Lecture 05
One Sector Money Model with Idio Risk
Markus Brunnermeier & Yuliy Sannikov
The 4 Roles of Money

- **Store of value**
  - “I Theory of Money without I”
    - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Unit of account**

- **Record keeping device**
  - Virtual ledger
<table>
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<tr>
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# Models on Money as Store of Value

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(money) bubbles if $r < g$
Abel et al. vs. Geerolf

$r^M = g$
One Sector Model with Money

- Agent $i$’s preferences
  \[ E \left[ \int_0^\infty e^{-\rho t} \frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt \right] \]

- Each agent operates one firm
  - Output
    \[ y_t^i = a k_t^i \]
  - Physical capital $k$
    \[ \frac{dk_t^i}{k_t^i} = (\Phi(i_t^i) - \delta) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^i \]

- Financial Friction: Incomplete markets: Agents cannot share $d\tilde{Z}_t^i$
One Sector Model with Money

- Agent $i$’s preferences
  \[ E \left[ \int_0^\infty e^{-\rho t} \frac{(c_t)^{1-\gamma}}{1-\gamma} dt \right] \]

- Each agent operates one firm
  - Output
    \[ y_t = a k_t \]
  - Physical capital $k$
    \[ \frac{d k_t^i}{k_t^i} = (\Phi(\theta_t) - \delta) dt + \sigma dZ_t + \sigma d\tilde{Z}_t^i \]

- Financial Friction: Incomplete markets: Agents cannot share $d\tilde{Z}_t^i$

- Outside money
  - Money supply growth rate $(\mu^M + \mu^{Mi})$
    - $\mu^{Mi}$ used to pay interest on money (reserves)
    - $\mu^M$ generates seignorage
      \[ \Rightarrow \text{transfers to agents proportional to networth } n_t^i \]
Postulate Aggregates and Processes

- $q_t K_t$ value of physical capital
- $p_t K_t$ value of nominal capital/outside money
  - $\frac{p_t K_t}{M_t}$ value of one unit of (outside) money
- $\vartheta_t = \frac{p_t}{q_t + p_t}$ fraction of nominal wealth
Postulate Aggregates and Processes

- $q_t K_t$ value of physical capital
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- $\theta_t = \frac{p_t}{q_t + p_t}$ fraction of nominal wealth

0. Postulate

- $q$-price process
  \[ dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t \]
- $p$-price process
  \[ dp_t / p_t = \mu_t^p dt + \sigma_t^p dZ_t, \]
- SDF for each $\tilde{i}$ agent
  \[ d\xi_t / \xi_t = -r_{\tilde{f}, \tilde{i}} dt - \zeta_t^i dZ_t - \tilde{\zeta}_t^i d\tilde{Z}_t \]

0. Return processes

\[ dr_t^{K, \tilde{i}} = \left( \frac{a - \iota_t^\tilde{i}}{q_t} + \Phi(\iota_t^\tilde{i}) - \delta + \mu_t^q + \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t + \tilde{\sigma} d\tilde{Z}_t^\tilde{i} \]

\[ dr_t^M = \left( \Phi(\iota_t) - \delta + \mu_t^p + \sigma \sigma_t^p - \mu_M - \mu_{Mi}^{Mi} + r \mu_{Mi}^{Mi} \right|_{=0} dt + (\sigma + \sigma_t^p) dZ_t \]
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

\[ \kappa \iota_t = q_t - 1 \]

\[
\frac{1}{q_t} = \Phi'(\iota_t) \quad \text{Tobin's } q
\]

All agents \( \iota_t = \iota \)

Special functional form:

\[ \Phi(\iota_t) = \frac{1}{\kappa} \log(\kappa \iota_t + 1) \Rightarrow \kappa \iota_t = q - 1 \]
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa \lambda_t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a - \iota)/q}{\gamma \tilde{\sigma}^2} + \frac{\mu^M}{\gamma \tilde{\sigma}^2} \]

\[
E[dr_t^K]/dt = \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu^q_t + \sigma \sigma^q_t = r^f_t + \zeta_t (\sigma + \sigma^q_t) + \tilde{\zeta}_t \tilde{\sigma} \\
E[dr_t^M]/dt = \Phi(\iota_t) - \delta + \mu^p_t + \sigma \sigma^p_t - \mu^M = r^f_t + \zeta_t (\sigma + \sigma^p_t) \\
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\]

Price of Risk: \[ \zeta_t = -\sigma_t^q + \sigma_t^{p+q} + \gamma \sigma, \quad \tilde{\zeta}_t = \gamma \tilde{\sigma}_t^n = \gamma (1 - \theta_t) \tilde{\sigma} \]
1b. Optimal Choices

