Financial and Monetary Economics
Eco529 Fall 2020
Lecture 06: Cash vs. Cashless Economy – The I Theory of Money

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Key Takeaways

- Real vs. Nominal Debt/Cashless vs. Cash
  - Inflation risk can improve risk sharing
- Intertemporal unit of account
  - State-contingent MoPo $\sigma^B$ as
- Equivalence of capital vs. risk allocation setting ($\kappa$ vs. $\chi$)
- Liquidity and Disinflationary Spiral

- Policy
  - Fiscal Policy
  - Monetary Policy
    - Stealth recapitalization of intermediaries
    - Macroprudential Policy

- Technical Takeaways
  - Two sector money models
The big Roadmap: Towards the I Theory of Money

- One sector model with idio risk - “The I Theory without I” (steady state focus)
  - Store of value
    - Insurance role of money within sector
  - Money as bubble or not
  - Fiscal Theory of the Price Level
  - Medium of Exchange Role ⇒ SDF-Liquidity multiplier ⇒ Money bubble

- 2 sector/type model with money and idio risk
  - Generic Solution procedure (compared to lecture 03)
  - Equivalence btw experts producers and intermediaries
  - Real debt vs. nominal debt/money
    - Implicit insurance role of money across sectors
  - I Theory

- Welfare analysis
- Optimal Monetary Policy and Macroprudential Policy
- International Monetary Model
The 4 Roles of Money

- Unit of account
  - Intratemporal: Numeraire
  - Intertemporal: Debt contract

- Store of value
  - “I Theory of Money without I”
    - Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- Medium of exchange
  - Overcome double-coincidence of wants problem

- Record keeping device – money is memory
  - Virtual ledger

- bounded rationality/price stickiness
- incomplete markets
Safe Assets $\supseteq$ (Narrow) Money

- Asset Price = $E[\text{PV(cash flows)}] + E[\text{PV(service flows)}]$
  
  dividends/interest

- Service flows/convenience yield
  
  1. Collateral: relax constraints (Lagrange multiplier)
  
  2. Safe asset: [good friend analogy]
    
    - When one needs funds, one can sell at stable price ... since others buy
    
    - Partial insurance through **retrading** - market liquidity!
  
  3. Money (narrow): relax double-coincidence of wants
    
    - Higher Asset Price = lower expected return

- Problem: safe asset + money status might burst like a bubble
  
  - Multiple equilibria: [safe asset tautology]
## Models on Money as Store of Value

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<th>Friction</th>
<th>OLG</th>
<th>Incomplete Markets + idiosyncratic risk</th>
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<td>Risk</td>
<td>deterministic</td>
<td>endowment risk borrowing constraint</td>
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<td></td>
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<td>return risk</td>
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<td>Risk tied up with Individual capital</td>
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<tr>
<td>Only money</td>
<td>Samuelson</td>
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<td>With capital</td>
<td>Diamond</td>
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“Theory without I” Brunnermeier-Sannikov (AER PP 2016)
<table>
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<tr>
<th>(New) Keynesian Demand Management</th>
<th>I Theory of Money Risk (Premium) Management</th>
</tr>
</thead>
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<tr>
<td>Stimulate aggregate consumption</td>
<td>Alleviate balance sheet constraints</td>
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<tr>
<td>Woodford (2003)</td>
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<td>Price stickiness &amp; ZLB</td>
<td>Both</td>
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<td>Perfect capital markets</td>
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<td></td>
<td>Incomplete markets</td>
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<td>Representative Agent</td>
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<td>Cut ( \hat{i} )</td>
<td>Cut ( \hat{i} )</td>
</tr>
<tr>
<td>Reduces ( r ) due to price stickiness</td>
<td>Changes bond prices</td>
</tr>
<tr>
<td>Consumption ( c ) rises</td>
<td>Redistributes from low MPC to high MPC consumers</td>
</tr>
</tbody>
</table>
“Money and Banking” (in macro-finance)

- Money
  - store of value/safe asset/Gov. bond
- Banking
  - “diversifier”
  - holds risky assets, issues inside money

Watch “Money and Banking”
YouTube Video Channel: “markus.economicus”
https://www.youtube.com/channel/UCV89KOTWjkuYhi9I/eRYigA?video?reload=10
“Money and Banking” (in macro-finance)

- Money: store of value/safe asset/Gov. bond
- Banking: “diversifier”
  holds risky assets, issues inside money

Amplification/endogenous risk dynamics
- Value of capital declines due to fire-sales
  Liquidity spiral
  Flight to safety
- Value of money rises
  Disinflation spiral a la Fisher
  - Demand for money rises
  - Supply for inside money declines
  - less idiosyncratic risk is diversified
  - less creation by intermediaries
  - Endogenous money multiplier = f(capitalization of critical sector)

Paradox of Prudence
Paradox of Thrift (in risk terms)

Monetary Policy (redistributive)
Roadmap

- Intro
- Equivalence btw experts producers and intermediaries
- Real vs. Nominal Debt
- I Theory of Money
- Policy
### Frictions:
- Household cannot diversify idio risk
- Limited risky claims issuance
Equivalence

- $a^e = a^h$
- $\tilde{\sigma}^e < \tilde{\sigma}^h$
Equivalence

- Why equivalence btw. Intermediaries $\chi$-risk allocation model and experts $\kappa$-capital allocation model?

  *Poll 13: Why are both models equivalent?*
  - a) Since $a^e = a^h$.
  - b) Intermediary sector doesn’t produce any output
  - c) Risk $\chi$ and capital allocation $\kappa$ are fundamentally different.

