The big Roadmap: Towards the I Theory of Money

- One sector model with idio risk - “The I Theory without I”
  (steady state focus)
  - Store of value
    - Insurance role of money within sector
  - Money as bubble or not
  - Fiscal Theory of the Price Level
  - Medium of Exchange Role $\Rightarrow$ SDF-Liquidity multiplier $\Rightarrow$ Money bubble

- 2 sector/type model with money and idio risk
  - Generic Solution procedure (compared to lecture 03)
  - Equivalence btw experts producers and intermediaries
  - Real debt vs. nominal debt/money
    - Implicit insurance role of money across sectors
  - I Theory

- Welfare analysis
- Optimal Monetary Policy and Macroprudential Policy
- International Monetary Model
Optimal Policy

- Finding the optimal policy is generally complicated, need
  1. precise definition of policy space
  2. analytical tools to characterize the optimum

- One side: inefficiencies / tradeoffs
  - insurance vs. investment (one sector/type)
  - allocation of assets / risk (across sectors/types)

- Other side: “large” policy space
  - controlling money growth rate
  - macroprudential tools / wealth redistribution
  - risk redistribution

- Approach
  - Start with simple model
  - Add step-by-step more model elements
Roadmap

- Expected Utility/Value function with log-utility

- One sector model with stochastic idiosyncratic volatility

- Two sector model
  - with exogenous net worth share $\eta$
  - With endogenous wealth share $\eta$
  - I theory (with two technologies)
Welfare with log utility

- The welfare for any agent $\tilde{i}$ of type $i$

$$E \left[ \int_0^\infty e^{-\rho t} \log(c_{\tilde{i}}^t) \, dt \right]$$

$$\tilde{\eta}_0^i = 1, \quad \frac{d\tilde{\eta}_t^i}{\tilde{\eta}_t^i} = \tilde{\sigma}_t^i \, d\tilde{Z}_t^i$$

- Recall from general model with log utility

  - $\frac{c_{t}^i}{n_{t}^i} = \rho$
  - $c_{t}^i = \rho \eta_{t}^i (A(\kappa_t) - \lambda_t) K_t \tilde{\eta}_t^i$ using goods market clearing
Welfare with log utility

The welfare of any agent $i$ is

$$E \left[ \int_0^\infty e^{-\rho t} \log(c_t) \, dt \right] = E \left[ \int_0^\infty e^{-\rho t} \log(\eta_t^i (A(k_t) - \lambda_t) K_t \tilde{\eta}_t^i) \, dt \right]$$

ignoring constant $\frac{\log \rho}{\rho}$

$$= E \left[ \int_0^\infty e^{-\rho t} \log \eta_t^i \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \log(A(k_t) - \lambda_t) \, dt \right]$$

$$+ E \left[ \int_0^\infty e^{-\rho t} \log K_t \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \log \tilde{\eta}_t^i \, dt \right]$$
Welfare with log utility

- Recall

\[ \log X_t - \log X_0 = \int_{0}^{t} d \log X_s \]

- Apply to Ito's lemma

\[
d \log X_t = \left( \mu_t^X - \frac{1}{2} \left( \sigma_t^X \right)^2 - \frac{1}{2} \left( \tilde{\sigma}_t^X \right)^2 \right) dt + \sigma_t^X dZ_t + \tilde{\sigma}_t^X d\tilde{Z}_t
\]

- Plug into Expected integral

\[
E \left[ \int_{0}^{\infty} e^{-\rho t} \log(X_t) \ dt \right] = \frac{1}{\rho} \log(X_0) + \frac{1}{\rho} E \left[ \int_{0}^{\infty} e^{-\rho t} \left( \mu_t^X - \frac{1}{2} \left( \sigma_t^X \right)^2 - \frac{1}{2} \left( \tilde{\sigma}_t^X \right)^2 \right) dt \right]
\]
Welfare with log utility