- Optimal investment rate
  \[ \kappa \iota_t = q_t - 1 \]

- Optimal portfolio
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\[ E[dr_t^K]/dt = \frac{a - \iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q + \sigma \sigma^q = r^f_t + \zeta_t(\sigma + \sigma^q) + \tilde{\zeta}_t \tilde{\sigma} \]
\[ E[dr_t^M]/dt = \Phi(\iota_t) - \delta + \mu^p + \sigma \sigma^p - \mu^M = r^f_t + \zeta_t(\sigma + \sigma^p) \]

In Steady State

\[ constant \; q, p \]
\[ \frac{a - \iota}{q_t} + \mu^q - \mu^p + \sigma (\sigma^q - \sigma^p) + \mu^M = \zeta_t(\sigma^q - \sigma^p) + \tilde{\zeta}_t \tilde{\sigma} \]

\[ Price \; of \; Risk: \quad \zeta_t = -\sigma^q + \sigma^p + \gamma \sigma, \quad \tilde{\zeta}_t = \gamma \tilde{\sigma}^n = \gamma (1 - \theta_t) \tilde{\sigma} \]
One Sector Model with Money

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In Steady State constant \( q, p \)

\[ \frac{a - \iota}{q} + \mu^M = \tilde{\zeta}_t \bar{\sigma} \]

Price of Risk:

\[ \zeta = \gamma \sigma, \quad \tilde{\zeta} = \gamma \bar{\sigma}^n = \gamma (1 - \theta) \bar{\sigma} \]

yields

\[ 1 - \theta = . . . \]
One Sector Model with Money

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In Steady State

- constant \( q, p \)
  \[ \frac{a-\iota}{q} + \mu^M = \tilde{\zeta}_t \bar{\sigma} \]

- Price of Risk: \[ \zeta = \gamma \sigma, \quad \zeta = \gamma \bar{\sigma}^n = \gamma (1 - \theta) \bar{\sigma} \]

- Risk-free rate: \[ r^f = \Phi(\iota) - \delta - \mu^M - \gamma \sigma^2 \]

\( \Rightarrow \) yields \( 1 - \theta = . . . \)
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa_t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a - \iota)}{q} \frac{\mu^M}{\gamma \bar{\sigma}^2} + \frac{\mu^M}{\gamma \bar{\sigma}^2} \]

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E[dr_t^K]/dt = \frac{a - \iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q + \sigma \sigma^q = r^f_t + \zeta_t (\sigma + \sigma^q_t) + \bar{\zeta}_t \bar{\sigma}
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In Steady State
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Price of Risk: \[ \zeta = \gamma \sigma, \quad \bar{\zeta} = \gamma \bar{\sigma}^n = \gamma (1 - \theta) \bar{\sigma} \]
Risk-free rate: \[ r^f = \Phi(\iota) - \delta - \mu^M - \gamma \sigma^2 \]

Poll 18: \( r^f \) is
a) Risk-free rate
b) Shadow risk-free rate
c) Differs across individuals
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa_t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a-\iota)/q + \mu^M}{\gamma \sigma^2} \]

Poll 18: why does real \( r^f \) decline with \( \mu^M \)

a) Because investment rate \( \iota \) changes
b) Insurance via money becomes more costly
c) Prices are not sticky, money is neutral, and hence the real rate should not be affected

\[
E[dr^K_t]/dt = \frac{a - \iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q_t + \sigma^q_t = r^f_t + \zeta_t(\sigma + \sigma^q_t) + \tilde{\zeta}_t \tilde{\sigma}
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In Steady State

constant \( q, p \)

\[ \frac{a - \iota}{q} + \mu^M = \tilde{\zeta}_t \tilde{\sigma} \]

\[ \text{Price of Risk: } \zeta = \gamma \sigma, \quad \tilde{\zeta} = \gamma \tilde{\sigma}^n = \gamma (1 - \theta) \tilde{\sigma} \]

\[ \text{Risk-free rate: } r^f = \Phi(\iota) - \delta - \mu^M - \gamma \sigma^2 \]
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
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\[ E[dr^M_t]/dt = \Phi(\iota_t) - \delta + \mu^p_t + \sigma \sigma^p_t - \mu^M = r^f_t + \zeta_t (\sigma + \sigma^p_t) \]

Poll 17: the real \( r^f \) does not depend on \( \tilde{\sigma} \)

- a) Because determined by growth rate of \( K \)
- b) Because it is a shadow price/rate
- c) Because return on money \( E[dr^M_t]/dt \) doesn’t

In Steady State

constant \( q, p \)

\[ \frac{a - \iota}{q} + \mu^M = \tilde{\zeta}_t \tilde{\sigma} \]

