Towards the I Theory of Money

- One sector model with idio risk - “The I Theory without I” (steady state focus)
  - Store of value
    - Insurance role of money within sector
  - Money as bubble or not
  - Fiscal Theory of the Price Level
  - Medium of Exchange Role ⇒ SDF-Liquidity multiplier ⇒ Money bubble

- 2 sector/type model with money and idio risk
  - Generic Solution procedure (compared to lecture 03)
  - Real debt vs. Money
    - Implicit insurance role of money across sectors
  - The curse of insurance
    - Reduces insurance premia and net worth gains

- I Theory with Intermediary sector
  - Intermediaries as diversifiers

- Welfare analysis

- Optimal Monetary Policy and Macroprudential Policy
Two Sector Model w/ Outside Equity & Money

- **Expert sector**

  - Capital: \( \psi_t q_t K_t \)
  - Money
  - Money/ Nominal Debt
  - Real Debt
  - Outside equity

- **Household sector**

  - Net worth: \( N_t \)
  - Money/ Nominal Debt
  - Real Debt Claims
  - Equity
  - Capital: \( (1 - \psi_t) q_t K_t \)

- **Experts must hold fraction** \( \chi_t \geq \alpha \psi_t \) (skin in the game constraint)

Expanded on Handbook of Macroeconomics 2017, Chapter 18
- Includes now money and idiosyncratic risk
Two Sector Model Setup

Expert sector
- Output: \( y_t = a k_t \)
- Consumption rate: \( c_t \)
- Investment rate: \( \ell_t \)

\[
\frac{dk_t^i}{k_t^i} = (\Phi(l_t) - \delta) dt + \sigma dZ_t + \bar{\sigma}d\tilde{Z}_t^i + d\Delta_t^{k,i}
\]

\[
E_0 \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]
\]

Friction: Can only issue
- Risk-free debt
- Equity, but most hold \( \chi_t \geq \alpha \)

Household sector
- Output: \( y_t = a k_t \)
- Consumption rate: \( c_t \)
- Investment rate: \( \ell_t \)

\[
\frac{dk_t^i}{k_t^i} = (\Phi(l_t) - \delta) dt + \sigma dZ_t + \bar{\sigma}d\tilde{Z}_t^i + d\Delta_t^{k,i}
\]

\[
E_0 \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]
\]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment \( \iota \), (portfolio \( \theta \), & consumption choice of each agent)
      - Toolbox 1: Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities \( \omega \)
      - Special cases
   b. De-scaled value fcn. as function of state variables \( \eta \)
      - Digression: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive \( \zeta \) price of risk, \( C/N \)-ratio from value fcn. envelop condition

3. Evolution of state variable \( \eta \)
   - Toolbox 3: Change in numeraire to total wealth (including SDF)
   - “Money evaluation equation” \( \mu^\theta \)

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
0. Postulate Aggregates

- Individual capital evolution:
  \[
  \frac{d k_t^{i,i}}{k_t^{i,i}} = (\Phi(t^i) - \delta) dt + \sigma dZ_t + \tilde{\sigma}^i d\tilde{Z}_t^{i,i} + d\Delta_t^{k,i,i}
  \]
  - Where \( \Delta_t^{k,i,i} \) is the individual cumulative capital purchase process

- Capital aggregation:
  - Within sector \( i \): \( K_t^i \equiv \int k_t^{i,i} d\tilde{\iota} \)
  - Across sectors: \( K_t \equiv \sum_i K_t^i \)
  - Capital share: \( \psi_t^i \equiv K_t^i / K_t \)
  \[
  \frac{dK_t}{K_t} = \int (\Phi(t^i) - \delta) dt \quad \text{for all } t
  \]

- Networth aggregation:
  - Within sector \( i \): \( N_t^i \equiv \int n_t^{i,i} d\tilde{\iota} \)
  - Across sectors: \( N_t \equiv \sum_i N_t^i \)
  - Wealth share: \( \eta_t^i \equiv N_t^i / N_t \)

- Value of capital: \( q_t K_t \)
- Value of money: \( p_t K_t \)
0. Postulate Processes

- Value of capital: \( q_t K_t \)
- Value of money: \( p_t K_t \)
- Postulate

\[
\frac{dq_t}{q_t} = \mu^q_t \, dt + \sigma^q_t \, dZ_t \\
\frac{dp_t}{p_t} = \mu^p_t \, dt + \sigma^p_t \, dZ_t
\]

\[
\frac{d\xi^i_t}{\xi^i_t} = \mu^\xi_t \, dt + \sigma^\xi^i_t \, dZ_t + \tilde{\sigma}^\xi^i_t \, d\tilde{Z}^i_t
\]

\[
\equiv -r_t \quad \equiv -\zeta^i_t \\
\equiv -\tilde{\zeta}^i_t
\]

- Derive return processes

\[
dr^K_{t,i,i} = \left( \frac{a^i - i_t}{q_t} + \Phi(i_t) - \delta + \mu^q_t + \sigma^q_t \right) dt + (\sigma + \sigma^q_t) dZ_t + \tilde{\sigma}^i d\tilde{Z}^i_t
\]

\[
dr^M_t = \left( \Phi(\xi_t) - \delta + \mu^p_t + \sigma^p_t - \mu^M \right) dt + (\sigma + \sigma^p_t) dZ_t
\]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

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3. Evolution of state variable \( \eta \)
   - Toolbox 3: Change in numeraire to total wealth (including SDF)
   - “Money evaluation equation” \( \mu^\theta \)

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
1a. Agent Choice of $\iota$, $\theta$, $c$

- Portfolio Choice: Martingale Approach
  - Let $x_t^A$ be the value of a “self-financing trading strategy” (reinvest dividends)
  - Theorem: $\xi_t x_t^A$ follows a Martingale, i.e. drift $= 0$.
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t + \bar{\sigma}_t^A d\bar{Z}_t$,
  - Recall $\frac{d\xi_t}{\xi_t} = -r_t dt - \zeta_t^i dZ_t - \bar{\zeta}_t^i d\bar{Z}_t$
  - By Ito product rule
    $\frac{d(\xi_t x_t^A)}{\xi_t x_t^A} = \left( -r_t + \mu_t^A - \zeta_t^i \sigma_t^A - \bar{\zeta}_t^i \bar{\sigma}_t^A \right) dt + \text{volatility terms}$
    $= 0$
  - Expected return: $\mu_t^A = r_t + \zeta_t^i \sigma_t^A + \bar{\zeta}_t^i \bar{\sigma}_t^A$

