LECTURE 06:
MEAN-VARIANCE ANALYSIS & CAPM
Overview

1. **Introduction:**
   Simple CAPM with quadratic utility functions
   (from beta-state price equation)

2. **Traditional Derivation of CAPM**
   - Demand: Portfolio Theory
   - Aggregation: Fund Separation Theorem
   - Equilibrium: CAPM

3. **Modern Derivation of CAPM**
   - Projections
   - Pricing Kernel and Expectation Kernel

4. **Testing CAPM**

5. **Practical Issues** – Black-Litterman
Recall State-price Beta model

Recall:

\[ E[R^h] - R^f = \beta^h E[R^* - R^f] \]

Where \( \beta^h := \frac{\text{cov}[R^*, R^h]}{\text{var}[R^*]} \)

very general – but what is \( R^* \) in reality?
Simple CAPM with Quadratic Expected Utility

1. All agents are identical
   - Expected utility \( U(x_0, x_1) = \sum_s \pi_s u(x_0, x_s) \Rightarrow m = \frac{\partial_1 u}{E[\partial_0 u]} \)
   - Quadratic \( u(x_0, x_1) = v_0(x_0) - (x_1 - \alpha)^2 \)
   - \( \Rightarrow \partial_1 u = [-2(x_{1,1} - \alpha), \ldots, -2(x_{S,1} - \alpha)] \)
   - Excess return
     \[
     E[R^h] - R^f = -\frac{\text{cov}[m, R^h]}{E[m]} = -\frac{R^f \text{cov}[\partial_1 u, R^h]}{E[\partial_0 u]} = R^f \frac{2\text{cov}[x_1, R^h]}{E[\partial_0 u]}
     \]
   - Also holds for market portfolio
     \[
     \frac{E[R^h]}{E[R^{mkt}]} - R^f = \frac{\text{cov}[x_1, R^h]}{\text{cov}[x_1, R^{mkt}]} \]
Simple CAPM with Quadratic Expected Utility

\[
\frac{E[R^h] - R^f}{E[R^{mkt}] - R^f} = \frac{\text{cov}[x_1, R^h]}{\text{cov}[x_1, R^{mkt}]} \]

2. Homogenous agents + Exchange economy
   \[ x_1 = \text{aggr. endowment and is perfectly correlated with } R^m \]

\[
\frac{E[R^h] - R^f}{E[R^{mkt}] - R^f} = \frac{\text{cov}[R^{mkt}, R^h]}{\text{var}[R^{mkt}]} \]

Since \( \beta^h = \frac{\text{cov}[R^h, R^{mkt}]}{\text{var}[R^{mkt}]} \)

**Market Security Line**

\[
E[R^h] = R^f + \beta^h \{E[R^{mkt}] - R^f\} \]

NB: \( R^* = R^f \frac{a + b_1 R^{mkt}}{a + b_1 R^f} \) in this case \((b_1 < 0)! \)
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5. Practical Issues – Black-Litterman
Definition: Mean-Variance Dominance & Efficient Frontier

- Asset (portfolio) A \textit{mean-variance dominates} asset (portfolio) B if \( \mu_A \geq \mu_B \) and \( \sigma_A < \sigma_B \) or if \( \mu_A > \mu_B \) while \( \sigma_A \leq \sigma_B \).

- \textit{Efficient frontier}: loci of all non-dominated portfolios in the mean-standard deviation space.

By definition, no ("rational") mean-variance investor would choose to hold a portfolio not located on the efficient frontier.
Expected Portfolio Returns & Variance

• Expected returns (linear)
  \[ -\mu^h := E[r^h] = w^h'\mu, \text{ where each } w^j = \frac{h^j}{\sum_j h^j} \]

• Variance
  \[ -\sigma_h^2 := \text{var}[r_h] = w'Vw \]
  \[ = (w_1 \quad w_2) \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]
  \[ = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2\sigma_{12} \geq 0 \]
Illustration of 2 Asset Case

• For certain weights: $w_1$ and $1 - w_1$
  \[
  \mu_h = w_1 \mu_1 + (1 - w_1) \mu_2
  \]
  \[
  \sigma_h^2 = w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2w_1(1 - w_1) \rho_{12} \sigma_1 \sigma_2
  \]
  (Specify $\sigma_h^2$ and one gets weights and $\mu_h$'s)

• Special cases $[w_1$ to obtain certain $\sigma_h]$
  \[
  - \rho_{12} = 1 \Rightarrow w_1 = \frac{\pm \sigma_h - \sigma_2}{\sigma_1 - \sigma_2}
  \]
  \[
  - \rho_{12} = -1 \Rightarrow w_1 = \frac{\pm \sigma_h + \sigma_2}{\sigma_1 + \sigma_2}
  \]
For $\rho_{12} = 1 \Rightarrow w_1 = \frac{\pm \sigma_h - \sigma_2}{\sigma_1 - \sigma_2}$

$$\sigma_h = |w_1 \sigma_1 + (1 - w_1)\sigma_2|$$

$$\mu_h = w_1 \mu_1 + (1 - w_1)\mu_2 = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} (\pm \sigma_h - \sigma_1)$$