Price of Risk: \[ \zeta = \gamma \sigma, \quad \tilde{\zeta} = \gamma \tilde{\sigma}^n = \gamma (1 - \theta) \tilde{\sigma} \]

Risk-free rate: \[ r^f = \Phi(\iota) - \delta - \mu^M - \gamma \sigma^2 \]
One Sector Model with Money

1b. Optimal Choices

▪ Optimal investment rate

\[ \kappa I_t = q_t - 1 \]

▪ Optimal portfolio

\[ 1 - \theta = \frac{(a-i)/q}{\gamma \bar{\sigma}^2} + \frac{\mu^M}{\gamma \bar{\sigma}^2} \]

▪ Optimal consumption

\[ \frac{c}{n} =: \zeta \quad \text{is a constant} \]

▪ Why a constant?

Recall \( \frac{c}{n} = \rho^{1/\gamma} \omega^{1-1/\gamma} \) and investment opportunity/networth multiplier is constant over time in steady state
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa \iota_t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a-i)/q}{\gamma \bar{\sigma}^2} + \frac{\mu^M}{\gamma \bar{\sigma}^2} = 1 - \vartheta = \frac{qK_t}{(p+q)K_t} \]

- Optimal consumption
  \[ \frac{c}{n} =: \zeta \Rightarrow C = \zeta (p + q)K_t = (a - i)K_t \]
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa l_t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a-\ell)/q}{\gamma \bar{\sigma}^2} + \frac{\mu^M}{\gamma \bar{\sigma}^2} \]

- Optimal consumption
  \[ \frac{c}{n} =: \zeta \Rightarrow C = \zeta \left( \frac{p + q}{q} \right) K_t / \left( 1/(1 - \theta) \right) \]

4. Market Clearing

\[ = 1 - \vartheta = \frac{qK_t}{(p+q)K_t} \]

\[ = \frac{(a - \ell)K_t}{q} \]
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa \kappa_t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a - \iota)/q}{\gamma \sigma^2} + \frac{\mu M}{\gamma \sigma^2} \]

- Optimal consumption
  \[ \frac{c}{n} = : \zeta \Rightarrow C = \zeta \left( \frac{p + q}{q} \right) \]
  \[ \Rightarrow \iota = \frac{1 - \vartheta}{1 - \vartheta + \kappa \zeta} \]

4. Market Clearing

\[ = 1 - \vartheta = \frac{qK_t}{(p + q)K_t} \]
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa I_t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a - \iota)/q}{\gamma \bar{\sigma}^2} + \frac{\mu^M}{\gamma \bar{\sigma}^2} \]

- Optimal consumption
  \[ \frac{c}{n} =: \zeta \Rightarrow C = \zeta \left( p + q \right) \]
  \[ = (a - \iota) \]
  \[ q \]
  \[ \Rightarrow \iota = \frac{(1 - \theta)a - \zeta}{1 - \theta + \kappa \zeta} \]
  \[ q = (1 - \theta) \frac{1 + \kappa a}{1 - \theta + \kappa \zeta} \]

4. Market Clearing

\[ = 1 - \vartheta = \frac{qK_t}{(p+q)K_t} \]
One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate
  \[ \kappa t = q_t - 1 \]

- Optimal portfolio
  \[ 1 - \theta = \frac{(a - i)/q}{\gamma \bar{\sigma}^2} + \frac{\mu^M}{\gamma \bar{\sigma}^2} = 1 - \vartheta = \frac{qK_t}{(p+q)K_t} \]

- Optimal consumption
  \[ \frac{c}{n} =: \zeta \Rightarrow C = \zeta \left( p + q \right) \frac{q}{1/(1 - \vartheta)} = \left( a - i \right) \frac{q}{q} \Rightarrow i = \frac{(1 - \vartheta)a - \zeta}{1 - \vartheta + \kappa \zeta} \]

4. Market Clearing

Let \( \hat{\mu}^M := (1 - \vartheta)\mu^M \) (monotone transformation)

\[ (1 - \vartheta) = \sqrt{\frac{\zeta + \hat{\mu}^M}{\gamma \bar{\sigma}^2}} = \frac{q}{q + p} \]

\[ q = (1 - \vartheta) \frac{1 + \kappa a}{1 - \vartheta + \kappa \zeta} \]
Two Stationary Equilibria

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<tr>
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<th>Money equilibrium</th>
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<td>( p_0 = 0 )</td>
<td>( p = \frac{(1 + \kappa \alpha)(\sqrt{\gamma \tilde{\sigma}} - \sqrt{\zeta} + \hat{\mu}_M)}{\sqrt{\zeta} + \hat{\mu}_M + \kappa \sqrt{\gamma \tilde{\sigma}} \zeta} )</td>
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<td>( q_0 = \frac{1 + \kappa \alpha}{1 + \kappa \zeta} )</td>
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<td>( \iota = \frac{\alpha - \zeta}{1 + \kappa \zeta} )</td>
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![Graph](image.png)
### Two Stationary Equilibria