- For risk-free asset, i.e. $\sigma_t^A = \bar{\sigma}_t^A = 0$:
  $r_t^f = r_t$
- Excess expected return to risky asset B:
  $\mu_t^A - \mu_t^B = \zeta_t^i (\sigma_t^A - \sigma_t^B) + \bar{\zeta}_t^i (\bar{\sigma}_t^A - \bar{\sigma}_t^B)$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes\hspace{1cm} \textit{static}
   a. Real investment $\iota$, (portfolio $\theta$, & consumption choice of each agent)
      - \textit{Toolbox 1}: Martingale Approach
   b. Asset/Risk Allocation \textit{across types/sectors} & asset market clearing
      - \textit{Toolbox 2}: “price-taking social planner approach” – Fisher separation theorem

2. Value functions\hspace{1cm} \textit{backward equation}
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - \textit{Special cases}
   b. De-scaled value fcn. as function of state variables $\eta$
      - \textit{Digression}: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive $\zeta$ price of risk, $C/N$-ratio from value fcn. envelop condition

3. Evolution of state variable $\eta$\hspace{1cm} \textit{forward equation}
   - \textit{Toolbox 3}: Change in numeraire to total wealth (including SDF)
   - “Money evaluation equation” $\mu^\theta$

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
1b. Asset/Risk Allocation across Types

- **Price-Taking Planner’s Theorem:**
  A social planner that takes prices as given chooses an physical asset and money allocation, $\psi_t$, and risk allocation $\chi_t, \tilde{\chi}_t$, that coincides with the choices implied by all individuals’ portfolio choices.

  \[ s_t = (s_t^1, \ldots, s_t^I) \]
  \[ \chi_t = (\chi_t^1, \ldots, \chi_t^I) \]
  \[ \sigma(\psi_t, \chi_t) = (\chi_t^1 \sigma^{\tilde{N}}(\psi_t), \ldots, \chi_t^I \sigma^{\tilde{N}}(\psi_t)) \]
  \[ \bar{\sigma}(\psi_t, \chi_t) = (\bar{\sigma}^{n_1}(\psi_t, \tilde{\chi}_t), \ldots, \bar{\sigma}^{n_I}(\psi_t, \tilde{\chi}_t)) \]

Return on total wealth (including money)

- **Planner’s problem**
  \[
  \max_{\{\psi_t, \chi_t, \tilde{\chi}_t\}} E_t[dr_t^{\tilde{N}}(\psi_t)] - s_t \sigma(\psi_t, \chi_t) - \zeta_t \bar{\sigma}(\psi_t, \tilde{\chi}_t) = dr^F \text{ in equilibrium}
  \]

  subject to friction: \( F(\psi_t, \chi_t, \tilde{\chi}_t) \leq 0 \)

- **Example:**
  1. \( \chi_t = \psi_t \) (if one holds capital, one has to hold risk)
  2. \( \chi_t \geq \alpha \psi_t \) (skin in the game constraint, outside equity up to a limit)

Note: By holding a portfolio of various experts’ outside equity HH can diversify idio risk away.
### 1b. Allocation of Capital, $\psi$, and Risk, $\chi$

If you shift one capital unit from HH to experts:
- Dividend yield rises by LHS,
- Change the aggregate required risk premium (alpha fraction due to skin of the game constraint)
- HH reduce their risk by one unit, sell back $(1 - \alpha)$ and diversified away

Note that HH, which hold a portfolio of different experts’ outside equity can diversify idiosyncratic risk away.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\chi_t \geq \alpha \psi_t$</th>
<th>$\psi_t \leq 1$</th>
<th>$(a - a)q_t \geq \alpha (\xi_t - \tilde{\xi}_t)(\sigma + \sigma^q_t) + (\alpha \tilde{\xi}_t - \tilde{\xi}_t)\tilde{\sigma}$</th>
<th>$\xi_t(\sigma + \sigma^q_t) + \tilde{\xi}_t\tilde{\sigma}$ $&gt; \xi_t(\sigma + \sigma^q_t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>$=$</td>
<td>$&lt;$</td>
<td>$=$</td>
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<td>1b</td>
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<td>2a</td>
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<tr>
<td>Impossible</td>
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Note: $a^q_t$ represents the dividend yield, $\xi_t$ is the required risk premium, and $\sigma^q_t$ is the idiosyncratic risk.
### 1b. Allocation of Capital, \( \psi \), and Risk, \( \chi \)

<table>
<thead>
<tr>
<th>Cases</th>
<th>( \chi_t \geq \alpha \psi_t )</th>
<th>( \psi_t \leq 1 )</th>
<th>( \frac{(a - \alpha)}{q_t} )</th>
<th>( \zeta_t (\sigma + \sigma_q^t) + \tilde{\zeta}_t \tilde{\sigma} )</th>
</tr>
</thead>
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<td>1a</td>
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HHs’ short-sale constraint of capital binds, \( \psi_t = 1 \)

Experts’ skin in the game constraint binds, \( \chi_t = \alpha \psi_t \)
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3. Evolution of state variable $\eta$
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   - “Money evaluation equation” $\mu^q$

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
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2b. CRRA Value Fcn: Isolating Idio. Risk

- Rephrase the conjecture value function as
  \[ V_t = \frac{1}{\rho} \frac{(\omega_t n_t)^{1-\gamma}}{1 - \gamma} = \frac{1}{\rho (1 - \gamma)} \left( \frac{N_t}{K_t} \right)^{1-\gamma} \left( \frac{n_t}{N_t} \right)^{1-\gamma} K_t^{1-\gamma} =: v_t \]

- \( v_t \) depends only on aggregate state \( \eta_t \)

- Ito’s quotation rule
  \[ \frac{d\hat{\eta}_t^i}{\hat{\eta}_t^i} = \frac{d(n_t/N_t)}{n_t/N_t} = (\mu_t^n - \mu_t^N + (\sigma_t^N)^2 - \sigma^N \sigma^n)dt + (\sigma_t^n - \sigma_t^N)dZ_t + \tilde{\sigma}^n d\tilde{Z}_t = \tilde{\sigma}^n d\tilde{Z}_t \]