- Next: Contrast Real Debt with Nominal Debt/Money Model
  - solve generic model and highlight the differences
Roadmap

- Intro
- Equivalence btw experts producers and intermediaries
- Real vs. Nominal Debt
- I Theory of Money
- Policy
Model with Intermediary Sector

Intermediary sector

- Hold equity up to $\bar{\chi} \leq 1$
- Diversify idio risk to $\phi \bar{\sigma}$
- Consumption rate: $c_t^l$
- $E_0[\int_0^\infty e^{-\rho t} \log c_t^l \, dt]$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Investment rate: $i_t^h$
  \[
  \frac{d k_{t+1}^h}{k_t^h} = (\Phi(i_t^h) - \delta^h) dt + \sigma^h d\tilde{Z}_t + \Delta_{t+1}^k
  \]
- Consumption rate: $c_t^h$
- $E_0[\int_0^\infty e^{-\rho t} \log c_t^h \, dt]$

- Friction: Can only issue debt
  - 2 Models:
    1. Real debt issuance only (and money has no value)
    2. Nominal debt issuance

- Bond/money supply $\frac{dB_t}{B_t} = (\mu_t^B + i_t) dt + \sigma_t^B dZ_t$

- seigniorage distribution as in Lecture 05 (no fiscal impact – per period balanced budget)
Solving MacroModels Step-by-Step

0. **Postulate aggregates, price processes & obtain return processes**
   
1. **For given $C/N$-ratio and SDF processes for each $i$ finance block**
   a. Real investment $\mathcal{U} +$ Goods market clearing *(static)*
      - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\mathcal{O} +$ Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - *Toolbox 2:* “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      - *Toolbox 3:* Change in numeraire to total wealth (including SDF)

2. **Evolution of state variable $\eta$ (and $K$)**

3. **Value functions**
   a. **Value fcn. as fcn. of individual investment opportunities $\omega$**
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $v^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $\rho = C/N$-ratio and $\zeta, \xi$ prices of risks

4. **Numerical model solution**
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. **KFE: Stationary distribution, Fan charts**
0. Postulate Aggregates and Processes

- \( q_t^K K_t \) value of physical capital
- \( q_t^B K_t \) value of nominal capital/outside money/gov. debt
  - \( \varphi_t := B_t / q_t^B K_t \) price level (inverse of “value of money”)
- \( N_t := (q_t^K + q_t^B)K_t \) is total wealth in the economy
- \( \vartheta_t := \frac{q_t^B}{q_t^K + q_t^B} \) fraction of nominal wealth
0. Postulate Aggregates and Processes

- $q_t^K K_t$ value of physical capital
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- $\nu_t := \frac{q_t^B}{q_t^K + q_t^B}$ fraction of nominal wealth

0. Postulate in the $N_t$-numeraire!

- $\varphi$-price process $d\varphi_t/\varphi_t = \mu_t^\varphi dt + \sigma_t^\varphi dZ_t$,
- SDF for each $\tilde{i}$ agent $\frac{d\xi_t^i}{\xi_t^i} = -r_t^i dt - c_t^i dZ_t - \tilde{\xi}_t^i d\tilde{Z}_t$
  - Change of notation compared to Lectures 02-05!
0. Postulate Aggregates and Processes

- \( q^K_t K_t \) value of physical capital
- \( q^B_t K_t \) value of nominal capital/outside money/gov. debt
  - \( \varrho_t := B_t / q^B_t K_t \) price level (inverse of “value of money”)
- \( N_t := (q^K_t + q^B_t)K_t \) is total wealth in the economy
- \( \vartheta_t := \frac{q^B_t}{q^K_t + q^B_t} \) fraction of nominal wealth

0. Postulate in the \( N_t \)-numeraire!

- \( \vartheta \)-price process \( d\vartheta_t / \vartheta_t = \mu^\vartheta_t dt + \sigma^\vartheta_t dZ_t, \)
- SDF for each \( i \) agent \( \frac{d\xi^i_t}{\xi^i_t} = -r^i_t dt - \zeta^i_t dZ_t - \tilde{\xi}^i_t d\tilde{Z}_t \)

Poll 19: Why is the drift \( -r^i_t \) and not simply \( -r^f_t \)?

a) With only nominal debt a real risk-free rate might not be in asset span.

b) Negative drift of the SDF in \( N_t \)-numeraire is not risk-free rate.
1a. Optimal \( \iota + \) Goods Market

- Use optional real investment \( \iota \) and goods market clearing
- Same as in Lecture 05
- Price of physical capital
  \[ q_t^K = (1 - \vartheta_t) \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho} \]
- Price of nominal capital
  \[ q_t^B = \vartheta_t \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho} \]
- Optimal investment rate
  \[ \iota_t = \frac{(1 - \vartheta_t) a - \rho}{(1 - \vartheta_t) + \phi \rho} \]
- Moneyless equilibrium with \( q_t^B = 0 \Rightarrow \vartheta_t = 0 \Rightarrow q_t^K = \frac{1 + \phi a}{1 + \phi \rho} \)
1b. Price-taking Planner’s Allocation

\[
\max \left\{ \kappa_t, \chi_t, \tilde{\chi}_t \right\} E_t[dr_t^N(\kappa_t)] - \zeta_t \sigma(\psi_t, \chi_t) - \tilde{\zeta}_t \bar{\sigma}(\psi_t, \tilde{\chi}_t)
\]

- In our model(s):
  - \( \kappa = 0 \) (households manage all physical capital)
  - \( \tilde{\chi}_t = \chi_t \)
  - \( E_t[dr_t^N(\kappa_t)] = 0 \)

Poll 21: Why is \( E_t[dr_t^N(\kappa_t)] = 0 ? \)
  a) Because capital is not reallocated, i.e. \( \kappa = 0 \) all the time.
  b) In the \( N_t \)-numeraire return of total wealth \( dr_t^N = 0 \).
1b. Price-taking Planner’s Allocation

- \[ \max_{\{\psi_t, \chi_t, \tilde{\chi}_t\}} E_t[dr_t^N(\kappa_t)] - \zeta_t \sigma(\psi_t, \chi_t) - \tilde{\zeta}_t \tilde{\sigma}(\psi_t, \tilde{\chi}_t) \]