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<td>( \lambda = \frac{\alpha - \zeta}{1 + \kappa \zeta} )</td>
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**Poll 25:** Why does aggregate risk \( \sigma \) not show up in solution

- a) We had to set it to zero to solve
- b) It scales everything in \( AK \)
- c) It is hidden in \( \zeta \)
- d) It is hidden in \( \hat{\mu}^M \)
### Two Stationary Equilibria

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<td>$q_0 = \frac{1 + \kappa a}{1 + \kappa \zeta}$</td>
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<td>$\iota = \frac{a - \zeta}{1 + \kappa \zeta}$</td>
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**Poll 26:** Why is $p$ moving in the opposite direction to $q$ in $\tilde{\sigma}$?

a) Flight to safety  

b) With high $\tilde{\sigma}$ insurance role of money is more important
Equilibrium consumption/networth ratio $\zeta$

- Recall $\zeta = \rho + \frac{\gamma - 1}{\gamma} \left( r_t^f - \rho + \frac{\gamma \sigma^2 + ((1-\vartheta)\bar{\sigma})^2}{2} \right)$
  and using $\tilde{\vartheta} = \gamma (1 - \vartheta)\bar{\sigma}$

  - Since $r_t^f = \Phi(u_t) - \delta - \mu^M - \gamma \sigma^2$ (from previous slide above)
    and using $u = \frac{\sqrt{\zeta} + \mu^M a - \sqrt{\gamma} \bar{\sigma} \zeta}{\sqrt{\zeta} + \mu^M + \kappa \sqrt{\gamma} \bar{\sigma} \zeta}$
  - ... we obtain $\zeta$

- Of course for log utility ($\gamma = 1$), simply $\zeta = \rho$

Poll 27: precautionary savings for $\gamma > 1$

a) Consumption-wealth ratio $\zeta$ decreases in $\sigma$, only for $\gamma < 1$

b) Risk $\sigma$ affects $r^f$

c) Risk $\bar{\sigma}$ affects $r^f$

d) Precautionary savings only exists with borrowing constraints
Two Stationary Equilibria for $\gamma = 1, \mu^M = 0$

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<tr>
<td>$p_0 = 0$</td>
<td>$p = \frac{(1 + \kappa a)(\tilde{\sigma} - \sqrt{\zeta})}{\sqrt{\zeta} + \kappa \tilde{\sigma} \zeta}$</td>
</tr>
<tr>
<td>$q_0 = \frac{1 + \kappa a}{1 + \kappa \zeta}$</td>
<td>$q = \frac{(1 + \kappa a)\sqrt{\zeta}}{\sqrt{\zeta} + \kappa \tilde{\sigma} \zeta}$</td>
</tr>
<tr>
<td>$\iota = \frac{a - \zeta}{1 + \kappa \zeta}$</td>
<td>$\iota = \frac{\sqrt{\zeta} a - \tilde{\sigma} \zeta}{\sqrt{\zeta} + \kappa \tilde{\sigma} \zeta}$</td>
</tr>
</tbody>
</table>

where $\zeta = \rho$
Welfare

- Value function for log utility

\[ V = \int_0^\infty e^{-\rho t} E[\log c_t] dt = \frac{1}{\rho} \log \rho + \int_0^\infty e^{-\rho t} E[\log n_t] dt \]

- By Ito:

\[ \log n_t = \log n_0 + \int_0^t \left( \frac{dn_s}{n_s} - \frac{1}{2} \frac{d<n>_s}{n_s^2} \right) \]

\[ = \log n_0 + \int_0^t \left( \mu^n_s - \frac{1}{2} (\sigma^n_s)^2 - \frac{1}{2} (\tilde{\sigma}^n_s)^2 \right) ds + \int_0^t \sigma^n_s dZ_s + \int_0^t \tilde{\sigma}^n_s d\tilde{Z}_s \]

\[ V = \frac{\log \rho}{\rho} + \frac{\log n_0}{\rho} + \int_0^\infty e^{-\rho t} \int_0^t E \left[ \mu^n_s - \frac{1}{2} (\sigma^n_s)^2 - \frac{1}{2} (\tilde{\sigma}^n_s)^2 \right] ds dt \]

