash in advance constraint
  - \( c_t \leq \sum_{j \in M} \nu^j \theta^j n_t \)  
    Lagrange multiplier \( \hat{\lambda}_t \)
  - Asset \( j \in M \) which relaxes liquidity/CIA constraint

- Price of liquid/money asset

\[
q^j_{t} \in M = E_t \left[ \frac{\xi_{t+\Delta}}{\xi_t} \left( x_{t+\Delta} + q_{t+\Delta}^j \right) \right] - \hat{\lambda}_t \nu^j p_t
\]

\[
q^j_{t} \in M = E_t \left[ \frac{\xi_{t+\Delta}}{\xi_t} \frac{1}{1 + \hat{\lambda}_t \nu^j} \left( x_{t+\Delta} + q_{t+\Delta}^j \right) \right]
\]

\[
q^j_{t} \in M = \lim_{T \to \infty} E_t \left[ (T-t)/\Delta \sum_{\tau=1} \frac{\xi_{t+\tau\Delta} \Lambda^j_{t+\tau\Delta}}{\xi_t \Lambda^j_t} x_{t+\tau\Delta} \right] + \lim_{T \to \infty} E_t \left[ \frac{\Lambda^j_T}{\xi_t} \Lambda^j_t q_T \right]
\]

As if SDF is multiplied by “liquidity multiplier” (Brunnermeier-Niepelt)
Cash in Advance

- Liquidity/cash in advance constraint
  - \( c_t \leq \sum_{j \in M} \nu^j \theta^j \eta_t \)
  - Lagrange multiplier \( \hat{\lambda}_t \)
  - Asset \( j \in M \) which relaxes liquidity/CIA constraint

\[
q_{t \in M}^j = \lim_{T \to \infty} E_t \left[ \int_t^T \frac{\xi_T \Lambda^j_T}{\xi_t \Lambda^j_t} x_t d\tau \right] + \lim_{T \to \infty} E_t \left[ \frac{\xi_T \Lambda^j_T}{\xi_t \Lambda^j_t} q_T \right]
\]

- "Money bubble" easier to obtain due to liquidity service
  - Condition absent aggregate risk: \( r^M < g \) easier to obtain since \( r^M < r^f \)

- HJB approach (Problem Set #3)
  \[
  \mu_{t}^{r,j} = r_t^f + \zeta_t \sigma_{t}^{r,j} + \tilde{\zeta}_t \tilde{\sigma}^{r,j} - \lambda_t \nu^j
  \]
  (Shadow) risk-free rate of illiquid asset

where \( \lambda_t = \hat{\lambda}_t / V'(n_t) \)
Add Cash in Advance to BruSan Model

- Return on money
  - Store of value – as before
  - Liquidity service
    \[
    \frac{E[dr_t^M]}{dt} = \Phi(\nu_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r^f_t + \zeta_t(\sigma + \sigma_t^p) - \lambda_t\nu^M
    \]
- In steady state
  \[
  \Phi(i) - \delta - \frac{(\mu^M - \lambda\nu^M)}{\bar{\mu}^M} = r^f + \zeta\sigma
  \]

  - Solving the model as before ...
    - By simply replace \( \mu^M \) with \( \mu^M - \lambda\nu^M \)
    - Special case: \( \bar{\mu}^M = 0 \), i.e. \( \mu^M = \lambda\nu^M, \gamma = 1 \) ⇒ explicit solution as fcn of \( \bar{\rho} \)
      - Same \( q \) and \( p \) as a function of \( \zeta \),
      - But \( \bar{\rho} \neq \rho \) if CIA constraint binds in steady state, otherwise \( \bar{\rho} = \rho \)
        1. Assume it binds, i.e. \( \zeta = \nu\theta \)
        2. Recall from slide 21 for \( \bar{\rho}^M = 0 \) and \( \gamma = 1 \), \( \theta = \frac{\bar{\sigma} - \sqrt{\zeta}}{\bar{\sigma}} \)
        3. Equate 1. and 2. to obtain quadratic solution for \( \bar{\rho} \)
          1. If \( \bar{\rho} < \rho \), then solution equals \( \bar{\rho} \)
          2. If \( \bar{\rho} > \rho \), then \( \bar{\rho} = \rho \) and hence CIA doesn’t bind, \( \lambda = 0 \), above solution

- “Occasionally” binding CIA constraint (outside of steady state)
  - for sufficiently high \( \bar{\sigma} \), store of value (insurance motive) ⇒ \( \lambda_t = 0 \)
Add Money in Utility to BruSan Model

- Money in utility function $u(c, M/\varphi) = u(c, \theta n)$

- Can be expressed as equality constraint
  - Difference to CIA inequality: No satiation point

- DiTella add MiU to BruSan 2016 AER PP
The 4 Roles of Money

- **Unit of account**
  - Intratemporal: Numeraire
  - Intertemporal: Debt contract
  - bounded rationality/price stickiness
  - incomplete markets

- **Store of value**
  - “I Theory of Money without I”
    - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Record keeping device – money is memory**
  - Virtual ledger
Roadmap

- Solve one-sector money model
  - Different ways to derive money evaluation equation
  - Value function with idiosyncratic risk
  - Bubble/Ponzi scheme, $r$ vs. $g$ vs. $\zeta$ and transfersality condition
  - Mining the bubble and MMT

- Fiscal Theory of the Price Level (with a bubble)
  - FTPL equation
  - Price Level Determination
  - Monetary vs. fiscal authority

- Medium of Exchange Role of Money
THE END