- In our model(s):
  - \( \kappa = 0 \) (households manage all physical capital)
  - \( \tilde{\chi}_t = \chi_t \)
  - \( E_t[dr_t^N(\kappa_t)] = 0 \)
  - \( \sigma = (\chi_t \sigma_t^{xK}, (1 - \chi_t) \sigma_t^{xK}) \),
    - where \( \sigma_t^{xK} \) = Risk of the excess return of capital beyond benchmark asset
  - \( \tilde{\sigma} = (\chi_t \varphi \tilde{\sigma}, (1 - \chi_t) \tilde{\sigma}) \)
    - \( \varphi < 1 \)
1b. Price-taking Planner’s Allocation

- Minimize weighted average cost of financing

\[
\min_{\chi_t \leq \bar{\chi}} \left( \zeta_t^l \chi_t + \zeta^h_t (1 - \chi_t) \right) \sigma_{t}^{xK} + \left( \tilde{\zeta}_t^l \varphi \chi_t + \tilde{\zeta}_t^h (1 - \chi_t) \right) \bar{\sigma}
\]

- FOC: (equality if \( \chi_t < \bar{\chi} \))

\[
\zeta_t^l \sigma_t^{xK} + \tilde{\zeta}_t^l \phi \bar{\sigma} \leq \zeta_t^h \sigma_t^{xK} + \tilde{\zeta}_t^h \bar{\sigma}
\]

- Real debt model:

- \( \sigma_t^{xK} = \sigma + \sigma^q \) (recall \( q_t \) is constant)

- Nominal debt model

- \( \sigma_t^{xK} = (-\sigma_t^g + \sigma_t^B)/(1 - \vartheta_t) \)
  - Risk of capital \( \sigma + \sigma_t^{qK} + \vartheta_t \sigma_t^B/(1 - \vartheta_t) - \sigma_t^N \) (in \( N_t \)-numeraire)
  - Risk of bond/money \( \sigma + \sigma_t^{qB} + \sigma_t^B - \sigma_t^N \) (in \( N_t \)-numeraire)
“Benchmark Asset Evaluation Equation”

- In $N_t$-numeraire $\eta_t^i$ takes on role of sector net worth $N_t^i$
- Return on individual agent’s net worth return (in $N_t$-numeraire)
  \[
  \frac{d\eta_t^i}{\eta_t^i} + \frac{d\tilde{\eta}_t^i}{\tilde{\eta}_t^i} + \frac{\rho dt}{\text{consumption}}
  \]
  \[\text{sector share} \quad \text{within sector share} \]
- Martingale condition relative to benchmark asset is
  \[
  \mu_t^i + \rho - r_{t}^{bm} = \zeta_t^i \left( \sigma_t^i - \sigma_t^{bm} \right) + \zeta_t \tilde{\sigma}_t^i
  \]
- Take $\eta_t^i$-weighted sum (across 2 types $i = I, h$ here)
  \[
  \rho - r_{t}^{bm} = \eta_t \zeta_t^I \left( \sigma_t^I - \sigma_t^{bm} \right) + (1 - \eta_t) \zeta_t^h \left( \frac{-\eta_t}{1 - \eta_t} \sigma_t^I - \sigma_t^{bm} \right) + \eta_t \tilde{\zeta_t} \tilde{\sigma}_t^I + (1 - \eta_t) \tilde{\zeta_t} \tilde{\sigma}_t^h
  \]
- For log utility:
  \[
  \zeta_t^I = \sigma_t^I, \zeta_t^h = \frac{-\eta_t}{1 - \eta_t} \sigma_t^I, \tilde{\zeta_t}^I = \tilde{\sigma}_t^I, \tilde{\zeta_t}^h = \tilde{\sigma}_t^h
  \]
  \[
  \rho - r_{t}^{bm} = \eta_t \left( \sigma_t^I \right)^2 + (1 - \eta_t) \left( \frac{-\eta_t}{1 - \eta_t} \sigma_t^I \right)^2 + \eta_t \left( \tilde{\sigma}_t^I \right)^2 + (1 - \eta_t) \left( \tilde{\sigma}_t^h \right)^2
  \]
- **Real** debt = benchmark asset \( bm \)
  - Redundant equation for allocation. Just useful for deriving risk-free rate in \( c \)-numeraire \( r_t^f \) (expressed in \( N_t \)-numeraire)

- **Nominal** debt/money = benchmark asset \( bm \)
  - Money evaluation equation
  - Replace \( r_t^{bm} = \mu_t^g/B := \mu_t^g - \mu_t^B - \sigma_t^B(\sigma_t^g - \sigma_t^B) \) (and \( \sigma_t^{bm} = \sigma_t^g \))

\[
\rho - \mu_t^{g/B} = \eta_t(\sigma_t^\eta)^2 + (1 - \eta_t)(-\frac{\eta_t}{1-\eta_t} \sigma_t^\eta)^2 + \eta_t\left(\tilde{\sigma}_t^{\tilde{\eta}}\right)^2 + (1 - \eta_t)\left(\tilde{\sigma}_t^{\tilde{\eta}}\right)^2
\]

excess return = (required) “net worth weighted risk premium”

of \( N_t \) (for holding risk in excess of money risk)
2. $\eta$-Evolution: Drift $\mu^\eta_t$ (in $N_t$-numeraire)

- Take difference from two earlier equations

$$\mu^\eta_t + \rho - r^bm_t = \zeta^l_t (\sigma^\eta_t - \sigma^{bm}_t) + \tilde{\zeta}^l_t \tilde{\sigma}^l_t$$

$$\rho - r^bm_t = \eta_t \zeta^l_t (\sigma^\eta_t - \sigma^{bm}_t) + (1-\eta_t) \zeta^h_t \left(-\frac{\eta_t}{1-\eta_t} \sigma^\eta_t - \sigma^{bm}_t\right) + \eta_t \tilde{\zeta}^l_t \tilde{\sigma}^l_t + (1-\eta_t) \zeta^h_t \tilde{\sigma}^h_t$$

$$\mu^\eta_t = (1-\eta_t) \left[ \zeta^l_t (\sigma^\eta_t - \sigma^{bm}_t) - \zeta^h_t \left(-\frac{\eta_t}{1-\eta_t} \sigma^\eta_t - \sigma^{bm}_t\right) + \tilde{\zeta}^l_t \tilde{\sigma}^l_t - \tilde{\zeta}^h_t \tilde{\sigma}^h_t \right]$$