- Ito’s Lemma
  \[ \frac{d(\hat{\eta}_t^i)^{1-\gamma}}{(\hat{\eta}_t^i)^{1-\gamma}} = -\frac{1}{2} \gamma (1 - \gamma) (\tilde{\sigma}^n)^2 dt + (1 - \gamma) \tilde{\sigma}^n d\tilde{Z}_t \]
2b. CRRA Value Function

\[ \frac{dV_t}{V_t} = d \left( \frac{v_t(\tilde{\eta}_t)^{1-\gamma} K_t^{1-\gamma}}{v_t(\tilde{\eta}_t)^{1-\gamma} K_t^{1-\gamma}} \right) \]

- By Ito’s product rule

\[ = \left( \mu_t^v + (1 - \gamma)(\Phi(u) - \delta) - \frac{1}{2} \gamma(1 - \gamma)(\sigma^2 + (\tilde{\sigma}^n)^2) + (1 - \gamma)\sigma \sigma_t^v \right) dt + \text{volatility terms} \]

- Recall by consumption optimality

\[ \frac{dV_t}{V_t} - \rho dt + \frac{c_t}{n_t} dt \text{ follows a martingale} \]

- Hence, drift above = \( \rho - \frac{c_t}{n_t} \)

Still have to solve for \( \mu_t^v, \sigma_t^v \)

Poll 16: Why martingale?

a) Because we can “price” networth with SDF
b) because \( \rho \) and \( c_t/n_t \) cancel out
2b. CRRA Value Func BSDE

- Only conceptual interim solution
  - We will transform it into a PDE in Step 4 below

- From last slide
  \[ \mu^v_t + (1 - \gamma)(\Phi(\nu) - \delta) - \frac{1}{2}\gamma(1 - \gamma)(\sigma^2 + (\tilde{\sigma}^n)^2) + (1 - \gamma)\sigma \sigma_t^v = \rho - \frac{c_t}{n_t} \]

- Can solve for \( \mu^v_t \), then \( v_t \) must follow
  \[ \frac{dv_t}{v_t} = f(\eta_t, v_t, \sigma_t^v)dt + \sigma_t^v dZ_t \]

- Together with terminal condition \( v_T \) (possibly a constant for 1000 periods ahead), this is a \textbf{backward stochastic differential equation (BSDE)}

- A solution consists of processes \( v \) and \( \sigma^v \)

- Can use numerical BSDE solution methods (as random objects, so only get simulated paths)

- To solve this via a PDE we also need to get state evolution
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

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      - **Toolbox 1:** Martingale Approach
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3. Evolution of state variable \( \eta \)
   - **Toolbox 3:** Change in numeraire to total wealth (including SDF)
   - “Money evaluation equation” \( \mu^g \)

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
2c. Get $\zeta$s from Value Function Envelop

- **Experts value function**

$$V_t(n) = \nu_t \frac{n_t^{1-\gamma}}{(\eta_t(q_t+p_t))^{1-\gamma}}$$

- To obtain $\frac{\partial V_t(n)}{\partial n_t}$ use $K_t = \frac{N_t}{\eta_t(q_t+p_t)} = \frac{1}{\tilde{\eta}_t} \frac{n_t}{\eta_t(q_t+p_t)}$

- Envelop condition

$$\nu_t \frac{n_t^{-\gamma}}{\eta_t(q_t+p_t)^{1-\gamma}} = c_t^{-\gamma}$$

- Using $K_t = \frac{1}{\tilde{\eta}_t} \frac{n_t}{\eta_t(q_t+p_t)}$, $C_t = \frac{1}{\tilde{\eta}_t} c_t$

$$\frac{\nu_t}{\eta_t(q_t+p_t)} K_t^{-\gamma} = C_t^{-\gamma}$$

- $\sigma_t^\nu - \sigma_t^\eta - \sigma_t^{q+p} - \gamma \sigma = -\gamma \sigma_c^c$

$$= -\zeta_t$$

- **HH’s value function**

$$V_t \frac{K_t^{1-\gamma}}{1-\gamma} \left( \tilde{\eta}_t \right)^{1-\gamma}$$

- To obtain $\frac{\partial V_t(n)}{\partial n_t}$ use $K_t = \frac{N_t}{\eta_t(q_t+p_t)} = \frac{1}{\tilde{\eta}_t} \frac{n_t}{\eta_t(q_t+p_t)}$

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$$\frac{\nu_t}{\eta_t(q_t+p_t)} K_t^{-\gamma} = C_t^{-\gamma}$$

- $\sigma_t^\nu - \sigma_t^\eta - \sigma_t^{q+p} - \gamma \sigma = -\gamma \sigma_c^c$

$$= -\zeta_t$$
## 2c. Get $\varsigma$s from Value Function Envelop

- **Experts value function**
  \[ v_t \frac{K_t^{1-\gamma}}{1-\gamma} \left( \tilde{\eta}_t \right)^{1-\gamma} \]
  
  - To obtain \( \frac{\partial V_t(n)}{\partial n_t} \) use \( K_t = \frac{N_t}{\eta_t(q_t+p_t)} = \frac{1}{\tilde{\eta}_t} \frac{n_t}{\eta_t(q_t+p_t)} \)
  \[ V_t(n) = v_t \frac{n_t^{1-\gamma}/(\eta_t(q_t+p_t))^{1-\gamma}}{1-\gamma} \]
  
  - Envelop condition \( \frac{\partial V_t(n)}{\partial n_t} = \frac{\partial u(c_t)}{\partial c_t} \)
  \[ v_t \frac{n_t^{-\gamma}}{(\eta_t(q_t+p_t))^{1-\gamma}} = c_t^{-\gamma} \]

- **HH’s value function**
  \[ v_t \frac{K_t^{1-\gamma}}{1-\gamma} \left( \tilde{\eta}_t \right)^{1-\gamma} \]
  
  - Using \( K_t = \frac{1}{\tilde{\eta}_t} \frac{n_t}{\eta_t(q_t+p_t)} \), \( C_t = \frac{1}{\tilde{\eta}_t} c_t \)
  \[ \frac{\nu_t}{\eta_t(q_t+p_t)} K_t^{-\gamma} = c_t^{-\gamma} \]