The Efficient Frontier: Two Perfectly Correlated Risky Assets
For $\rho_{12} = -1 \Rightarrow w_1 = \frac{\pm \sigma_p + -\sigma_2}{\sigma_1 + \sigma_2}$

$\sigma_h = |w_1 \sigma_1 - (1 - w_1) \sigma_2|$

$\mu_h = w_1 \mu_1 + (1 - w_1) \mu_2 = \frac{\sigma_2}{\sigma_1 + \sigma_2} \mu_1 + \frac{\sigma_1}{\sigma_1 + \sigma_2} \mu_2 \pm \frac{\mu_2 - \mu_1}{\sigma_1 + \sigma_2} \sigma_p$

The Efficient Frontier: Two Perfectly Negative Correlated Risky Assets
For $\rho_{12} \in (-1,1)$
For $\sigma_1 = 0$

The Efficient Frontier: One Risky and One Risk-Free Asset
Efficient frontier with \( n \) risky assets

- A frontier portfolio is one which displays minimum variance among all feasible portfolios with the same expected portfolio return.

- \[
\begin{align*}
\min_w \frac{1}{2} w' V w \\
\lambda: w' \mu = \mu^h, \quad (\sum_j w_j \mathbb{E}[\tilde{r}_i] = \mu^h) \\
\gamma: w' 1 = 1, \quad (\sum_j w_j = 1)
\end{align*}
\]

- **Result:** Portfolio weights are linear in expected portfolio return
  \[
  w_h = g + h \mu^h
  \]
  - If \( \mu^h = 0, w_h = g \)
  - If \( \mu^h = 1, w_h = g + h \)

  • Hence, \( g \) and \( g + h \) are portfolios on the frontier.
The first FOC can be written as:

\[ Vw = \lambda \mu + \gamma 1 \]
\[ w = \lambda V^{-1} \mu + \gamma V^{-1} 1 \]
\[ \mu'w = \lambda (\mu' V^{-1} \mu) + \gamma (\mu' V^{-1} 1) \]
• Noting that $\mu'w_h = w_h'$, combining 1st and 2nd FOC

$$
\mu_h = \mu'w_h = \lambda \left( \mu'V^{-1}\mu \right) + \gamma \left( \mu'V^{-1}1 \right)
$$

• Pre-multiplying the 1st FOC by 1 yields

$$
1'w_h = w_h'1 = \lambda (1'V^{-1}\mu + \gamma (1'V^{-1}1) = 1
$$

• Solving for $\lambda, \gamma$

$$
\lambda = \frac{C\mu^h - A}{D}, \quad \gamma = \frac{B - A\mu^h}{D}
$$

$$
D = BC - A^2
$$
• Hence, \( w_h = \lambda V^{-1}\mu + \gamma V^{-1}\mathbf{1} \) becomes

\[
w_h = \frac{C\mu^h - A}{D} V^{-1}\mu + \frac{B - A\mu^h}{D} V^{-1}\mathbf{1}
\]

\[
= \frac{1}{D} [B(V^{-1}\mathbf{1}) - A(V^{-1}\mu)] + \frac{1}{D} [C(V^{-1}\mu) - A(V^{-1}\mathbf{1})] \mu^h
\]

• **Result:** Portfolio weights are linear in expected portfolio return \( w_h = g + h\mu^h \)
  - If \( \mu^h = 0 \), \( w_h = g \)
  - If \( \mu^h = 1 \), \( w_h = g + h \)
    • Hence, \( g \) and \( g + h \) are portfolios on the frontier
Characterization of Frontier Portfolios

• **Proposition:** The entire set of frontier portfolios can be generated by ("are convex combinations" $g$ of) and $g + h$.

• **Proposition:** The portfolio frontier can be described as convex combinations of any two frontier portfolios, not just the frontier portfolios $g$ and $g + h$.

• **Proposition:** Any convex combination of frontier portfolios is also a frontier portfolio.
...Characterization of Frontier Portfolios...

- For any portfolio on the frontier,
  \[ \sigma^2(\mu^h) = [\mathbf{g} + \mathbf{h}\mu^h]'V[\mathbf{g} + \mathbf{h}\mu^h] \]
  with \( \mathbf{g} \) and \( \mathbf{h} \) as defined earlier.

Multiplying all this out and some algebra yields:

\[ \sigma^2(\mu^h) = \frac{C}{D} \left[ \mu^h - \frac{A}{C} \right]^2 + \frac{1}{C} \]
...Characterization of Frontier Portfolios...

i. the expected return of the minimum variance portfolio is $\frac{A}{C}$;

ii. the variance of the minimum variance portfolio is given by $\frac{1}{C}$;

iii. Equation $\sigma^2(\mu^h) = \frac{C}{D} \left[ \mu^h - \frac{A}{C} \right]^2 + \frac{1}{C}$ is a
   - parabola with vertex $\left( \frac{1}{C}, \frac{A}{C} \right)$ in the expected return/variance space
   - hyperbola in the expected return/standard deviation space.
Figure 6-3  The Set of Frontier Portfolios: Mean/Variance Space

\[ E[\tilde{r}_n] = \frac{A}{C} \pm \sqrt{\frac{D}{C}} \left( \sigma^2 - \frac{1}{C} \right) \]
Figure 6-4  The Set of Frontier Portfolios: Mean/SD Space
Figure 6-5  The Set of Frontier Portfolios: Short Selling Allowed
Efficient Frontier with risk-free asset