- **Real** Debt
  - $\sigma^{bm}_t = -\sigma^N_t = -\sigma$ \quad (Recall $\sigma^q_t = 0$)

- **Nominal** Debt/Money
  - $\sigma^{bm}_t = \sigma^\vartheta_t - \sigma^B$
2. $\eta$-Evolution: $\eta$-Aggregate Risk

- $\sigma_t^\eta = \sigma_t^{r^{bm}} + (1 - \theta_t^I) \left( \sigma_t^{r^K} - \sigma_t^{r^{bm}} \right)$
  - Where portfolio share $1 - \theta_t^I = \chi_t \eta_t (1 - \vartheta_t)$

- **Real Debt**
  - Note $\sigma_t^{r^K} = 0$ given $N_t = q_t^K K_t$-numeraire
  - $\sigma_t^\eta = \frac{\chi_t - \eta_t}{\eta_t} \sigma$ (recall $\vartheta_t = 0$)

- No amplification since $q^K$ is constant

- Imperfect risk-sharing for $\chi_t \neq \eta_t$
- **Nominal Debt**
  - Note $\sigma_t^{r^K} = \sigma_t^{1-\vartheta} = -\frac{\vartheta_t}{1-\vartheta_t}\sigma_t^\vartheta$
  - $\sigma_t^\eta = \sigma_t^\vartheta - \sigma_t^M + \frac{\chi_t}{\eta_t}(1-\vartheta_t)\left(-\frac{\vartheta_t}{1-\vartheta_t}\sigma_t^\vartheta - \sigma_t^\vartheta + \sigma^B\right)$
  - Use $\sigma_t^\vartheta = \vartheta'(\eta_t)\eta_t\sigma_t^\eta$ and solve for $\eta_t\sigma_t^\eta$ yields
    $$\eta_t\sigma_t^\eta = \frac{(\chi_t - \eta_t)\sigma_t^B}{1-\chi_t - \eta_t\left(-\frac{\vartheta'(\eta_t)\eta_t}{\vartheta(\eta_t)}\right)}$$

- Intermediaries’ balance sheet perfectly hedges agg. risk for $\sigma^B = 0$!

- **Proposition**: Aggregate risk is perfectly shared for $\sigma^B = 0$!
  - Via inflation risk
  - Stable inflation (targeting) would ruin risk-sharing
    - Example: Brexit uncertainty. Use inflation reaction to share risks within UK
2. Within Type $\tilde{\eta}$-Risk

- Within intermediary sector
  \[
  \tilde{\sigma}_t^{\tilde{\eta}_i} = (1 - \theta_t^I) \varphi \tilde{\sigma} = \frac{\chi_t}{\eta_t} (1 - \vartheta_t) \varphi \tilde{\sigma}
  \]

- Within household sector
  \[
  \tilde{\sigma}_t^{\tilde{\eta}_h} = (1 - \theta_t^h) \tilde{\sigma} = \frac{1 - \chi_t}{1 - \eta_t} (1 - \vartheta_t) \tilde{\sigma}
  \]
Solving for $\chi_t$

- Recall planner condition: (equality if $\chi_t < \tilde{\chi}$)
  \[
  \zeta_t^l \sigma_t^{xK} + \tilde{\zeta}_t^l \phi \tilde{\sigma} \leq \zeta_t^h \sigma_t^{xK} + \tilde{\zeta}_t^h \tilde{\sigma}
  \]

### Price of Risks

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<th></th>
<th>Real Debt</th>
<th>Nominal Debt with $\sigma^B = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta_t^l = \sigma_t^{\eta}$</td>
<td>$\frac{\chi_t - \eta_t}{\eta_t}$</td>
<td>$= 0$</td>
</tr>
<tr>
<td>$\zeta_t^h = -\frac{\eta_t}{1 - \eta_t} \sigma_t^{\eta}$</td>
<td>$\frac{\chi_t - \eta_t}{1 - \eta_t}$</td>
<td>$= 0$</td>
</tr>
<tr>
<td>$\tilde{\zeta}_t^l = \frac{\chi_t}{\eta_t} (1 - \vartheta_t) \phi \tilde{\sigma}$</td>
<td>$\frac{\chi_t \phi \tilde{\sigma}}{\eta_t}$</td>
<td>$= \frac{\chi_t}{\eta_t} (1 - \vartheta_t) \phi \tilde{\sigma}$</td>
</tr>
<tr>
<td>$\tilde{\zeta}_t^h = \frac{1 - \chi_t}{1 - \eta_t} (1 - \vartheta_t) \tilde{\sigma}$</td>
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<td>$= \frac{1 - \chi_t}{1 - \eta_t} (1 - \vartheta_t) \tilde{\sigma}$</td>
</tr>
</tbody>
</table>
Solving for $\chi_t$

- **Real debt**

  \[
  \chi_t = \min \left\{ \frac{\eta_t (\sigma^2 + \bar{\sigma}^2)}{\sigma^2 + [(1 - \eta_t) \phi^2 + \eta_t] \bar{\sigma}^2}, \bar{\chi} \right\}
  \]