- By Ito’s Lemma
  \[ (q_t + p_t)\sigma_t^{q+p} = q_t\sigma_t^q + p_t\sigma_t^p \]

\[ \sigma_t^v - \sigma_t^\eta - \sigma_t^{q+p} - \gamma \sigma = -\gamma \sigma_t^c, \quad \sigma_t^v - \frac{\eta}{\sigma_t} - \sigma_t^{q+p} - \gamma \sigma = -\gamma \sigma_t^c \]

\[ = -\varsigma_t \quad = -\varsigma_t \]
2c. Get $\varsigma$s from Value Function Envelope

- **Experts risk-premia**
  
  \[
  \nu_t \frac{K_t^{1-\gamma}}{1-\gamma} (\tilde{\eta}_t)_{1-\gamma} \\
  \varsigma_t = \gamma \sigma^c_t = -\sigma^v_t + \sigma^\eta_t + \sigma^{q+p}_t + \gamma \sigma
  \]

- **HH’s risk premia**
  
  \[
  \nu_t \frac{K_t^{1-\gamma}}{1-\gamma} (\tilde{\eta}_t)_{1-\gamma} \\
  \varsigma_t = \gamma \sigma^c_t = -\sigma^v_t - \frac{\eta_t \sigma^\eta_t}{1-\eta_t} + \sigma^{q+p}_t + \gamma \sigma
  \]

- For $\varsigma_t$
  
  note that from
  
  \[
  \nu_t \frac{n_t^{-\gamma}}{n_t (q_t+p_t))^{1-\gamma}} = c_t^{-\gamma} \text{ follows } \tilde{\sigma}_t^n = \tilde{\sigma}_t^c
  \]
  
  Hence, $\tilde{\varsigma}_t = \gamma \tilde{\sigma}_t^n$
  
  $\tilde{\varsigma}_t = \gamma \tilde{\sigma}_t^n$
2c. Get $\frac{C_t}{N_t'} \frac{C_t}{N_t}$ from Value Function Envelop

- **Experts**

- **Recall** $v_t \frac{n_t^{-\gamma}}{(\eta_t(q_t+p_t))^{1-\gamma}} = c_t^{-\gamma}$

- $c_t = \frac{(\eta_t(q_t+p_t))^{1/\gamma-1}}{v_t^{1/\gamma}}$

- $C_t = \frac{(\eta_t(q_t+p_t))^{1/\gamma-1}}{v_t^{1/\gamma}}$

- **Households**

- $\frac{C_t}{N_t} = \frac{((1-\eta_t)(q_t+p_t))^{1/\gamma-1}}{v_t^{1/\gamma}}$

- $\frac{C_t}{N_t} + \frac{C_t}{N_t'} = \eta_t \frac{C_t}{N_t} + (1 - \eta_t) \frac{C_t}{N_t}$

Plug in from above
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   b. De-scaled value fcn. as function of state variables $\eta$
      ▪ *Digression:* HJB-approach (instead of martingale approach & envelop condition)
   c. Derive $\zeta$ price of risk, $C/N$-ratio from value fcn. envelop condition

3. Evolution of state variable $\eta$
   ▪ *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   ▪ “Money evaluation equation” $\mu^\theta$

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
3. $\mu^n\eta$ Drift of Wealth Share: Two Types

- Asset pricing formula (relative to benchmark asset)
  \[ \mu^n_t + \frac{C_t}{N_t} - \vartheta_t \mu^M_t - r^M_t = (\varsigma - \sigma^N)(\sigma^n_t - \sigma^M_t) + \tilde{\varsigma}\tilde{\sigma}^n_t \]
  Seigniorage due to money supply growth leads to transfers

- Add up across types (weighted),
  (capital letters without superscripts are aggregates for total economy)
  \[ \eta_t(\varsigma_t - \sigma^N_t)(\sigma^n_t - \sigma^M_t) + (1 - \eta_t)(\varsigma_t - \sigma^N_t)(\sigma^n_t - \sigma^M_t) + \eta_t\tilde{\varsigma}\tilde{\sigma}^n_t + (1 - \eta_t)\tilde{\varsigma}\tilde{\sigma}^n_t = 0 \]

Subtract from each other yields wealth share drift
\[ \mu^n_t = (1 - \eta_t)(\varsigma_t - \sigma^N_t)(\sigma^n_t - \sigma^M_t) \]
\[ + (1 - \eta_t)\tilde{\varsigma_t}\tilde{\sigma}^n_t - (1 - \eta_t)\tilde{\varsigma_t}\tilde{\sigma}^n_t - \left( \frac{C_t}{N_t} - \frac{C_t + C_q}{(q_t + p_t)K_t} \right) \]
3. $\sigma^\eta$ Volatility of Wealth Share

- Since $\eta_t^i = N_t^i / \bar{N}_t$,

$$\sigma_t^\eta = \sigma_t^{N_i} - \sigma_t^{\bar{N}} = \sigma_t^{N_i} - \sum \eta_t^{i'} \sigma_t^{N_i'}$$

$$= (1 - \eta_t^i) \sigma_t^{N_i} - \sum_{i' \neq i} \eta_t^{i-} \sigma_t^{N_i'^{-}}$$

- Note for 2 types example

$$\sigma_t^\eta = (1 - \eta_t) (\sigma_t^n - \sigma_t^\eta)$$

$$\sigma_t^n = (\sigma + \sigma_t^p) + \frac{\chi_t}{\eta_t} (1 - \vartheta) (\sigma_t^q - \sigma_t^p), \quad \sigma_t^\eta = (\sigma + \sigma_t^p) + \frac{1 - \chi_t}{1 - \eta_t} (1 - \vartheta) (\sigma_t^q - \sigma_t^p)$$

- Hence,

$$\sigma_t^\eta = \frac{\chi_t - \eta_t}{\eta_t} (1 - \vartheta) (\sigma_t^q - \sigma_t^p)$$

- Note also,

$$\eta_t \sigma_t^\eta + (1 - \eta_t) \sigma_t^\eta = 0 \Rightarrow \sigma_t^\eta = -\frac{\eta_t}{1 - \eta_t} \sigma_t^\eta$$