- The welfare of any agent $\tilde{i}$ is

$$E \left[ \int_0^\infty e^{-\rho t} \log(c_t^{\tilde{i}}) \, dt \right] = E \left[ \int_0^\infty e^{-\rho t} \log(\eta_t^i (A(\kappa_t) - \iota_t) K_t \tilde{\eta}_t^i) \, dt \right]$$

ignoring constant $\frac{\log \rho}{\rho}$

$$= E \left[ \int_0^\infty e^{-\rho t} \log \eta_t^i \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \log(A(\kappa_t) - \iota_t) \, dt \right]$$

$$\frac{\log \eta_0^i}{\rho} + E \left[ \int_0^\infty e^{-\rho t} \left( \frac{\mu_t^i}{\rho} - \frac{(\sigma_t^i)^2}{2\rho} \right) \, dt \right]$$

$$+ E \left[ \int_0^\infty e^{-\rho t} \log K_t \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \log \tilde{\eta}_t^i \, dt \right]$$

$$\frac{\log K_0}{\rho} + E \left[ \int_0^\infty e^{-\rho t} \left( \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{(\sigma_t^K)^2}{2\rho} \right) \, dt \right]$$

$$- E \left[ \int_0^\infty e^{-\rho t} \left( \frac{(\tilde{\eta}^i_t)^2}{2\rho} \right) \, dt \right]$$
Welfare of Intermediaries $I$ and HH $h$

- Intermediaries (Pareto weight $\lambda$)

$$E \left[ \int_0^\infty e^{-\rho t} \left( \log \eta_t + \log(A(\kappa) - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \vartheta_t)^2}{2\rho} \frac{\kappa^2 \varphi^2 \sigma^2}{\eta^2} \right) dt \right]$$

- Households (Pareto weight $1 - \lambda$)

$$E \left[ \int_0^\infty e^{-\rho t} \left( \log(1 - \eta_t) + \log(A(\kappa) - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \vartheta_t)^2}{2\rho} \frac{(1 - \kappa)^2 \sigma^2}{(1 - \eta)^2} \right) dt \right]$$
Roadmap

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  - With endogenous wealth share $\eta$
  - I theory (with two technologies)
One Sector Model with Money

- Agent $i$'s preferences
  \[ E \left[ \int_0^\infty e^{-\rho t} \log c_t^i \; dt \right] \]

- Each agent operates one firm
  - Output
    \[ y_t^i = a k_t^i \]
  - Physical capital $k$
    \[ \frac{dk_t^i}{k_t^i} = (\Phi(i_t^i) - \delta)dt + \tilde{\sigma}_t d\tilde{Z}_t^i \]

- Idiosyncratic risk $\tilde{\sigma}$ is stochastic (hence a state variable)
  \[ d\tilde{\sigma}_t = \mu(\tilde{\sigma}_t)dt + \nu(\tilde{\sigma}_t)dZ_t^\nu \]
  e.g. CIR process
  \[ d\tilde{\sigma}_t = \alpha \left( \sigma^{SS} - \tilde{\sigma}_t \right)dt + \nu \sqrt{\tilde{\sigma}_t} dZ_t^\nu \]

- Financial Friction: Incomplete markets: Agents cannot share $d\tilde{Z}_t^i$
One Sector Model with Money

- Agent $i$’s preferences
  \[ E \left[ \int_0^\infty e^{-\rho t} \log c_t^i \, dt \right] \]

- Each agent operates one firm
  - Output
    \[ y_t^i = a k_t^i \]
  - Physical capital $k$
    \[ \frac{dk_t^i}{k_t^i} = (\Phi(\iota_t^i) - \delta) \, dt + \tilde{\sigma}_t \, d\tilde{Z}_t \]

- Financial Friction: Incomplete markets: Agents cannot share $d\tilde{Z}_t$

- Outside money/Gov. bond
  \[ \frac{dM_t^i}{M_t} = \mu_t^M \, dt + \nu_t^M \, dZ_t^\gamma. \]

State variable is $\tilde{\sigma}$: -- Monetary policy $\mu^M(\tilde{\sigma}_t), \nu^M(\tilde{\sigma}_t)$
One Sector Model with Money