The Efficient Frontier: One Risk Free and n Risky Assets
Efficient Frontier with risk-free asset

- $\min_{w} \frac{1}{2} w'Vw$
  - s.t. $w'\mu + (1 - w^T 1)r^f = \mu^h$
  - FOC
    - $w_h = \lambda V^{-1}(\mu - r^f 1)$
    - Multiplying by $(\mu - r^f 1)^T$ yields $\lambda = \frac{\mu^h - r^f}{(\mu - r^f 1)^TV^{-1}(\mu - r^f 1)}$
  - Solution
    - $w_h = \frac{V^{-1}(\mu - r^f 1)(\mu^h - r^f)}{H^2}$, where $H = \sqrt{B - 2Ar^f + C(r^f)^2}$
Efficient frontier with risk-free asset

- **Result 1:** Excess return in frontier excess return

\[
\text{cov}[r_h, r_p] = w'_h V w_p = w'_h (\mu - r^f 1) \frac{E[r_p] - r^f}{H^2}
\]

\[
= \frac{(E[r_h] - r^f)(E[r_p] - r^f)}{H^2}
\]

\[
\text{var}[r_p] = \frac{(E[r_p] - r^f)^2}{H^2}
\]

\[
E[r_h] - r^f = \frac{\text{cov}[r_h, r_p]}{\text{var}[r_p]} (E[r_p] - r^f)
\]

\[
\beta_{h,p} = \frac{\text{cov}[r_h, r_p]}{\text{var}[r_p]}
\]

(Holds for any frontier portfolio \( p \), in particular the market portfolio)
Efficient Frontier with risk-free asset

• **Result 2:** Frontier is linear in \((E[r], \sigma)\)-space

\[
\text{var}[r_h] = \frac{(E[r_h] - r_f)^2}{H^2}
\]

\[
E[r_h] = r_f + H\sigma_h
\]

where \(H\) is the Sharpe ratio

\[
H = \frac{E[r_h] - r_f}{\sigma_h}
\]
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5. Practical Issues – Black-Litterman
Aggregation: Two Fund Separation

• Doing it in two steps:
  – First solve frontier for $n$ risky asset
  – Then solve tangency point

• Advantage:
  – Same portfolio of $n$ risky asset for different agents with different risk aversion
  – Useful for applying equilibrium argument (later)

Recall HARA class of preferences
Two Fund Separation

Price of Risk = highest Sharpe ratio

Optimal Portfolios of Two Investors with Different Risk Aversion
Mean-Variance Preferences

- $U(\mu_h, \sigma_h)$ with $\frac{\partial U}{\partial \mu_h} > 0, \frac{\partial U}{\partial \sigma_h^2} < 0$
  - Example: $E[W] - \frac{\rho}{2} \text{var}[W]$

- Also in expected utility framework
  - Example 1: Quadratic utility function (with portfolio return $R$)
    - $U(R) = a + bR + cR^2$
    - vNM: $E[U(R)] = a + bE[R] + cE[R^2] = a + b\mu_h + c\mu_h^2 + c\sigma_h^2 = g(\mu_h, \sigma_h)$
  - Example 2: CARA Gaussian
    - asset returns jointly normal $\Rightarrow \sum_i w^i r^i$ normal
    - If $U$ is CARA $\Rightarrow$ certainty equivalent is $\mu_h - \frac{\rho}{2} \sigma_h^2$
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5. Practical Issues – Black-Litterman
Equilibrium leads to CAPM

• Portfolio theory: only analysis of demand
  – price/returns are taken as given
  – composition of risky portfolio is same for all investors

• Equilibrium Demand = Supply (market portfolio)

• CAPM allows to derive
  – equilibrium prices/returns.
  – risk-premium
The CAPM with a risk-free bond

- The market portfolio is efficient since it is on the efficient frontier.
- All individual optimal portfolios are located on the half-line originating at point \((0, r_f)\).
- The slope of Capital Market Line (CML):

\[
E[R_{mkt}] - R_f \over \sigma_{mkt}
\]

\[
E[R_h] = R_f + \frac{E[R_{mkt}] - R_f}{\sigma_{mkt}} \sigma_h
\]
The Capital Market Line

\[ M \]

\[ r_M \]

\[ r_f \]

\[ \sigma_M \]

\[ \sigma_p \]

CML
The Security Market Line

\[ \text{slope SML} = \frac{E(r_i) - r_f}{\beta_i} \]

\[ E(r) \]

\[ E(r_i) \]

\[ E(r_M) \]

\[ r_f \]

\[ \beta_{M=1} \]

\[ \beta_i \]
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4. Practical Issues
Projections

- States $s = 1, \ldots, S$ with $\pi_s > 0$
- Probability inner product

$$[x, y]_{\pi} = \sum_s \pi_s x_s y_s = \sum_s \sqrt{\pi_s} x_s \sqrt{\pi_s} y_s$$

- $\pi$-norm $\|x\| = \sqrt{[x, x]_{\pi}}$ (measure of length)
  1. $\|x\| > 0 \ \forall x \neq 0$ and $\|x\| = 0$ if $x = 0$
  2. $\|\lambda x\| = |\lambda| \|x\|$
  3. $\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in \mathbb{R}^S$
x and y are \( \pi \)-orthogonal iff \([x, y]_\pi = 0\), i.e. \( \mathbb{E}[xy] = 0\)
...Projections...