- **Nominal debt**

  \[
  \chi_t = \min \left\{ \frac{\eta_t}{(1 - \eta_t) \phi^2 + \eta_t}, \bar{\chi} \right\}
  \]
<table>
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<th>( \chi_t )</th>
<th>( \min \left{ \frac{\eta_t (\sigma^2 + \tilde{\sigma}^2)}{\sigma^2 + [(1 - \eta_t) \varphi^2 + \eta_t] \tilde{\sigma}^2}, \bar{\chi} \right} )</th>
<th>( \min \left{ \frac{\eta_t}{(1 - \eta_t) \varphi^2 + \eta_t}, \bar{\chi} \right} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_t^{\eta} )</td>
<td>( \frac{\chi_t - \eta_t \chi_t - 2 \chi_t \eta_t + \eta_t^2}{\eta_t (1 - \eta_t)} \sigma^2 + (1 - \eta_t) \left( \frac{\chi_t}{\eta_t} \right)^2 \varphi^2 - \frac{1 - \chi_t}{1 - \eta_t} \right)^2 \tilde{\sigma}^2 )</td>
<td>( (1 - \eta_t)(1 - \vartheta)^2 \left( \frac{(\chi_t}{\eta_t} \right)^2 \varphi^2 - \left( \frac{1 - \chi_t}{1 - \eta_t} \right)^2 \right) \tilde{\sigma}^2 )</td>
</tr>
<tr>
<td>( \sigma_t^{\eta} )</td>
<td>( \frac{\chi_t - \eta_t \sigma}{\eta_t} )</td>
<td>0</td>
</tr>
<tr>
<td>( q_t^K )</td>
<td>( \frac{1 + \phi a}{1 + \phi \rho} )</td>
<td>( (1 - \vartheta_t) \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho} )</td>
</tr>
<tr>
<td>( q_t^B )</td>
<td>0</td>
<td>( \vartheta_t \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho} )</td>
</tr>
<tr>
<td>( \vartheta_t )</td>
<td>0</td>
<td>( \rho - \mu_t^{\vartheta} + \mu_t^B )</td>
</tr>
<tr>
<td>( \lambda_t )</td>
<td>( \frac{a - \rho}{1 + \phi \rho} )</td>
<td>( \frac{(1 - \vartheta_t) a - \rho}{(1 - \vartheta_t) + \phi \rho} )</td>
</tr>
</tbody>
</table>
Example: Nominal Debt/Money with $\bar{\chi} = 1$

- $a = .15, \rho = .03, \sigma = .1, \phi = 2, \delta = .03, \tilde{\sigma}^e = .2, \tilde{\sigma}^h = .3, \bar{\chi} = 1$

Blue: real debt model
Red: nominal model
Contrasting Real with Nominal Debt

- **Real** debt model:
  - Changes in $\eta$ are absorbed by risk-free rate moves
  - Aggregate risk
  - $u(\eta)$ and $q^K(\eta)$ are constant

- **Nominal** debt/money model
  - Inflation risk completes markets
  - Perfect aggregate risk sharing
    - Banks balance sheet is perfectly hedged!!!
  - Risk-free rate is high
  - $u(\eta)$ and $q^K(\eta)$ are functions of $\eta$
Example: Nominal Debt with Limit on Risk Offloading

- $\rho = .05, \kappa = 2, \tilde{\sigma} = .5, \phi = .4, \bar{\chi} = .8$
Combining Nominal & Real Debt

- Adding real debt to money model does not alter the equilibrium, since
  - Markets are complete w.r.t. to aggregate risk
    (perfect aggregate risk sharing)
  - Markets are incomplete w.r.t. to idiosyncratic risk only

- Note: Result relies on absence of price stickiness

- Both Settings: Real Debt and Money/Nominal Debt converge in the long-run to the “I Theory without I” steady state model of Lecture 05 if $\bar{X} = 1$. 
\( \theta \) Minimized at Stochastic Steady State

- With \( \sigma_t^B = 0 \ \forall t \)
  - \( \sigma_t^n = 0 \),
  - \( \mu_t^n = (\tilde{\sigma}_t^l)^2 - \eta_t(\tilde{\sigma}_t^l)^2 - (1 - \eta_t)(\tilde{\sigma}_t^h)^2 = (1 - \eta_t)(1 - \vartheta_t)^2 \left( \frac{\chi_t^2 \phi^2}{\eta_t^2} - \frac{(1 - \chi_t)^2}{(1 - \eta_t)^2} \right) \tilde{\sigma}^2 \)

- Money valuation equation

\[
\rho - \mu_t^{\theta/B} = (1 - \vartheta_t)^2 \frac{\chi_t^2 \phi^2}{\eta_t^2} - (1 - \eta_t) \left( \frac{1 - \chi_t}{1 - \eta_t} \right)^2 \tilde{\sigma}^2
\]

\[
\sqrt{\eta_t (\tilde{\sigma}_t^l)^2 + (1 - \eta_t)(\tilde{\sigma}_t^h)^2}
\]

where \( \chi_t = \min \left( \frac{\eta_t}{\eta_t + (1 - \eta_t) \phi^2}, \bar{\chi} \right) \)

- Average idiosyncratic risk exposure (before the effect of money is minimized at the stochastic steady state of \( \eta \))
Cashless/Bondless Limit with Jump

- Removing cash/nominal gov. bonds (comparative static)
  - \( B > 0 \) vs. \( B = 0 \)
    - Price flexibility \( \Rightarrow \) Neutrality of money
  - Discontinuity at \( \lim_{B \to 0} \)

- Remark:
  - Different from Woodford (2003) – medium of exchange role of money
    - CIA becomes relevant for fewer and fewer goods

- Inflation on nominal claims (bond/cash)
  - Change \( \mu^B \) and subsidize capital
  - Continuous process
I Theory of Money

- Aim: intermediary sector is not perfectly hedged
- Idiosyncratic risk that HH have to bear is time-varying
- Needed: Intermediaries’ aggregate risk ≠ aggregate risk of economy
- One way to model: 2 technologies \( a \) and \( b \)

<table>
<thead>
<tr>
<th>Technology</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital share</td>
<td>( 1 - \kappa )</td>
<td>( \kappa )</td>
</tr>
<tr>
<td>Risk</td>
<td>( \frac{dk_t}{k_t} = (\cdot)dt + \sigma^a dZ_t + \tilde{\sigma} d\tilde{Z}_t )</td>
<td>( \frac{dk_t}{k_t} = (\cdot)dt + \sigma^b dZ_t + \tilde{\sigma} d\tilde{Z}_t )</td>
</tr>
<tr>
<td>Intermediaries</td>
<td>No</td>
<td>Yes, reduce ( \tilde{\sigma} ) to ( \varphi \tilde{\sigma} )</td>
</tr>
<tr>
<td>Excess risk</td>
<td>( -\kappa (\sigma^b - \sigma^a) - \frac{\sigma^\vartheta - \sigma^B}{1 - \vartheta} )</td>
<td>( (1 - \kappa)(\sigma^b - \sigma^a) - \frac{\sigma^\vartheta - \sigma^B}{1 - \vartheta} )</td>
</tr>
</tbody>
</table>
I Theory: Balance Sheets