Change in notation in 2 type setting
Type-networth is $n = N^i$

Apply Ito’s Lemma on $\vartheta$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment $\iota$, (portfolio $\theta$, & consumption choice of each agent)
      - Toolbox 1: Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases
   b. De-scaled value fcn. as function of state variables $\eta$
      - Digression: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive $\zeta$ price of risk, $C/N$-ratio from value fcn. envelop condition

3. Evolution of state variable $\eta$
   - Toolbox 3: Change in numeraire to total wealth (including SDF)
   - “Money evaluation equation” $\mu^9$

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
3. “Money evaluation equation” \( \mu^\vartheta \)

- Recall \( \frac{\tilde{C}_t}{N_t} - r_t^M - \vartheta_t \mu^M = \)
  \[ \eta_t \left( \varsigma_t - \sigma_t \overline{N} \right) \left( \sigma_t^\eta - \sigma_t^M \right) + (1 - \eta_t) \left( \varsigma_t - \sigma_t \overline{N} \right) \left( \sigma_t^\eta - \sigma_t^M \right) + \eta_t \tilde{\varsigma} \tilde{\sigma}_t^n + (1 - \eta_t) \tilde{\varsigma}_t \tilde{\sigma}_t^n \]

- If benchmark asset is money
  - Replace \( r_t^M = \mu_t^\vartheta - \mu^M \) and \( \sigma_t^M = \sigma_t^\vartheta \) (in the total wealth numeraire)
  \[ -\mu_t^\vartheta = -(1 - \vartheta) \mu^M - \frac{\tilde{C}_t}{N_t} + \eta_t \left( \varsigma_t - \sigma_t \overline{N} \right) \left( \sigma_t^\eta - \sigma_t^\vartheta \right) + (1 - \eta_t) \left( \varsigma_t - \sigma_t \overline{N} \right) \left( \sigma_t^\eta - \sigma_t^\vartheta \right) + \eta_t \tilde{\varsigma} \tilde{\sigma}_t^n + (1 - \eta_t) \tilde{\varsigma}_t \tilde{\sigma}_t^n \]

- Why is return on money in new numeriare \( \frac{d\vartheta_t}{\vartheta_t} - \mu^M \)?
  - \( \vartheta_t = p_t K_t / \overline{N}_t \) is the value of money stock in total networth units
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given SDF processes
   a. Real investment $\iota$, (portfolio $\theta$, & consumption choice of each agent)
      - Toolbox 1: Martingale Approach
   b. Asset/Risk Allocation across types/sectors & asset market clearing
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem

2. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases
   b. De-scaled value fcn. as function of state variables $\eta$
      - Digression: HJB-approach (instead of martingale approach & envelop condition)
   c. Derive $\zeta$ price of risk, $C/N$-ratio from value fcn. envelop condition

3. Evolution of state variable $\eta$
   - Toolbox 3: Change in numeraire to total wealth (including SDF)
   - “Money evaluation equation” $\mu^\vartheta$

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE
4. PDE Value Function Iteration

- Postulate $v_t = v(\eta_t, t)$
- By Ito’s Lemma

\[
\frac{dv_t}{v_t} = \frac{\partial_t v_t + \partial_\eta v_t \eta \mu_t^\eta + \frac{1}{2} \partial_{\eta \eta} v_t (\eta \sigma_t^\eta)^2}{v_t} dt + \frac{\partial_\eta v_t \eta \sigma_t^\eta}{v_t} dZ_t
\]

That is,

\[
\mu_t^v v_t = \partial_t v_t + \partial_\eta v_t \eta \mu_t^\eta + \frac{1}{2} \partial_{\eta \eta} v_t (\eta \sigma_t^\eta)^2
\]

\[
\sigma_t^v v_t = \partial_\eta v_t \eta \sigma_t^\eta
\]

- Plugging in previous slides drift equation $\Rightarrow$ “growth equation”

\[
\partial_t v_t + (\eta \mu_t^\eta + (1 - \gamma) \sigma_t \sigma_t^\eta v_t) \partial_\eta v_t + \frac{1}{2} \partial_{\eta \eta} v_t (\eta \sigma_t^\eta)^2 = \\
\left(\rho - (1 - \gamma)(\Phi(l) - \delta) + \frac{1}{2} \gamma (1 - \gamma) (\sigma^2 + (\tilde{\sigma}_n^\eta)^2)\right) v_t - \frac{c_t}{n_t} v_t
\]
4a. Algorithm

- **Dynamic steps** involves now iterating \(v(\eta), \underline{v}(\eta), \) and \(\vartheta(\eta)\)

- **Static step** only involve planner’s conditions (which implicitly includes asset market clearing),
  - Solve everything in terms of \(\vartheta(\eta)\)
    - \(q(\eta)\) and \(p(\eta)\) can be easily derived since we have it as a function of \(\vartheta\) and \(\psi\) in closed form
    - \(q_t = (1 - \vartheta_t) \frac{1 + \kappa A(\psi_t)}{1 - \vartheta_t + \kappa \zeta_t}\), where \(\zeta_t := \frac{C_t}{N_t}\)
    - \(p_t = \vartheta_t \frac{1 + \kappa A(\psi_t)}{1 - \vartheta_t + \kappa \zeta_t}\)

- **Remark:**
  One can obtain the moneyless equilibrium with \(\vartheta(\eta) = 0\) by setting \(\sigma^p = -\sigma\) (in models with real risk-free debt)
  - Why? recall \(dr^M = [(\Phi(\iota) - \delta) - \mu^M]dt + (\sigma + \sigma^p)dz\)
    - We never used the drift to solve the model.
    - To make money, risk-free asset we have to set \(\sigma^p = -\sigma\)
Roadmap

- Changes in solution procedure in a setting with idiosyncratic risk and money
  - Compare to lecture 03 without idiosyncratic risk and money

- Simple two sector model
  1. Real Debt
  2. Money/Nominal Debt
Two Sector Model Setup

**Expert sector**
- **Output:** \( y_t = a k_t \)
- **Consumption rate:** \( c_t \)
- **Investment rate:** \( \ell_t \)

\[
\frac{d k_t}{k_t} = (\Phi(\ell_t) - \delta) dt + \sigma d Z_t + \tilde{\sigma} d \tilde{Z}_t + d \Delta_t^{k, i}
\]

\[
\exp{E_0[\int_0^\infty e^{-\rho t} \log c_t dt]} \]