- $\mathcal{Z}$ space of all linear combinations of vectors $z_1, \ldots, z_n$
- Given a vector $y \in \mathbb{R}^S$ solve
  $$\min_{\alpha \in \mathbb{R}^n} E \left[ y - \sum_j \alpha^j z^j \right]^2$$
- FOC: $\sum_s \pi_s (y_s - \sum_j \alpha^j z^j_s)z^j = 0$
  - Solution $\hat{\alpha} \Rightarrow y^Z = \sum_j \hat{\alpha}^j z^j, \epsilon := y - y^Z$
- [smallest distance between vector $y$ and $\mathcal{Z}$ space]
$E[\varepsilon z^j] = 0$ for each $j = 1, \ldots, n$ (FOC)
$\varepsilon \perp z$
$y^Z$ is the (orthogonal) projection on $Z$
$y = y^Z + \varepsilon'$, $y^Z \in Z, \varepsilon \perp z$
Expected Value and Co-Variance...

squeeze axis by $\sqrt{\pi_s}$

\[
x, y = E[xy] = \text{cov}[x,y] + E[x]E[y]
\]
\[
x, x = E[x^2] = \text{var}[x] + E[x]^2
\]
\[
\|x\| = \sqrt{E[x^2]}
\]

\[x = \hat{x} + \tilde{x}\]
...Expected Value and Co-Variance

- $x = \hat{x} + \tilde{x}$ where
  - $\hat{x}$ is a projection of $x$ onto $\langle 1 \rangle$
  - $\tilde{x}$ is a projection of $x$ onto $\langle 1 \rangle^\perp$

- $E[x] = [x, 1]_\pi = [\hat{x}, 1]_\pi = \hat{x}[1, 1]_\pi = \hat{x}$
- $\text{var}[x] = [\tilde{x}, \tilde{x}]_\pi = \text{var}[\tilde{x}]$
  - $\sigma_x = ||\tilde{x}||_\pi$

- $\text{cov}[x, y] = \text{cov}[\tilde{x}, \tilde{y}] = [\tilde{x}, \tilde{y}]_\pi$

- Proof: $[x, y]_\pi = [\hat{x}, \hat{y}]_\pi + [\tilde{x}, \tilde{y}]_\pi$
  - $[\hat{y}, \tilde{x}]_\pi = [\tilde{y}, \hat{x}]_\pi = 0$, $[x, y]_\pi = E[\hat{y}]E[\hat{x}] + \text{cov}[\tilde{x}, \tilde{y}]$
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Pricing Kernel $m^*$...

- $\langle X \rangle$ space of feasible payoffs.
- If no arbitrage and $\pi \gg 0$ there exists SDF $m \in \mathbb{R}^S, m \gg 0$, such that $q(z) = E[mz]$.
- $m \in \mathbb{R}^S$ – SDF need not be in asset span.
- A pricing kernel is a $m^* \in \langle X \rangle$ such that for each $z \in \langle X \rangle$, $q(z) = E[m^*z]$
...Pricing Kernel - Examples...

• Example 1:
  
  – $S = 3, \pi^S = \frac{1}{3}$
  
  – $x_1 = (1,0,0), x_2 = (0,1,1)$ and $p = \left(\frac{1}{3}, \frac{2}{3}\right)$
  
  – Then $m^* = (1,1,1)$ is the unique pricing kernel.

• Example 2:

  – $x_1 = (1,0,0), x_2 = (0,1,0), p = \left(\frac{1}{3}, \frac{2}{3}\right)$
  
  – Then $m^* = (1,2,0)$ is the unique pricing kernel.
...Pricing Kernel – Uniqueness

• If a state price density exists, there exists a **unique** pricing kernel.
  – If $\dim \langle X \rangle = S$ (markets are complete), there are exactly $m$ equations and $m$ unknowns
  – If $\dim \langle X \rangle < S$, (markets may be incomplete)
    For any state price density (=SDF) $m$ and any $z \in \langle X \rangle$
    $E[(m - m^*)z] = 0$
    $m = (m - m^*) + m^* \Rightarrow m^*$ is the “projection” of $m$ on $\langle X \rangle$
  • Complete markets $\Rightarrow m^* = m$ (SDF=state price density)
**Expectations Kernel** $k^*$

- An expectations kernel is a vector $k^* \in \langle X \rangle$
  - Such that $E[z] = E[k^*z]$ for each $z \in \langle X \rangle$
- Example
  - $S = 3, \pi^s = \frac{1}{3}, x_1 = (1,0,0), x_2 = (0,1,0)$
  - Then the unique $k^* = (1,1,0)$
- If $\pi \gg 0$, there exists a unique expectations kernel.
- Let $I = (1, \ldots, 1)$ then for any $z \in \langle X \rangle$
  $$E[(I - k^*)z] = 0$$
  - $k^*$ is the “projection” of $I$ on $\langle X \rangle$
  - $k^* = I$ if bond can be replicated (e.g. if markets are complete)
Mean Variance Frontier