- **Frictions:**
  - Household cannot diversify idio risk
  - Limited risky claims issuance
  - Only nominal deposits
Overview Slide that Explains the Role of Each Model Ingredient

- $$\tilde{\xi}$$ -- avoid degenerated distribution (households dying out)
- $$\kappa$$
  - if $$\kappa = 1$$ intermediaries would die out,
  - if $$\kappa = 0$$ don’t earn risk premium (except for aggregate risk)
- $$\sigma^b > \sigma^a$$ – avoid perfect hedging for intermediaries
  - (except $$\sigma^B \neq 0$$ – for example risk-free asset is in zero net supply)
    (like AER paper/handbook chapter)
- Fraction $$\tilde{\kappa}$$ of $$K$$ has aggregate risk of sigma rest has risk of zero (it’s exogenous) (allocation does not determine total risk in aggregate economy)
  (To keep it clean (taste choice): price-taking planner’s choice is less involved)
- …
1b. Price-taking Planner’s Allocation

- Minimize weighted average cost of financing

\[
\min_{\chi_t \leq \bar{\chi}} (1 - \bar{\kappa}) \zeta_t^h \sigma_t^{xK^a} + (\zeta_t^l \chi_t + \zeta_t^h (\bar{\kappa} - \chi_t)) \sigma_t^{xK^b} + (\zeta_t^l \phi_t + \zeta_t^h (1 - \chi_t)) \bar{\sigma}
\]

- FOC: (equality if \( \chi_t < \bar{\chi} \))

\[
\zeta_t^l \sigma_t^{xK^b} + \zeta_t^l \phi_t \bar{\sigma} \leq \zeta_t^h \sigma_t^{xK^b} + \zeta_t^h \bar{\sigma}
\]

\[
\sigma_t^{xK^b} = (1 - \bar{\kappa}) \sigma - \frac{\sigma^g - \sigma^M}{1 - \vartheta}
\]

- Price of risk with log-utility in total wealth numeraire:

<table>
<thead>
<tr>
<th>Intermediaries</th>
<th>Households</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate risk: ( \zeta_t^l = \sigma_t^\eta )</td>
<td>( \zeta_t^h = -\eta_t \sigma_t^\eta / (1 - \eta_t) )</td>
</tr>
</tbody>
</table>
| Idiosyncratic risk \( \zeta_t^l = (1 - \vartheta_t) \frac{\chi_t}{\eta_t} \phi_t \bar{\sigma} \) | \( \zeta_t^h = (1 - \vartheta_t) \frac{(1 - \chi_t)}{(1 - \eta_t)} \bar{\sigma} \)

\[
\sigma_t^\eta \left( (1 - \bar{\kappa}) \sigma - \frac{\sigma^g - \sigma^p}{1 - \vartheta_t} \right) + \left[ (1 - \vartheta_t) \frac{\chi_t}{\eta_t} \phi_t \bar{\sigma} \right] \phi_t \bar{\sigma} \leq \frac{-\eta_t \sigma_t^\eta}{1 - \eta_t} \left( (1 - \bar{\kappa}) \sigma - \frac{\sigma^g - \sigma^p}{1 - \vartheta_t} \right) + \left[ (1 - \vartheta_t) \frac{(1 - \chi_t)}{(1 - \eta_t)} \bar{\sigma} \right] \bar{\sigma}
\]
1c. Money Evaluation + 2. $\eta$-Drift

- As before in money/nominal debt model

- Money evaluation

$$\rho - \mu_t^{\theta/B} = \eta_t \left( (\sigma_t^\eta)^2 + (\tilde{\sigma}_t^\eta)^2 \right) + (1 - \eta_t) \left( \left( \frac{\eta_t \sigma_t^\eta}{1 - \eta_t} \right)^2 + (\tilde{\sigma}_t^\eta)^2 \right)$$

- $\eta$-drift

$$\mu_t^\eta = (1 - \eta_t) \left( (\sigma_t^\eta)^2 + (\tilde{\sigma}_t^\eta)^2 - \left( \frac{\eta_t \sigma_t^\eta}{1 - \eta_t} \right)^2 - (\tilde{\sigma}_t^\eta)^2 \right) - \sigma_t^\eta \frac{\sigma_t^{\theta/B}}{\sigma_t^\theta - \sigma_B}$$
\( \eta \)-Volatility and Amplification

\[ \sigma_t^\eta = \sigma_t^{rM} + (1 - \theta_t^l)\sigma_t^{xK^b} \]
- Where portfolio share \( 1 - \theta_t^l = \frac{\chi_t}{\eta_t} (1 - \vartheta_t) \)

\[ \sigma_t^\eta = \sigma_t^\vartheta - \sigma^B + \frac{\chi_t (1 - \vartheta_t)}{\eta_t} \left( (1 - \bar{\kappa})\sigma - \frac{\sigma_t^\vartheta - \sigma^B}{1 - \vartheta_t} \right) \]

\[ \Rightarrow \eta_t \sigma_t^\eta = \frac{(1 - \vartheta_t)\chi_t (1 - \bar{\kappa})\sigma + (\chi_t - \eta_t)\sigma^M}{1 - \chi_t - \eta_t \left( \frac{-\vartheta'(\eta_t)\eta_t}{\vartheta(\eta_t)} \right)} \]
- Note that \( \frac{-\vartheta'(\eta_t)\eta_t}{\vartheta(\eta_t)} = (1 - \vartheta_t) \left( \frac{q^{K'}(\eta_t)\eta_t}{q^K(\eta_t)} + \frac{-q^{B'}(\eta_t)\eta_t}{q^B(\eta_t)} \right) \)

Policy removes endog. amplification

Liquidity Spiral - Disinflationary Spiral
I Theory: Balance Sheets

- Frictions:
  - Household cannot diversify idio risk
  - Limited risky claims issuance
  - Only nominal deposits
Consequences of a Shock in 4 Steps