**Household sector**
- **Output:** \( y_t = a k_t \)
- **Consumption rate:** \( c_t \)
- **Investment rate:** \( \ell_t \)

\[
\frac{d k_t}{k_t} = (\Phi(\ell_t) - \delta) dt + \sigma d Z_t + \tilde{\sigma} d \tilde{Z}_t + d \Delta_t^{k, i}
\]

\[
\exp{E_0[\int_0^\infty e^{-\rho t} \log c_t dt]} \]

**Friction:** Can only issue
- Risk-free debt
Two Sector Model with & without Money

- Idio risk without money and real debt

Poll 33: Increasing experts idiosyncratic risk $\tilde{\sigma}$
- a) Lowers experts wealth share drift $\mu$ $\eta$
- b) Increases experts wealth share drift $\mu$ $\eta$, as they earn some extra risk premium
- c) Hurts the households, as it depresses $r_f$
Two Sector Model with & without Money

- **Idio risk without money and real debt**

- **Idio risk with money (and nominal short-term debt)**

- Value of money covaries with $K$-shocks $\Rightarrow$ implicit insurance
Solution Procedure for Both Settings

**Goods market clearing**

\[ \rho(p_t + q_t) = a - \vartheta_t \]

\[ \rho \frac{1}{1 - \vartheta_t} = \frac{a - i}{1 + \kappa i} \]

- \( \vartheta_t = \frac{(1 - \vartheta_t)a - \rho}{1 - \vartheta_t + \kappa \rho} \)
- \( q_t = 1 + \kappa \vartheta_t = (1 - \vartheta_t) \frac{1 + \kappa a}{1 - \vartheta_t + \kappa \rho} \)
- \( p_t = q_t \frac{\vartheta_t}{1 - \vartheta_t} = \vartheta_t \frac{1 + \kappa a}{1 - \vartheta_t + \kappa \rho} \)
Solution Procedure for Both Settings

- **Goods market clearing**

\[
\rho(p_t + q_t) = a - \iota_t
\]

**divide by** \(q\) **and use** \(q = 1 + \kappa \iota\)

\[
\rho \frac{1}{1 - \vartheta_t} = \frac{a - i}{1 + \kappa \iota}
\]

- \(\iota_t = \frac{(1 - \vartheta_t)a - \rho}{1 - \vartheta_t + \kappa \rho}\)

- \(q_t = 1 + \kappa \iota_t = (1 - \vartheta_t) \frac{1 + \kappa a}{1 - \vartheta_t + \kappa \rho}\)

- \(p_t = q_t \frac{\vartheta_t}{1 - \vartheta_t} = \vartheta_t \frac{1 + \kappa a}{1 - \vartheta_t + \kappa \rho}\)

**Poll 36:**

How would equations change if \(a \neq \bar{a}\)

- **a)** Replace \(a\) with \(A(\psi_t)\)
- **b)** Nothing
- **c)** Whole approach has to be different.
Solution Procedure for Both Settings

- **Goods market clearing**

\[ \rho(p_t + q_t) = a - \iota_t \]

\[ \rho \frac{1}{1 - \vartheta_t} = \frac{a - i}{1 + \kappa \iota} \]

\[ \iota_t = \frac{(1 - \vartheta_t)a - \rho}{1 - \vartheta_t + \kappa \rho} \]

\[ q_t = 1 + \kappa \iota_t = (1 - \vartheta_t) \frac{1 + \kappa a}{1 - \vartheta_t + \kappa \rho} \]

\[ p_t = q_t \frac{\vartheta_t}{1 - \vartheta_t} = \vartheta_t \frac{1 + \kappa a}{1 - \vartheta_t + \kappa \rho} \]

- **Capital market clearing**

\[ (1 - \theta_t) = \frac{\psi_t}{\eta_t} (1 - \vartheta_t) \]

\[ (1 - \theta_t) = \frac{1 - \psi_t}{1 - \eta_t} (1 - \vartheta_t) \]

- **Money market clearing** by Walras law
Solution Procedure for Both Settings

- Price-taking Social Planner Problem

\[ \max_{\psi_t} \vartheta_t E[r_t^M] + (1 - \vartheta_t) E[r_t^K] \]

\[ - (\varsigma_t \psi_t + \varsigma_t (1 - \psi_t)) (\sigma + \sigma_t^{p+q}) \]

\[ - (\tilde{\varsigma}_t \psi_t \tilde{\sigma} + \tilde{\varsigma}_t (1 - \psi_t) \tilde{\sigma}) \]

Poll 38: Does \( E_t[r_t^K] \) depend on \( \psi \)?

a) Yes
b) No
Solution Procedure for Both Settings

- Price-taking Social Planner Problem
- \[ \max_{\psi_t} \psi_t E[r_t^M] + (1 - \psi_t) E[r_t^K] \]

\[
-(\zeta_t \psi_t + \zeta_t (1 - \psi_t)) (\sigma + \sigma_t^{p+q}) \\
-(\tilde{\zeta}_t \psi_t \tilde{\sigma} + \tilde{\zeta} (1 - \psi_t) \tilde{\sigma})
\]

- FOC: \[ \zeta_t \sigma + \tilde{\zeta}_t \tilde{\sigma} = \zeta_t \sigma + \tilde{\zeta} \tilde{\sigma} \quad \text{(prices of risks adjust for interior solution)} \]
1. Real Debt Setting: $q, \zeta$, Planner’s prob.