- **Definition 1:** \( z \in \langle X \rangle \) is in the mean variance frontier if there exists no \( z' \in \langle X \rangle \) such that \( E[z'] = E[z] \), \( q(z') = q(z) \) and \( \text{var}[z'] < \text{var}[z] \)

- **Definition 2:** Let \( \mathcal{E} \) be the space generated by \( m^* \) and \( k^* \)
  - Decompose \( z = z^\varepsilon + \varepsilon \) with \( z^\varepsilon \in \mathcal{E} \) and \( \varepsilon \perp \mathcal{E} \)
  - Hence, \( E[\varepsilon] = E[\varepsilon k^*] = 0 \), \( q(\varepsilon) = E[\varepsilon m^*] = 0 \)
  - \( \text{cov}[\varepsilon, z^\varepsilon] = E[\varepsilon z^\varepsilon] = 0 \), since \( \varepsilon \perp \mathcal{E} \)
  - \( \text{var}[z] = \text{var}[z^\varepsilon] + \text{var}[\varepsilon] \) (price of \( \varepsilon \) is zero, but positive variance)

- \( z \) is in mean variance frontier \( \Rightarrow \) \( z \in \mathcal{E} \).
  - Every \( z \in \mathcal{E} \) is in mean variance frontier.
Frontier Returns...

- Frontier returns are the returns of frontier payoffs with non-zero prices.
  [Note: R indicates Gross return]

\[ R_{k^*} = \frac{k^*}{q(k^*)} = \frac{k^*}{E[m^*]} \]
\[ R_{m^*} = \frac{m^*}{q(m^*)} = \frac{m^*}{E[m^*m^*]} \]

- If \( z = \alpha m^* + \beta k^* \) then
  \[ R_z = \frac{\alpha q(m^*)}{\alpha q(m^*) + \beta q(k^*)} R_{m^*} + \frac{\beta q(k^*)}{\alpha q(m^*) + \beta q(k^*)} R_{k^*} \]

- graphically: payoffs with price of p=1.
\( \langle X \rangle = R^S = R^3 \)

Mean-Variance Return Frontier

\( p=1 \)-line = return-line (orthogonal to \( m^* \) )

Mean-Variance Payoff Frontier
Mean-Variance (Payoff) Frontier

NB: graphical illustrated of expected returns and standard deviation changes if bond is not in payoff span.
Mean-Variance (Payoff) Frontier

- Efficient (return) frontier
- Inefficient (return) frontier
- Expected return
- Standard deviation

\[ (1,1,1) \]

\[ m^* \]
...Frontier Returns

(if agent is risk-neutral)

- If \( k^* = \alpha m^* \), frontier returns \( R_{k^*} \)
- If \( k^* \neq \alpha m^* \), frontier returns can be written as:
  \[
  R_\lambda = R_{k^*} + \lambda (R_{m^*} - R_{k^*})
  \]
- Expectations and variance are
  \[
  E[R_\lambda] = E[R_{k^*}] + \lambda (E[R_{m^*}] - E[R_{k^*}])
  \]
  \[
  \text{var}[R_\lambda] = \text{var}[R_{k^*}] + 2\lambda \text{cov}[R_{k^*}, R_{m^*} - R_{k^*}] + \lambda^2 \text{var}[R_{m^*} - R_{k^*}]
  \]
- If risk-free asset exists, these simplify to:
  \[
  E[R_\lambda] = R_f + \lambda (E[R_{m^*}] - R_f) = R_f \pm \sigma(R_\lambda) \frac{E[R_{m^*}] - R_f}{\sigma(R_{m^*})}
  \]
  \[
  \text{var}[R_\lambda] = \lambda^2 \text{var}[R_{m^*}] , \sigma(R_\lambda) = |\lambda| \sigma(R_{m^*})
  \]
Minimum Variance Portfolio

• Take FOC w.r.t. \( \lambda \) of
  
  \[
  \text{var}[R_{\lambda}] = \text{var}[R_{k^*}] + 2\lambda \text{cov}[R_{k^*}, R_{m^*} - R_{k^*}] + \lambda^2 \text{var}[R_{m^*} - R_{k^*}]
  \]

• Hence, MVP has return of
  
  \[
  R_{k^*} + \lambda_0 (R_{m^*} - R_{k^*})
  \]
  
  \[
  \lambda_0 = -\frac{\text{cov}[R_{k^*}, R_{m^*} - R_{k^*}]}{\text{var}[R_{m^*} - R_{k^*}]}
  \]
Illustration of MVP

\[
\langle X \rangle = \mathbb{R}^2 \text{ and } S = 3
\]

Expected return of MVP

Minimum standard deviation

\((1,1,1)\)
Mean-Variance Efficient Returns

- **Definition:** A return is **mean-variance efficient** if there is no other return with same variance but greater expectation.