- Dynamics of $\tilde{\eta}_t$:
  \[ \frac{d\tilde{\eta}_t}{\tilde{\eta}_t} = d \left( \frac{n^i_t}{N^i_t} \right) / d\tilde{\eta}_t = \frac{(1 - \vartheta_t)}{\tilde{\sigma}_t} d\tilde{Z}^i_t \]

- Total wealth as numeraire has return $\rho$, $dr^N_t = \rho dt$

- Money has return
  \[ dr^\vartheta_t/M_t = \frac{d(\vartheta_t/M_t)}{\vartheta_t/M_t} = \left( \frac{\mu^\vartheta_t - \mu^M_t + \nu^M_t (\nu^M_t - \sigma^\vartheta_t)}{\mu^\vartheta/M_t} \right) dt + \left( \frac{\sigma^\vartheta_t - \nu^M_t}{\sigma^\vartheta/M_t} \right) d\tilde{Z}^\nu_t \]

- Money valuation equation
  \[ \rho - \mu^\vartheta_t/M^t = \left( \tilde{\sigma}_t \tilde{\eta}^i_t \right)^2 = (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 \]

- Without policy, equation
  \[ \rho - \mu^\vartheta_t = (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 \]

has a unique solution in $\vartheta(\tilde{\sigma}_t) \in (0,1)$ (if $\tilde{\sigma}_t$ sufficiently large)
Recall Equilibrium

- Price of physical capital
  \[ q^K_t = (1 - \vartheta_t) \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho} \]

- Price of nominal capital
  \[ q^M_t = \vartheta_t \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho} \]

- Optimal investment rate
  \[ \iota_t = \frac{(1 - \vartheta_t) a - \rho}{(1 - \vartheta_t) + \phi \rho} \]

- Fraction of nominal wealth \( \vartheta_t \)
  \[ 1 - \vartheta_t = \sqrt{\rho + \mu^M_t - (\sigma^M_t)^2 - \mu^\vartheta_t + \sigma^\vartheta_t \sigma^M_t} \]

-One Sector Model with Money/Gov. Bond
Welfare is
\[
\frac{\log K_0}{\rho} - \frac{\delta}{\rho^2} + E \left[ \int_0^\infty e^{-\rho t} \log(A(\kappa_t) - \iota_t) \, dt \right] \\
= a
\]

\[E \left[ \int_0^\infty e^{-\rho t} \log \left( \frac{\alpha \rho + 1}{\rho \phi + 1 - \vartheta_t} \right) dt \right]
\]

\[+ E \left[ \int_0^\infty e^{-\rho t} \frac{\Phi(\iota_t)}{\rho} dt \right] - E \left[ \int_0^\infty e^{-\rho t} \frac{(1 - \vartheta_t)^2 \bar{\sigma}_t^2}{2\rho} dt \right]
\]

\[\frac{1}{\rho \phi} E \left[ \int_0^\infty e^{-\rho t} \log \left( \frac{(\alpha \phi + 1)(1 - \vartheta_t)}{\rho \phi + 1 - \vartheta_t} \right) dt \right]
\]
Optimal Policy

- Welfare is

$$E \left[ \int_0^\infty e^{-\rho t} \left[ \log \left( \frac{a\phi + 1}{\rho\phi + 1 - \vartheta_t} \right) + \frac{1}{\rho\phi} \log \left( \frac{(a\phi + 1)(1 - \vartheta_t)}{\rho\phi + 1 - \vartheta_t} \right) - \frac{(1 - \vartheta_t)^2 \tilde{\sigma}_t^2}{2\rho} \right] \, dt \right]$$

- **Lemma:** Problem collapses to a static problem for each $t$

- Let $\vartheta^*(\tilde{\sigma}_t^2)$ be the maximizer of welfare (optimal policy)

$$\max_{\vartheta} \frac{1}{\rho\phi} \log(1 - \vartheta_t) - \frac{1}{\rho\phi} \log(\rho\phi + 1 - \vartheta_t) - \frac{(1 - \vartheta_t)^2 \tilde{\sigma}_t^2}{2\rho}$$
Optimal Policy