- Set $\vartheta_t = 0, \Rightarrow p = 0$
- $q = \frac{1+\kappa a}{1+\kappa \rho}$ $\forall t \Rightarrow \sigma^q = \sigma^{p+q} = 0$ (as in Basak Cuoco)
- Prices of Risk

$$\zeta_t = \sigma^n_t = (1 - \theta_t) \sigma = \frac{\psi_t}{\eta_t} \sigma, \quad \zeta_t = \sigma^n_t = (1 - \theta_t) \sigma = \frac{1-\psi_t}{1-\eta_t} \sigma$$

$$\tilde{\zeta}_t = \tilde{\sigma}^n_t = (1 - \theta_t) \tilde{\sigma} = \frac{\psi_t}{\eta_t} \tilde{\sigma}, \quad \tilde{\zeta}_t = \tilde{\sigma}^n_t = (1 - \theta_t) \tilde{\sigma} = \frac{1-\psi_t}{1-\eta_t} \tilde{\sigma}$$

- Plug in planners FOC: $\zeta_t \sigma^2 + \tilde{\zeta}_t \tilde{\sigma} = \bar{\zeta}_t \sigma + \tilde{\zeta} \tilde{\sigma}$

$$\psi_t = \frac{\eta_t (\sigma^2 + \tilde{\sigma}^2)}{\sigma^2 + (1 - \eta_t) \tilde{\sigma}^2 + \eta_t \tilde{\sigma}^2}$$

- $\mu_t^\eta = \cdots, \sigma_t^\eta = \cdots$
1. Real Debt Setting: $\eta$-Evolution

$$\sigma^\eta_t = (1 - \eta_t) \left( \sigma^N_t - \sigma^N_{\bar{N}} \right) = (1 - \eta_t) \left( \frac{\psi_t}{\eta_t} - \frac{1 - \psi_t}{1 - \eta_t} \right) \sigma = \frac{\psi_t - \eta_t}{\eta_t} \sigma$$

$$\mu^\eta_t = (1 - \eta_t) (\zeta_t - \sigma^N_t) (\sigma^\eta_t - \sigma^M_t) - (1 - \eta_t) \left( \zeta_t - \sigma^N_t \right) \left( \sigma^\eta_t - \sigma^M_t \right)$$

$$+ (1 - \eta_t) \tilde{\zeta}_t \tilde{\sigma}^\eta_t - (1 - \eta_t) \tilde{\zeta}_t \tilde{\sigma}^\eta_t - \left( \frac{C_t}{N_t} - \frac{C_t + C_t}{(q_t + p_t)K_t} \right)$$

Benchmark asset is risk-free asset in $\bar{N}$-numeraire

- $\sigma^M_t = -\sigma$ because $\sigma^\bar{N}_t = \sigma$ (since $\sigma^q = 0$), $\frac{C}{N} = \rho$
- $\mu^\eta_t = (1 - \eta_t) (\zeta_t - \sigma) (\sigma^\eta_t + \sigma) - (1 - \eta_t) \left( \zeta_t - \sigma \right) \left( \sigma^\eta_t + \sigma \right)$

$$+ (1 - \eta_t) \tilde{\zeta}_t \tilde{\sigma}^\eta_t - (1 - \eta_t) \tilde{\zeta}_t \tilde{\sigma}^\eta_t$$

- $\mu^\eta_t = \frac{\psi_t - \eta_t}{\eta_t} \frac{\psi_t - 2 \eta_t \psi_t + \eta_t^2}{\eta_t (1 - \eta_t)} \sigma^2 + (1 - \eta_t) \left[ \left( \frac{\psi_t}{\eta_t} \right)^2 \tilde{\sigma}^2 - \left( \frac{1 - \psi_t}{1 - \eta_t} \right)^2 \tilde{\sigma}^2 \right]$
1. Real Debt Setting: risk free rate

\[ \frac{a - \lambda}{q} + \Phi(t) - \delta = r^f_t + \zeta_t \sigma + \tilde{\zeta}_t \tilde{\sigma} \]

\[ r^f_t = \rho + (\Phi(t) - \delta) - \frac{\psi_t}{\eta_t} (\sigma^2 + \tilde{\sigma}^2), \text{ where } \psi_t = \frac{(\sigma^2 + \tilde{\sigma}^2)}{\sigma^2 + (1 - \eta_t)\tilde{\sigma}^2 + \eta_t\tilde{\sigma}^2} \]

\[ r^f_t = \rho + (\Phi(t) - \delta) - \frac{(\sigma^2 + \tilde{\sigma}^2)(\sigma^2 + \tilde{\sigma}^2)}{\sigma^2 + (1 - \eta_t)\tilde{\sigma}^2 + \eta_t\tilde{\sigma}^2} \]

**Proposition:** \( r^f_t \) is decreasing in \( \tilde{\sigma}^2 \)

- HH suffer from experts’ idiosyncratic risk exposure via a lower \( r^f_t \)
- Experts have more idio risk, but benefit from lower \( r^f_t \) (since they have to earn risk premium for idio risk)

**Difference to**

- Basak-Cuoco: limited participation \( \psi = 1 \), HH fully at mercy of experts’ ability to hedge idio risk
- Here: HH participate in capital holding
2. Money/Nominal Debt Setting: $\varsigma$s

- **Experts’ price of risk**

\[ \varsigma_t = \sigma_t^n = \sigma + \sigma_t^p + (1 - \theta_t)(\sigma_t^q - \sigma_t^p) \]
\[ = \sigma + \sigma_t^p + \frac{\psi_t}{\eta_t} (1 - \vartheta_t)(\sigma_t^q - \sigma_t^p) \]
\[ \tilde{\varsigma}_t = \sigma_t^{\tilde{n}} = (1 - \theta_t)\tilde{\sigma} = \frac{\psi_t}{\eta_t} (1 - \vartheta_t)\tilde{\sigma} \]

- **Households’ price of risk**

\[ \underline{\varsigma}_t = \sigma_t^n = \sigma + \sigma_p + (1 - \theta_t)(\sigma_t^q - \sigma_p) \]
\[ = \sigma + \sigma_p + \frac{1 - \psi_t}{1 - \eta_t} (1 - \vartheta_t)(\sigma_t^q - \sigma_p) \]
\[ \underline{\tilde{\varsigma}}_t = \tilde{\sigma}_t^n = (1 - \theta_t)\tilde{\sigma} = \frac{1 - \psi_t}{1 - \eta_t} (1 - \vartheta_t)\tilde{\sigma} \]
2. Money Setting: Planner’s Problem

- Conjecture: $\sigma_t^q = \sigma_t^p = 0 \forall t$

  \[ \Rightarrow \zeta = \sigma = \underline{\zeta} = \sigma \]

- Proposition: Aggregate risk is perfectly shared!
  - Via inflation risk
  - Stable inflation (targeting) would ruin risk-sharing
    - Example: Brexit uncertainty. Use inflation reaction to share risks within UK