- Mean variance efficient returns are frontier returns with $E[R_\lambda] \geq E[R_{\lambda_0}]$

- If risk-free asset can be replicated
  - Mean variance efficient returns correspond to $\lambda_0$.
  - Pricing kernel (portfolio) is not mean-variance efficient, since $E[R_{m^*}] = \frac{E[m^*]}{E[(m^*)^2]} < \frac{1}{E[m^*]} = R_f$
Zero-Covariance Frontier Returns

• Take two frontier portfolios with returns
  \[ R_\lambda = R_{k*} + \lambda (R_{m*} - R_{k*}) \]  and  \[ R_\mu = R_{k*} + \mu (R_{m*} - R_{k*}) \]

• \[ \text{cov} [R_\mu, R_\lambda] = \text{var} [R_{k*}] + (\lambda + \mu) \text{cov} [R_{k*}, R_{m*} - R_{k*}] + \lambda \mu \text{var} [R_{m*} - R_{k*}] \]

• The portfolios have zero co-variance if
  \[ \mu = -\frac{\text{var} [R_{k*}] + \lambda \text{cov} [R_{k*}, R_{m*} - R_{k*}]}{\text{cov} [R_{k*}, R_{m*} - R_{k*}]} + \lambda \text{var} [R_{m*} - R_{k*}] \]

• For all \( \lambda \neq \lambda_0 \), \( \mu \) exists
  \[ -\mu = 0 \text{ if risk-free bond can be replicated} \]
Illustration of ZC Portfolio...

\[ \langle X \rangle = \mathbb{R}^2 \text{ and } S = 3 \]

Recall:

\[ \text{cov}[x, y] = [\tilde{x}, \tilde{y}]_\pi \]
Illustration of ZC Portfolio

Green lines do not necessarily cross.

Arbitrary portfolio p
Beta Pricing...

- Frontier Returns (are on linear subspace). Hence
  \[ R_{\beta} = R_\mu + \beta (R_\lambda - R_\mu) \]
- Consider any asset with payoff \( x_j \)
  - It can be decomposed in \( x_j = x_j^\varepsilon + \varepsilon_i \)
  - \( q(x_j) = q(x_j^\varepsilon) \) and \( E[x_j] = E[x_j^\varepsilon] \), since \( \varepsilon \perp \varepsilon \)
  - Return of \( x_j \) is \( R_j = R_j^\varepsilon + \frac{\varepsilon_j}{q(x_j)} \)
  - Using above and assuming \( \lambda \neq \lambda_0 \) and \( \mu \) is ZC-portfolio of \( \lambda \),
    \[ R_j = R_\mu + \beta_j (R_\lambda - R_\mu) + \frac{\varepsilon_j}{q(x_j)} \]
...Beta Pricing

• Taking expectations and deriving covariance

\[ E[R_j] = E[R_\mu] + \beta_j (E[R_\lambda] - E[R_\mu]) \]

• \( \text{cov}[R_\lambda, R_j] = \beta_j \text{var}[R_\lambda] \Rightarrow \beta_j = \frac{\text{cov}[R_\lambda, R_j]}{\text{var}[R_\lambda]} \)

  \[ \text{Since } R_\lambda \perp \frac{\epsilon_j}{q(x_j)} \]

• If risk-free asset can be replicated, beta-pricing equation simplifies to

\[ E[R_j] = R_f + \beta_j (E[R_\lambda] - R_f) \]

• Problem: How to identify frontier returns
Capital Asset Pricing Model...

• CAPM = market return is frontier return
  – Derive conditions under which market return is frontier return
  – Two periods: 0,1.
  – Endowment: individual $w_i$ at time 1, aggregate $\bar{w}_1 = \bar{w}_1^{\langle X \rangle} + \bar{w}_1^{\langle Y \rangle}$, where $\bar{w}_1^{\langle X \rangle}, \bar{w}_1^{\langle Y \rangle}$ are orthogonal and $\bar{w}_1^{\langle X \rangle}$ is the orthogonal projection of $\bar{w}_1$ on $\langle X \rangle$.
  – The market payoff is $\bar{w}_1^{\langle X \rangle}$
  – Assume $q(\bar{w}_1^{\langle X \rangle}) \neq 0$, let $R_{mkt} = \frac{\bar{w}_1^{\langle X \rangle}}{q(\bar{w}_1^{\langle X \rangle})}$, and assume that $R_{mkt}$ is not the minimum variance return.
...Capital Asset Pricing Model

• If $R_0$ is the frontier return that has zero covariance with $R_{mkt}$ then, for every security $j$,  
  $E[R_j] = E[R_0] + \beta_j (E[R_{mkt}] - E[R_0])$ with  
  $\beta_j = \frac{\text{cov}[R_j,R_{mkt}]}{\text{var}[R_{mkt}]}$

• If a risk free asset exists, equation becomes,  
  $E[R_j] = R_f + \beta_j (E[R_{mkt}] - R_f)$

• N.B. first equation always hold if there are only two assets.
Overview

1. Introduction: Simple CAPM with quadratic utility functions
2. Traditional Derivation of CAPM
   - Demand: Portfolio Theory
   - Aggregation: Fund Separation Theorem
   - Equilibrium: CAPM
3. Modern Derivation of CAPM
   - Projections
   - Pricing Kernel and Expectation Kernel
4. Testing CAPM
5. Practical Issues – Black-Litterman
Practical Issues