- in steady state \( \mu^n_s = \mu^n = \Phi(\iota) - \delta \), \( \sigma^n_s = \sigma^n = \sigma \), \( \tilde{\sigma}^n_s = \tilde{\sigma}^n = (1 - \vartheta)\tilde{\sigma} \)

\[ \int_0^t E[...] ds = \left( \mu^n_s - \frac{1}{2} (\sigma^n_s)^2 - \frac{1}{2} (\tilde{\sigma}^n_s)^2 \right) t = \left( \Phi(\iota) - \delta - \frac{1}{2} \sigma^2 - \frac{1}{2} (1 - \vartheta)^2 \tilde{\sigma}^2 \right) t \]

- Hence, \( \int_0^\infty e^{-\rho t} \int_0^t E[...] ds dt = \int_0^\infty e^{-\rho t} \left( \Phi(\iota) - \delta - \frac{1}{2} \sigma^2 - \frac{1}{2} (1 - \vartheta)^2 \tilde{\sigma}^2 \right) t dt \)

\[ = \frac{1}{\rho} \int_0^\infty e^{-\rho t} dt \left( \Phi(\iota) - \delta - \frac{1}{2} \sigma^2 - \frac{1}{2} (1 - \vartheta)^2 \tilde{\sigma}^2 \right) \quad \text{(integration by parts)} \]

\[ = \frac{1}{\rho^2} \left( \Phi(\iota) - \delta - \frac{1}{2} \sigma^2 - \frac{1}{2} (1 - \vartheta)^2 \tilde{\sigma}^2 \right) \]
Welfare

- **Value function**

\[
V = \frac{\log \rho}{\rho} - \frac{\delta + \frac{1}{2} \sigma^2}{\rho^2} + \frac{\log K_0}{\rho} + \frac{\log (p+q)}{\rho} + \frac{\Phi(\iota) - \frac{1}{2}(1-\vartheta)^2 \tilde{\sigma}^2}{\rho^2}
\]

\( V_0 := \)

(\text{does not depend on } \hat{\mu}^M)  

Effect of \( \hat{\mu}^M \) on total (initial) wealth  

Growth-risk trade-off

- **Plug in model solution for \( p + q, \Phi(\iota), \text{and } \vartheta \)**

\[
V = V_0 + \frac{1}{\rho} \left( \frac{1}{\kappa} \log \frac{(1 + \kappa a) \tilde{\sigma}}{\kappa p \tilde{\sigma} + \sqrt{\rho + \hat{\mu}^M}} \right) + \frac{1}{\rho^2} \left( \frac{1}{\kappa} \log \frac{(1 + \kappa a) \sqrt{\rho + \hat{\mu}^M}}{\kappa p \tilde{\sigma} + \sqrt{\rho + \hat{\mu}^M}} \right) - \frac{1}{2} (\rho + \mu^M)
\]

Closed form!  
(up to \( \hat{\mu}^M \)-transformation)
Optimal Inflation Rate

- Money growth $\mu^M$ affects
  - Shadow risk-free rate
  - (Steady state) inflation in two ways
    \[
    \pi = \mu^M + \mu^{Mi} - \left(\Phi(\iota(\mu^M)) - \delta\right) \frac{g}{g}
    \]

- Proposition:
  - For sufficiently large $\tilde{\sigma}$ and $\kappa < \infty$ welfare maximizing $\mu^{M*} > 0$.
    - Laissez-faire Market outcome is not even constrained Pareto efficient
    - Economic growth rate $g$ is also higher
  - Growth maximizing $\mu^{g*} \geq \mu^{M*}$, s.t. $p^{g*} = 0$, Tobin (1965)

- Corollary: No super-neutrality of money
  - $\mu^{Mi}$: Super-neutrality only w.r.t. part of money growth rate that is used to pay interest on money
  - $\mu^M$: Nominal money growth rate affects real economic growth by distorting portfolio choice if $\kappa < \infty$
    - No price/wage rigidity, no monopolistic competition
Optimal Inflation Rate

- Pecuniary Externalities
  - Individual agent takes prices, including interest rate as given
  - Tilt portfolio towards (physical capital)
  - $q$ rises
    - Investment rate $\iota$ rises, growth rate is higher $r^M$ increases
    - Idiosyncratic risk increases reduces welfare
      - After negative shock, replacing lost capital is cheaper
  - due to “capital shocks”
  - Not with “cash flow shock” (in consumption units) as in Brunnermeier & Sannikov (2016) AER P&P
Proposition: (comparative static) 
\( \mu^M \) and optimal inflation target 
- does not depend on depreciation rate \( \delta \), but inflation does 
- is strictly increasing in idiosyncratic risk \( \tilde{\sigma} \)

“Emerging markets should have higher inflation target”
In sum..