- Planner’s FOC: $\bar{\zeta} \bar{\sigma} = \bar{\zeta} \bar{\sigma}$
  \[ \frac{\psi_t}{\eta_t} (1 - \vartheta_t) \bar{\sigma}^2 = \frac{1 - \psi_t}{1 - \eta_t} (1 - \vartheta_t) \bar{\sigma}^2 \]

  - $\psi(\eta, \vartheta)$ does not depend on $\vartheta$

  \[ \psi(\eta) = \frac{\eta \bar{\sigma}^2}{(1 - \eta) \bar{\sigma}^2 + \eta \bar{\sigma}^2} \]
2. Money Setting: $\eta$-Evolution

- $\sigma_t^{\eta} = (1 - \eta_t) \left( \frac{\psi_t}{\eta_t} - \frac{1 - \psi_t}{1 - \eta_t} \right) \left( 1 - \vartheta_t \right) \left( \sigma_t^q - \sigma_p \right)$

  - If $\sigma^q = \sigma^p = 0$, then $\sigma_t^{\eta} = 0 \ \forall t$, if $\sigma_t^{\eta} = 0$, then $\sigma^q = \sigma^p = 0$

  By Ito's lemma on $q(\eta)$ and $p(\eta)$

- $\mu_t^{\eta} = (1 - \eta_t) (\zeta_t - \sigma_t^{\bar{N}}) \left( \sigma_t^{\eta} - \sigma_t^M \right) - (1 - \eta_t) \left( \zeta_t - \sigma_t^{\bar{N}} \right) \left( \sigma_t^{\eta} - \sigma_t^M \right)$

  $+ (1 - \eta_t) \tilde{\zeta}_t \tilde{\sigma}_t^{\eta} - (1 - \eta_t) \tilde{\zeta}_t \tilde{\sigma}_t^{\eta} - \left( \frac{C_t}{N_t} - \frac{C_t + C_t}{(q_t + p_t)K_t} \right)$

- Benchmark asset is risk-free asset in $\bar{N}$-numeraire
  
  - $\sigma_t^M = 0$ and $\sigma_t^{\bar{N}} = \sigma, \frac{C}{\bar{N}} = \rho$
  
  - $\mu_t^{\eta} / (1 - \eta_t) = (\zeta_t - \sigma) \sigma_t^{\eta} - (\zeta_t - \sigma) \sigma_t^{\eta} + \tilde{\zeta}_t \frac{\psi_t}{\eta_t} (1 - \vartheta_t) \tilde{\sigma} - \tilde{\zeta}_t \frac{1 - \psi_t}{1 - \eta_t} (1 - \vartheta_t) \tilde{\sigma}$

  - $\mu_t^{\eta} = (1 - \eta_t) (1 - \vartheta_t)^2 \left[ \left( \frac{\psi_t}{\eta_t} \right)^2 \tilde{\sigma}^2 - \left( \frac{1 - \psi_t}{1 - \eta_t} \right)^2 \tilde{\sigma}^2 \right]$
2. Money Setting: Money Evaluation

- Recall $-\mu^g_t = -(1 - \vartheta)\mu^M - \frac{\bar{c}_t}{\bar{N}_t} + \eta_t (\zeta_t - \sigma_{\bar{N}}) (\sigma_t^\eta - \sigma_t^g) + (1 - \eta_t) (\zeta_t - \sigma_{\bar{N}}) (\sigma_t^\eta - \sigma_t^g) + \eta_t \tilde{\zeta} \tilde{\sigma}^n + (1 - \eta_t) \tilde{\zeta} \tilde{\sigma}^n$

- Plug in $\mu^M = 0, \frac{\bar{c}_t}{\bar{N}_t} = \rho, \zeta_t = \zeta_t = \sigma, \sigma_{\bar{N}} = \sigma$

$\tilde{\zeta} \tilde{\sigma}^n = (1 - \vartheta_t)^2 \left( \frac{\psi_t}{\eta_t} \right)^2 \tilde{\sigma}^2, \quad \tilde{\zeta} = \tilde{\sigma}^n_t = (1 - \vartheta_t)^2 \left( \frac{1 - \psi_t}{1 - \eta_t} \right)^2 \tilde{\sigma}^2$

$-\mu^g_t = -\rho + (1 - \vartheta_t)^2 \left[ \eta_t \left( \frac{\psi_t}{\eta_t} \right)^2 \tilde{\sigma}^2 + (1 - \eta_t) \left( \frac{1 - \psi_t}{1 - \eta_t} \right)^2 \tilde{\sigma}^2 \right]$

- where $\psi_t = \frac{\eta_t \tilde{\sigma}^2}{(1 - \eta_t) \tilde{\sigma}^2 + \eta_t \tilde{\sigma}^2}$
2. Money Setting: Adding Real Debt

- Adding Real Debt does not alter the equilibrium, since
  - Markets are complete w.r.t. to aggregate risk (perfect aggregate risk sharing)
  - Markets are incomplete w.r.t. to idiosyncratic risk only

- Note: Result relies on absence of price stickiness

- Both Settings: Real Debt and Money/Nominal Debt converge in the long-run to the “I Theory without I” steady state model of Lecture 05.
Example: **Real vs. Nominal Debt/Money**

- \( a = 0.15, \rho = 0.03, \sigma = 0.1, \kappa = 2, \delta = 0.03, \tilde{\sigma} = 0.2, \tilde{\delta} = 0.3 \)

*Blue: real debt model*
*Red: nominal model*
Towards the I Theory of Money

- One sector model with idio risk - “The I Theory without I” (steady state focus)
  - Store of value
    - Insurance role of money \( \textit{within sector} \)
  - Money as bubble or not
  - Fiscal Theory of the Price Level
  - Medium of Exchange Role \( \Rightarrow \) SDF-Liquidity multiplier \( \Rightarrow \) Money bubble

- 2 sector/type model with money and idio risk
  - Generic Solution procedure (compared to lecture 03)
  - Real debt vs. Money
    - Implicit insurance role of money \( \textit{across sectors} \)
  - The curse of insurance
    - Reduces insurance premia and net worth gains

- I Theory with Intermediary sector
  - Intermediaries as diversifiers

- Welfare analysis

- Optimal Monetary Policy and Macroprudential Policy