• Testing of CAPM
• Jumping weights
  – Domestic investments
  – International investment
• Black-Litterman solution
Testing the CAPM

• Take CAPM as given and test empirical implications

• Time series approach
  – Regress individual returns on market returns
    \[ R_{it} - R_{ft} = \hat{\alpha}_i + \hat{\beta}_{im}(R_{mt} - R_{ft}) + \varepsilon_{it} \]
  – Test whether constant term \( \alpha_i = 0 \)

• Cross sectional approach
  – Estimate betas from time series regression
  – Regress individual returns on betas
    \[ R_i = \lambda \hat{\beta}_{im} + \alpha_i \]
  – Test whether regression residuals \( \alpha_i = 0 \)
Empirical Evidence

• Excess returns on high-beta stocks are low
• Excess returns are high for small stocks
  – Effect has been weak since early 1980s
• Value stocks have high returns despite low betas
• Momentum stocks have high returns and low betas
Reactions and Critiques

- **Roll Critique**
  - The CAPM is not testable because composition of true market portfolio is not observable

- **Hansen-Richard Critique**
  - The CAPM could hold *conditionally* at each point in time, but fail unconditionally

- Anomalies are result of “data mining”

- Anomalies are concentrated in small, illiquid stocks

- Markets are inefficient – “joint hypothesis test”
Practical Issues

• Estimation
  – How do we estimate all the parameters we need for portfolio optimization?

• What is the market portfolio?
  – Restricted short-sales and other restrictions
  – International assets & currency risk

• How does the market portfolio change over time?
  – Empirical evidence
  – More in dynamic models
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5. Practical Issues – Black Litterman
MV Portfolio Selection in Real Life

• An investor seeking to use mean-variance portfolio construction has to
  – Estimate N means,
  – N variances,
  – N*(N-1)/2 co-variances

• Estimating means
  – For any partition of [0,T] with N points (Δt=T/N):
    \[ E[r] = \frac{1}{\Delta t} \cdot \left( \frac{1}{N} \cdot \sum_{i=1}^{N} r_i \cdot \Delta t \right) = \frac{p_T-p_0}{T} \] (in log prices)
  – Knowing the first and last price is sufficient
Estimating Means

- Let $X_k$ denote the logarithmic return on the market, with $k = 1, \ldots, n$ over a period of length $h$
  - The dynamics to be estimated are:
    $$X_k = \mu \cdot \Delta + \sigma \cdot \sqrt{\Delta} \cdot \epsilon_k$$
    where the $\epsilon_k$ are i.i.d. standard normal random variables.
  - The standard estimator for the expected logarithmic mean rate of return is:
    $$\hat{\mu} = \frac{1}{h} \cdot \sum_{1}^{n} X_k$$
  - The mean and variance of this estimator
    $$E[\hat{\mu}] = \frac{1}{h} \cdot E \left[ \sum_{1}^{n} X_k \right] = \frac{1}{h} \cdot n \cdot \mu \cdot \Delta = \mu$$
    $$Var[\hat{\mu}] = \frac{1}{h^2} \cdot Var \left[ \sum_{1}^{n} X_k \right] = \frac{1}{h^2} \cdot n \cdot \sigma^2 \cdot \Delta = \frac{\sigma^2}{h}$$
    where
    - $h$ is length of observation
    - $n$ number of observations
    - $\Delta = n/h$

- The accuracy of the estimator depends only upon the total length of the observation period ($h$), and not upon the number of observations ($n$).
**Estimating Variances**

- Consider the following estimator:
  \[ \hat{\sigma}^2 = \frac{1}{h} \cdot \sum_{i=1}^{n} X_k^2 \]

- The mean and variance of this estimator:
  \[
  E[\hat{\sigma}^2] = \frac{1}{h} \cdot \sum_{i=1}^{n} E[X_k^2] = \frac{1}{h} \cdot n \cdot (\mu^2 \cdot \Delta^2 + \sigma^2 \cdot \Delta) = \sigma^2 + \mu^2 \cdot \frac{h}{n}
  \]
  \[
  Var[\hat{\sigma}^2] = \frac{1}{h^2} \cdot Var \left[ \sum_{i=1}^{n} X_k^2 \right] = \frac{1}{h^2} \cdot \sum_{i=1}^{n} Var[X_k^2] = \frac{n}{h^2} \cdot (E[X_k^4] - E[X_k^2]^2) = \frac{2 \cdot \sigma^4}{n} + \frac{4 \cdot \mu^2 \cdot h}{n^2}
  \]

  - The estimator is biased b/c we did not subtract out the expected return from each realization.
  - Magnitude of the bias declines as \( n \) increases.
  - For a fixed \( h \), the accuracy of the variance estimator can be improved by sampling the data more frequently.
Estimating variances: Theory vs. Practice

• For any partition of $[0, T]$ with $N$ points ($\Delta t = T/N$):
  \[
  \text{Var}[r] = \frac{1}{N} \sum_{i=1}^{N} (r_{i\cdot \Delta t} - E[r])^2 \rightarrow \sigma^2 \text{ as } N \rightarrow \infty
  \]

• Theory: Observing the same time series at progressively higher frequencies increases the precision of the estimate.