- What should the (long-run) optimal inflation rate be?
  - Competitive market outcome is constrained Pareto inefficient.
  - Inflation is Pigouvian & internalizes pecuniary externality!
    - HH take real interest rate as given, but
    - Portfolio choice affects economic growth and real interest rate

- What role do financial frictions play?
  - incomplete markets ⇒ no superneutrality of money
    - No price/wage rigidity needed

- Emerging markets, with less developed financial markets, should have higher inflation rate/target
  - Higher idiosyncratic risk ⇒ higher pecuniary externality
The 4 Roles of Money

- **Store of value**
  - “I Theory of Money without I”
    - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Unit of account**

- **Record keeping device**
  - Virtual ledger
Fiscal Theory of the Price Level

- Money in a broad sense (includes government debt)
  - store of value emphasis!
- Suppose one can pay taxes with money (fiscal backing)
  - HH can pay with money instead of real goods
- Central bank might “print money” to pay expenditures and dilute real value of government debt
- FTPL equation: What is the real value of government debt
  - Like asset pricing equation (in discrete time)

\[
\frac{M_t + B_t}{\varphi_t} = E \left[ \sum_{t=\tau}^{\infty} \frac{\xi_{\tau}}{\xi_t} s_{\tau} K_{\tau} \right]
\]

- \( B_t \) all nominal government debt (long-term government bond \( B_t = 0 \))
- \( s_{\tau} K_{\tau} \) is primary surplus
  - (tax revenue minus government expenditure (without interest payments))
- \( \varphi_t = M_t/p_t K_t \) price level (inverse of “value of money”)
FTPL Equation

- Fiscal budget with $B_t = 0 \ \forall t$

$$p_t K_t \mu^M M_t dt + \tau a K_t = g K_t$$

- $p_t K_t \mu^M dt$ seignorage (Recall $\mu^M$ is money growth rate that excludes the part used to pay interest)

- $\tau$ tax minus transfers per unit of output

- $g$ government expenditures per unit of $K_t$ (totally wasted)

- If $g = 0$, then $\tau a K_t$ is primary surplus, denoted by $sK_t$

FTPL equation:

$$\frac{M_t + B_t}{\sigma_t} = E \left[ \sum_{\tau=t}^{\infty} \frac{\xi_\tau}{\xi_t} S_\tau K_\tau \right]$$

$$p_t K_t = \lim_{T \to \infty} \int_t^T E_t \left[ \frac{\xi_\tau}{\xi_t} S_\tau K_\tau \right] d\tau + \lim_{T \to \infty} E_t \left[ \frac{\xi_T}{\xi_t} p_T K_T \right]$$

Bubble
FTPL and Money Bubbles

- FTPL equation:

\[
p_t K_t = \lim_{T \to \infty} \int_t^T E_t \left[ \frac{\xi_\tau}{\xi_t} s_\tau K_\tau \right] d\tau + \lim_{T \to \infty} E_t \left[ \frac{\xi_T}{\xi_t} r_T K_T \right]
\]

- w/o aggregate risk, \( \sigma = 0 \):

\[
\Rightarrow \frac{\xi_\tau}{\xi_t} = e^{-r^f(\tau-t)} \text{ and } r^f = (\Phi(\nu) - \delta) - \mu^M \]

- If \( \mu^M = 0 \)

\[
\Rightarrow s = 0 \quad r^f = g, \text{ bubble can exist}
\]
FTPL and Money Bubbles

**FTPL equation:**

\[ p_t K_t = \lim_{T \to \infty} \int_t^T E_t \left[ \frac{\xi_\tau}{\xi_t} s_\tau K_\tau \right] d\tau + \lim_{T \to \infty} E_t \left[ \frac{\xi_T}{\xi_t} p_T K_T \right] \]

**w/o aggregate risk, \( \sigma = 0 \):**

\[ \Rightarrow \frac{\xi_\tau}{\xi_t} = e^{-r_f(\tau-t)} \text{ and } r_f = (\Phi(\nu) - \delta) - \mu^M \]

- If \( \mu^M = 0 \)
  \[ \Rightarrow s = 0 \quad r_f = g, \text{ bubble can exist} \]

Poll 39: What pins down the size of the money bubble?

a) For \( r_f = g \) bubble can take on any size
b) Asset pricing/Euler equation
c) Output good market clearing equation
FTPL and Money Bubbles

**FTPL equation:**

\[
p_tK_t = \lim_{T \to \infty} \int_t^T E_t \left[ \frac{\xi_t}{\xi_t} s_t K_t \right] d\tau + \lim_{T \to \infty} E_t \left[ \frac{\xi_T}{\xi_t} p_T K_T \right]
\]