1. **Shock:** destruction of some capital
   - % loss in intermediaries net worth > % loss in assets
   - Leverage shoots up
   - Intermediaries %-loss > Household %-losses
   - $\eta$-derivative shifts losses to intermediaries

2. **Response:** shrink balance sheet / delever
   - For given prices no impact

3. Asset side: asset price $q^K$ shrinks
   - Further losses, leverage ↑, further deleveraging

4a. Liability side: money supply declines
   - value of money $q^B$ rises
   - Disinflationary spiral

4b. Households’ money demand rises
   - HH face more idiosyncratic risk (can’t diversify)
Policy

- Fiscal policy
- Monetary policy without fiscal implications
- Macroprudential policy
Fiscal policy

- Includes monetary policy that has fiscal implications
- ...
Monetary Policy

- No fiscal implications, no seigniorage $\tau^{i,\bar{i}} = 0 \ \forall i, \bar{i}$
- Any seigniorage is paid out to government debt/money holders in form of interest
- Introducing interest rates on bond/reserves $i_t$.

\[
\begin{align*}
\text{To study monetary policy without fiscal implications, we let } \sigma^B_t &= 0, \text{ so } \\
\frac{dr^B_t}{dt} &= \left\{ i_t - \mu_t^B + \Phi(t_t) - \delta + \mu_t^qB \right\} dt + \sigma^qB_t dZ_t.
\end{align*}
\]
Monetary Policy: Super-neutrality

- If interest paid on bond holdings is simply financed by issuing new bonds (issuing money), then money is
  - Neutral
  - Super-neutral

\[ \frac{dB_t}{B_t} = i_t \, dt \]

- Fisher equation

\[ dr_t^B = i_t \, dt - d\pi_t \]
Introducing Long-term Gov. Bond

- Introduce long-term (perpetual) bond
  - No default ... held by intermediaries in equilibrium
  - Value of long-term bond is endogenous

\[
\frac{dB_t}{B_t} = \mu_t^B \, dt + \sigma_t^B \, dZ_t
\]

Perpetual bonds:
- pay in money (at unit rate)
- endogenous price \( B_t \) (in money)
Redistributive MoPo: Ex-post perspective

- Adverse shock $\rightarrow$ value of risky claims drops
- Monetary policy
  - Interest rate cut $\Rightarrow$ long-term bond price
  - Asset purchase $\Rightarrow$ asset price
  - $\Rightarrow$ “stealth recapitalization” - redistributive
  - $\Rightarrow$ risk premia
- Liquidity & Deflationary Spirals are mitigated
Introducing long-term bonds

- Long-term bond
  - yields fixed coupon interest rate on face value $F(i,m)$
  - Matures at random time with arrival rate $1/m$
  - Nominal price of the bond $P_t(B(i,m))$
  - Nominal value of all bonds outstanding of a certain maturity
    $$B_t^{(m)} = P_t(B(i,m))F(i,m)$$
  - Nominal value of all bonds $B_t = \sum_m B_t^{(m)}$

- Special bonds
  - Reserves: $B_t^{(0)}$ and note $P_t^{B(0)} = 1$
  - Consol bond: $B_t^{(\infty)}$
Debt evolution w/o fiscal implications

- \( dB_t^{(0)} = i_t B_t^{(0)} dt \)
  \[ + \sum_{(i,m)} \left[ \left( i + \frac{1}{m} \right) F_t^{(i,m)} dt - \frac{B_t^{(i,m)}}{F_t^{(i,m)}} (dF_t^{i,m} + \frac{1}{m} F_t^{(i,m)} dt) \right] \]

- Money \( B_t^{(0)} \) is different since it pays floating interest rate

- If we have only reserves and consol bond, then
  \[ dB_t^{(0)} = i_t B_t^{(0)} dt + iF_t^{(i,\infty)} dt - \frac{B_t^{(i,\infty)}}{F_t^{(i,\infty)}} dF_t^{i,\infty} \]
  \[ dM_t = i_t M_t dt + i^L F_t^L dt - P_t^L dF_t^L \]
Define fraction of value of bonds that are not in short-term reserves

\[ \vartheta_t^L = \frac{p_t^L F_t^L}{B_t} \]

Let’s postulate the price of a single long-term consol bond

\[ \frac{dP_t^L}{P_t^L} = \mu_t^{PL} dt + \sigma_t^{PL} dZ_t \]

In the total net worth numeraire the

\[ E_t[dr_t^L - dr_t^M] = \sigma_t^{PL} \sigma_t^n \] (for now assuming that only intermediaries find it worthwhile to hold consul bonds)

\[ \sigma_t^n = \ldots \text{(in net worth numeraire) \hspace{1cm} (5.3)} \]

\[ dr_t^L = dr_t^M + \sigma_t^{PL} \eta_t dt + \sigma_t^{PL} dZ_t \] (CHECK)
- Return of total bond portfolio (in total net worth numeraire)
  \[ \frac{d}{dt} r_t^B = \mu_t^g dt + \sigma_t^g dZ_t \]  
  (since no fiscal implications)
- \[ \frac{d}{dt} r_t^B = \left(1 - \vartheta_t^L \right) \frac{d}{dt} r_t^M + \vartheta_t^L \frac{d}{dt} r_t^L \]
- \[ \frac{d}{dt} r_t^B = \frac{d}{dt} r_t^M + \vartheta_t^L \left( \sigma_t^{PL} \sigma_t^\eta dt + \sigma_t^{PL} dZ_t \right) \]

- Return of a single coin (reserve unit/short-term bond)
  \[ \frac{d}{dt} r_t^M = \left( \mu_t^g - \vartheta_t^L \sigma_t^{PL} \sigma_t^\eta \right) dt + \left( \sigma_t^g - \vartheta_t^L \sigma_t^{PL} \right) dZ_t \]
- \[ \vartheta_t^L \sigma_t^{PL} \] shows importance of long-term bond price variation
  - the dZ-term is a “risk-transfer”.
  - The dt-term shows that it also affects risk premia
\( \eta^\sigma \)-Volatility and Amplification

\[ \sigma_t^\eta = \sigma_t^rM + (1 - \theta_t^M \cdot 1 - \theta_t^L \cdot 1) \sigma_t^xK^b + \theta_t^L (\sigma_t^L - \sigma_t^rM) \]