• Practice:
  – Over shorter interval increments are non-Gaussian
  – Volatility is time-varying (GARCH, SV-models)
  – Market microstructure noise
Estimating covariances: Theory vs. Practice

• In theory, the estimation of covariances shares the features of variance estimation.

• In practice:
  – Difficult to obtain synchronously observed time-series -> may require interpolation, which affects the covariance estimates.
  – The number of covariances to be estimated grows very quickly, such that the resulting covariance matrices are unstable (check condition numbers!).
  – Shrinkage estimators (Ledoit and Wolf, 2003, “Honey, I Shrank the Covariance Matrix”)

Unstable Portfolio Weights

• Are optimal weights statistically different from zero?
  – Properly designed regression yields portfolio weights
  – Statistical tests for significance of weight

• Example: Britton-Jones (1999) for international portfolio
  – Fully hedged USD Returns
    • Period: 1977-1966
    • 11 countries
  – Results
    • Weights vary significantly across time and in the cross section
    • Standard errors on coefficients tend to be large
# Britton-Jones (1999)

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Black-Litterman Approach

• Since portfolio weights are very unstable, we need to discipline our estimates somehow
  – Our current approach focuses only on historical data

• Priors
  – Unusually high (or low) past return may not (on average) earn the same high (or low) return going forward
  – Highly correlated sectors should have similar expected returns
  – A “good deal” in the past (i.e. a good realized return relative to risk) should not persist if everyone is applying mean-variance optimization.

• Black Litterman Approach
  – Begin with “CAPM prior”
  – Add views on assets or portfolios
  – Update estimates using Bayes rule
Black-Litterman Model: Priors

• Suppose the returns of $N$ risky assets (in vector/matrix notation) are
  \[ r \sim \mathcal{N}(\mu, \Sigma) \]

• CAPM: The equilibrium risk premium on each asset is given by:
  \[ \Pi = \gamma \cdot \Sigma \cdot w_{eq} \]
  - $\gamma$ is the investors coefficient of risk aversion.
  - $w_{eq}$ are the equilibrium (i.e. market) portfolio weights.

• The investor is assumed to start with the following Bayesian prior (with imprecision):
  \[ \mu = \Pi + \epsilon^{eq} \text{ where } \epsilon^{eq} \sim \mathcal{N}(0, \tau \cdot \Sigma) \]
  - The precision of the equilibrium return estimates is assumed to be proportional to the variance of the returns.
  - $\tau$ is a scaling parameter
Black-Litterman Model: Views

- Investor views on a single asset affect many weights.
- “Portfolio views”
  - Investor views regarding the performance of K portfolios (e.g. each portfolio can contain only a single asset)
  - P: K x N matrix with portfolio weights
  - Q: K x 1 vector of views regarding the expected returns of these portfolios
- Investor views are assumed to be imprecise:
  \[ P \cdot \mu = Q + \epsilon^v \] where \( \epsilon^v \sim N(0, \Omega) \)
  - Without loss of generality, \( \Omega \) is assumed to be a diagonal matrix
  - \( \epsilon^{eq} \) and \( \epsilon^v \) are assumed to be independent
Black-Litterman Model: Posterior

• Bayes rule:

\[
f(\theta|x) = \frac{f(\theta, x)}{f(x)} = \frac{f(x|\theta) \cdot f(\theta)}{f(x)}
\]

• Posterior distribution:

  – If \( X_1, X_2 \) are normally distributed as:

    \[
    \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N \left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)
    \]

  – Then, the conditional distribution is given by

    \[
    X_1 | X_2 = x \sim N(\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x - \mu_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21})
    \]
Black-Litterman Model: Posterior

• The Black-Litterman formula for the posterior distribution of expected returns

\[
E[R|Q] = \left[ (\tau \cdot \Sigma)^{-1} + P' \cdot \Omega^{-1} \cdot P \right]^{-1} \cdot \left[ (\tau \cdot \Sigma)^{-1} \cdot \Pi + P' \cdot \Omega^{-1} \cdot Q \right]
\]

\[
\text{var}[R|Q] = \left[ (\tau \cdot \Sigma)^{-1} + P' \cdot \Omega^{-1} \cdot P \right]^{-1}
\]
Black Litterman: 2-asset Example

• Suppose you have a view on the equally weighted portfolio \( \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2 = q + \varepsilon^v \)

• Then

\[
E[R|Q] = \left[ (\tau \cdot \Sigma)^{-1} + \frac{1}{2\Omega} \right]^{-1} \cdot \left[ (\tau \cdot \Sigma)^{-1} \cdot \Pi + \frac{q}{2\Omega} \right]
\]

\[
\text{var}[R|Q] = \left[ (\tau \cdot \Sigma)^{-1} + \frac{1}{2\Omega} \right]^{-1}
\]
Advantages of Black-Litterman

• Returns are adjusted only partially toward the investor’s views using Bayesian updating
  – Recognizes that views may be due to estimation error
  – Only highly precise/confident views are weighted heavily.
• Returns are modified in way that is consistent with economic priors
  – Highly correlated sectors have returns modified in the same direction.
• Returns can be modified to reflect absolute or relative views.
• Resulting weight are reasonable and do not load up on estimation error.