- w/o aggregate risk, \( \sigma = 0 \):
  \[
  \Rightarrow \frac{\xi_t}{\xi_t} = e^{-r_f(\tau-t)} \quad \text{and} \quad r_f = \left( \Phi(\mu) - \delta \right) - \mu_M
  \]
  - If \( \mu_M = 0 \) \( \Rightarrow s = 0 \quad r_f = g \), bubble can exist
  - If \( \mu_M > 0 \) \( \Rightarrow \) transfers \( \Rightarrow s < 0 \quad r_f < g \), fundamental<0, bubble>0
  - If \( \mu_M < 0 \) \( \Rightarrow \) taxes \( \Rightarrow s > 0 \quad r_f > g \), fundamental only

- w/ aggregate risk similar
  - homework
FTPL and Money Bubbles

**FTPL equation:**

\[
p_t K_t = \lim_{T \to \infty} \int_t^T E_t \left[ \frac{\xi_{\tau}}{\xi_t} s_{\tau} K_{\tau} \right] d\tau + \lim_{T \to \infty} E_t \left[ \frac{\xi_T}{\xi_t} p_T K_T \right]
\]

- w/o aggregate risk, \( \sigma = 0 \):
  
  \[
  \Rightarrow \frac{\xi_{\tau}}{\xi_t} = e^{-r_f (\tau-t)} \text{ and } r_f = (\Phi(\nu) - \delta) - \mu^M
  \]
  
  - If \( \mu^M = 0 \) \( \Rightarrow s = 0 \) \( r_f = g \), bubble can exist
  - If \( \mu^M > 0 \) \( \Rightarrow \) transfers \( \Rightarrow s < 0 \) \( r_f < g \), fundamental < 0, bubble > 0
  - If \( \mu^M < 0 \) \( \Rightarrow \) taxes \( \Rightarrow s > 0 \) \( r_f > g \), fundamental only

- w/ aggregate risk similar
  - homework

Poll 41: Suppose gov. \( g > 0 \) (and wasted)
  a) Analysis doesn’t change
  b) Only goods market clearing changes
  c) SDF \( \xi_t \) is different, and so is \( r_f \)
FTPL: Resolving Equilibrium Multiplicity

- **Equilibria**
  - Moneyless steady state with $p^0 = 0$
  - Price $p_t$ converges over time to zero (hyperinflation)

- With $\varepsilon > 0$ fiscal backing $p_t > \varepsilon$, these equilibria are eliminated
  $\Rightarrow$ only steady state money equilibrium remains

- Off equilibrium fiscal backing suffices to rule out moneyless and hyperinflation equilibria
  - If after a hypothetical jump into the moneyless equilibrium, one can pay (a small amount) of taxes with money.
  Hence, money is not worthless and the moneyless equilibrium does not exist.
FTPL: Who controls inflation?

- **Monetary dominance**
  - Fiscal authority is forced to adjust budget deficits

- **Fiscal dominance**
  - Inability or unwillingness of fiscal authorities to control long-run expenditure/GDP ratio
  - Limits monetary authority to raise interest rates

- **0/1 Dominance vs. battle:** “dynamic game of chicken”

See [YouTube video 4](https://www.youtube.com/watch?v=4), minute 4:15
The 4 Roles of Money

- **Store of value**
  - “I Theory of Money without I”
    - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Unit of account**

- **Record keeping device**
  - Virtual ledger
Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants

- Quantity equation: \( \varphi_t T_t = \nu M_t \)
Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants

Quantity equation: \( \varphi_t T = \nu M_t \)

- \( \nu \) (nu) is velocity (Monetarism: \( \nu \) exogenous, constant)
- \( T \) transactions
  - Consumption
  - New investment production
  - Transaction of physical capital
  - Transaction of financial claims

\[
\begin{align*}
C & \quad Y \\
iK & \\
d\Delta^k & \\
d\theta^{j \in M} & 
\end{align*}
\]
Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants

Quantity equation: \( \varphi_t T_t = \nu M_t \)

- \( \nu \) (nu) is velocity (Monetarism: \( \nu \) exogenous, constant)
- \( T \) transactions
  - Consumption \( C \)
  - New investment production \( iK \) produce own machines
  - Transaction of physical capital \( d\Delta^k \) infinite velocity
  - Transaction of financial claims \( d\theta^j_{i\in M} \) infinite velocity
Models of Medium of Exchange

- Reduced form models
  - Cash in advance
  - Shopping time models
  - Money in the utility function
    - New Keynesian Models
    - No satiation point
  - New Monetary Economics

\[ T_t = \nu \frac{M_t}{\theta_t} \]

\[ c_t \leq \sum_{j \in M} \nu^j \theta^j n_t \]

\[ c = (c^c, l) \]

Only asset with money-like features

For general setting:
see Brunnermeier-Niepelt 2018
Cash in Advance

- **Liquidity/cash in advance constraint**
  - \( c_t \leq \sum_{j \in M} \nu^j \theta^j n_t \)  
    - Lagrange multiplier \( \hat{\lambda}_t \)
  - Asset \( j \in M \) which relaxes liquidity/CIA constraint