- Where portfolio share \( 1 - \theta_t^M \cdot 1 - \theta_t^L \cdot 1 = \frac{1}{\eta} (1 - \theta_t) \) and \( \theta_t = \theta_t / \eta_t \)

\[ \sigma_t^\eta = \sigma_t^\theta - \theta_t^L \sigma_t^{PL} + \frac{\chi_t (1 - \theta_t)}{\eta_t} \left( (1 - \bar{k}) \sigma - \frac{\sigma_t^\theta}{1 - \theta_t} + \theta_t^L \sigma_t^{PL} \right) + \frac{\theta_t^L \theta_t}{\eta_t} \sigma_t^{PL} \]

Collect \( \sigma_t^{PL} \)-terms

\[ \sigma_t^\eta = \sigma_t^\theta + \frac{\chi_t (1 - \theta_t)}{\eta_t} \left( (1 - \bar{k}) \sigma - \frac{\sigma_t^\theta}{1 - \theta_t} \right) + \frac{\chi_t (1 - \theta_t) + \theta_t - \eta_t}{\eta_t} \theta_t^L \sigma_t^{PL} \]

Replace \( \sigma_t^\theta = \frac{\varphi' (\eta_t) \eta_t}{\vartheta (\eta_t)} \sigma_t^\eta \) and \( \sigma_t^{PL} = \frac{p_t^{PL} (\eta_t) \eta_t}{p_t (\eta_t)} \sigma_t^\eta \)

\[ \Rightarrow \eta_t \sigma_t^\eta = \frac{(1 - \theta_t) \chi_t (1 - \bar{k}) \sigma}{1 - \frac{\chi_t - \eta_t}{\eta_t} + \theta_t^L \left( \frac{p_t^{PL} (\eta_t) \eta_t}{p_t (\eta_t)} \right) \chi_t (1 - \theta_t) + \theta_t - \eta_t} \]

- Note that \( -\frac{\varphi' (\eta_t) \eta_t}{\vartheta (\eta_t)} = (1 - \theta_t) \left( \frac{q_L (\eta_t) \eta_t}{q_K (\eta_t)} + \frac{-q_P' (\eta_t) \eta_t}{q_K (\eta_t)} \right) \) ... and is the mitigation term due to policy

\[ \begin{align*}
\text{Liquidity Spiral} & \quad \text{Disinflationary Spiral}
\end{align*} \]
Derive $\mu_t^n$

- Same steps as before
Monetary Policy: Ex-post perspective

- Money view Friedman-Schwartz
  - Restore money supply
    - Replace missing inside money with outside money
  - Aim: Reduce deflationary spiral
    - ... but banks extent less credit & diversify less idiosyncratic risk away
    - ... as households have to hold more idiosyncratic risk, money demand rises
  - Undershoots inflation target

- Credit view Tobin
  - Restore credit
  - Aim: Switch off deflationary spiral & liquidity spiral

- Theory: “Stealth” recapitalization of impaired sector
  - Interest policy and OMO affect asset prices
MoPo Benchmark 1: Removing endogenous Risk

- The policy that removes endogenous risk, $\sigma_t^M = \sigma_t^\vartheta$
- FOC gives (in closed form)

$$\chi_t = \min \left( \frac{\eta_t}{\eta_t + (1 - \eta_t)\phi^2 + (1 - \psi)^2 (\sigma^b)^2 / \bar{\sigma}^2}, \bar{\psi} \right)$$

- $\eta$-Evolution
  - $\sigma^\eta = (1 - \vartheta_t) \frac{\chi_t}{\eta_t} (1 - \bar{\psi}) \sigma^b$
  - $\eta_t \mu_t^\eta = \eta_t (1 - \eta_t)(1 - \vartheta_t)^2 \left( \frac{1 - 2\eta_t}{(1 - \eta_t)^2} \frac{\chi_t^2}{\eta_t^2} (1 - \bar{\psi})^2 (\sigma^b)^2 + \frac{\chi_t^2 \phi^2 \bar{\sigma}^2}{\eta_t^2} - \frac{(1 - \chi_t)^2 \phi^2 \bar{\sigma}^2}{\eta_t^2} \right)$

Closed form up to $\vartheta_t$ (which is choice of planner)
Numerical Example

- \( \rho = .05, \kappa = 2, \tilde{\sigma} = .5, \phi = .4, \bar{\chi} = .8, \sigma^a = 0, \sigma^b = .1 \)
Numerical Example

- $\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.5, \phi = 0.4, \bar{\chi} = 0.8, \sigma^a = 0, \sigma^b = 0.1$

\[ \theta \text{ falls (because low } \eta \text{-regions with high idiosyncratic risk are visited less frequently)}] \]

Policy: $\mu^M = 0, \sigma^M = \sigma^\eta$

Volatility goes down/amplification removed

With policy, risk is lower, but recovery is faster

No amplification when $\eta_t = \kappa_t = \bar{\kappa}$
Optimal Policy

- Next lecture
Recall

- Unified macro “Money and Banking” model to analyze
  - Financial stability - Liquidity spiral
  - Monetary stability - Fisher disinflation spiral

- Exogenous risk &
  - Sector specific
  - Idiosyncratic

- Endogenous risk
  - Time varying risk premia – flight to safety
  - Capitalization of intermediaries is key state variable

- Monetary policy rule
  - Risk transfer to undercapitalized critical sectors
  - Income/wealth effects are crucial instead of substitution effect
  - Reduces endogenous risk – better aggregate risk sharing
    - Self-defeating in equilibrium – excessive idiosyncratic risk taking

“Paradox of Prudence”
Flipped Classroom Experience

Series of 4 YouTube videos, each about 10 minutes
Thank you!

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