- **Price of liquid/money asset**

\[
\begin{align*}
p_t^{j \in M} &= E_t \left[ \frac{\xi_t + \Delta}{\xi_t} \left( x_{t+\Delta} + p_{t+\Delta} \right) \right] - \hat{\lambda}_t \nu^j p_j^{t \in M} \\
p_t^{j \in M} &= E_t \left[ \frac{1}{\xi_t} \frac{1}{1 + \hat{\lambda}_t \nu^j} \left( x_{t+\Delta} + p_{t+\Delta} \right) \right] \\
p_t^{j \in M} &= \lim_{T \to \infty} E_t \left[ \sum_{\tau=1}^{(T-t)/\Delta} \frac{\xi_t + \tau \Delta}{\xi_t} \frac{\Lambda^j_{t+\tau \Delta}}{\Lambda^j_t} x_{t+\tau \Delta} \right] + \lim_{T \to \infty} E_t \left[ \frac{\xi_T}{\xi_t} \frac{\Lambda^j_T}{\Lambda^j_t} p_T \right]
\end{align*}
\]

As if SDF is multiplied by “liquidity multiplier” (Brunnermeier Niepelt)
Cash in Advance

- Liquidity/cash in advance constraint
  - \( c_t \leq \sum_{j \in M} \nu^j \theta^j n_t \)
  - Lagrange multiplier \( \hat{\lambda}_t \)
  - Asset \( j \in M \) which relaxes liquidity/CIA constraint

\[
p^j_{t\in M} = \lim_{T \to \infty} E_t \left[ \int_t^T \frac{\xi_\tau \Lambda^j_{\tau}}{\xi_t \Lambda^j_{t}} x_\tau d\tau \right] + \lim_{T \to \infty} E_t \left[ \frac{\xi_T \Lambda^j_T}{\xi_t \Lambda^j_t} p_T \right]
\]

- “Money bubble” easier to obtain due to liquidity service
  - Condition absent aggregate risk: \( r^M < g \) easier to obtain since \( r^M < r^f \)

- HJB approach
  (Problem Set #3)

\[
\mu_t^r,^j = r_t^f + \zeta_t \sigma_t^r,^j + \tilde{\zeta}_t \tilde{\sigma}^r,^j - \lambda_t \nu^j
\]

(Shadow) risk-free rate of illiquid asset

where \( \lambda_t = \hat{\lambda}_t / V'(n_t) \)
Add Cash in Advance to BruSan Model

- Return on money
  - Store of value – as before
  - Liquidity service
    \[ \frac{E[dr_t^M]}{dt} = \Phi(\iota_t) - \delta + \mu_t^p + \sigma \sigma_t^p - \mu^M = r^f_t + \zeta_t (\sigma + \sigma_t^p) - \lambda_t v^M \]

- In steady state
  \[ \Phi(i) - \delta - \left( \frac{\mu^M - \lambda v^M}{\bar{\mu}^M} \right) = r^f + \zeta \sigma \]

- Solving the model as before ...
  - By simply replace \( \mu^M \) with \( \mu^M - \lambda_t v^M_t \)
  - Special case: \( \bar{\mu}^M = 0 \), i.e. \( \mu^M = \lambda v^M \), \( \gamma = 1 \) ⇒ explicit solution as fcn of \( \zeta \)
    - Same \( q \) and \( p \) as a function of \( \zeta \),
    - But \( \zeta \neq \rho \) if CIA constraint binds in steady state
      - Check:
        1. Assume it binds, i.e. \( \zeta = v \theta \)
        2. Recall from slide 21 for \( \bar{\mu}^M = 0 \) and \( \gamma = 1 \), \( \theta = \frac{\sigma - \sqrt{\zeta}}{\sigma} \)
        3. Equate 1. and 2. to obtain quadratic solution for \( \zeta \)
          1. If \( \zeta < \rho \), then solution equals \( \zeta \)
          2. If \( \zeta > \rho \), then \( \zeta = \rho \) and hence CIA doesn’t bind, \( \lambda = 0 \), above solution

- “Occasionally” binding CIA constraint (outside of steady state)
  since for sufficiently high \( \bar{\sigma} \) agents hold money as store of value (insurance motive)
  \( \Rightarrow \lambda_t = 0 \)

- Money in the utility function is as if constraint always binds, see DiTella (2018)
The 4 Roles of Money

- **Store of value**
  - “I Theory of Money without I”
    - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Unit of account**
  - Benchmark price to have agreed upon/fewer relative prices
  - Price stickiness in New Keynesian Models

- **Record keeping device**
  - Virtual ledger
Extra Slides