Red: equilibrium $\theta$ in the baseline model
Black: optimal policy $\theta^*$

\[
\theta^* (\tilde{\sigma}^2) = \max_{\theta} \frac{1}{\rho \phi} \log(1 - \theta) - \frac{\rho \phi + 1}{\rho \phi} \log(\rho \phi + 1 - \theta) - \frac{(1 - \theta)^2 \tilde{\sigma}^2}{2\rho}.
\]
Pecuniary Externality Explanation

- Money growth $\mu^M$ affects
  - Shadow risk-free rate
  - (Steady state) inflation in two ways
    \[ \pi = \mu^M + i - \left( \Phi \left( i(\mu^M) \right) - \delta \right) / g \]

- Proposition:
  - For sufficiently large $\tilde{\sigma}$ and $\phi < \infty$ welfare maximizing $\mu^M > 0$.
    - Laissez-faire Market outcome is not even constrained Pareto efficient
    - Economic growth rate $g$ is also higher
    - Growth maximizing $\mu^g \geq \mu^M$, s.t. $p^g = 0$, Tobin (1965)

- Corollary: No super-neutrality of money
  - $i$: Super-neutrality only w.r.t. part of money growth rate that is used to pay interest on money
  - $\mu^M$: Nominal money growth rate affects real economic growth by distorting portfolio choice if $\phi < \infty$
    - No price/wage rigidity, no monopolistic competition
Optimal Policy

- If the planner can control $\vartheta_t$ directly, she would set $\vartheta_t = \vartheta^*(\tilde{\sigma}_t^2)$
  \[
d\tilde{\sigma}_t = \mu^{\tilde{\sigma}}(\tilde{\sigma}_t)dt + \nu^{\tilde{\sigma}}(\tilde{\sigma}_t)dZ^\nu_t
  \]
  \[
d\vartheta_t = \mu^{\vartheta}(\tilde{\sigma}_t)\vartheta_t dt + \nu^{\vartheta}(\tilde{\sigma}_t)\vartheta_t dZ^\nu_t
  \]

- The planner can choose instruments $\mu^M(\tilde{\sigma}_t), \nu^M(\tilde{\sigma}_t)$ to achieve any function $\vartheta_t$
  - How to find the instruments $\mu^M(\tilde{\sigma}_t), \nu^M(\tilde{\sigma}_t)$ that achieve $\vartheta^*(\tilde{\sigma}_t^2)$?
    -- solving money valuation equation
    \[
    \rho - (\mu^\vartheta_t + \nu_t(\nu_t - \sigma^\vartheta_t)) = (1 - \vartheta_t)^2 \tilde{\sigma}_t^2
    \]

- Optimal policy is easier to find than equilibrium outcome
  - differentiation vs. integration (or solve PDEs)
Roadmap

- Expected Utility/Value function with log-utility

- One sector model with stochastic idiosyncratic volatility

- Two sector model
  - with exogenously fixed net worth share $\eta$
  - With endogenous wealth share $\eta$
  - I theory (with two technologies)
Two Switching Sector model with Exogenous wealth dist.

- **Model Setup**

  - $\lambda^s$ Switching infinitely fast
  - $\lambda^e$

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<td>Share of agents = net worth share</td>
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<td>Idiosyncratic risk of capital</td>
<td>$\varphi\tilde{\sigma}, \varphi \in (0,1)$ diversification</td>
<td>$\tilde{\sigma}$</td>
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<td>$a$ the same, independently of the allocation</td>
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Policy marker can choose the money growth rate $\mu^M$. 
Remark

- Policy-marker cannot affect the wealth shares
- Welfare Pareto weights
  - $\lambda = \eta$ for intermediaries and
  - $1 - \lambda = 1 - \eta$ for households from the setup
- Optimal monetary
  (with or without macroprudential policy – controlling capital allocation)
  - Perfect commitment – Ramsey problem
Equilibrium capital allocation

- Fraction $\chi$ of risk ($\kappa$ of capital) is held by the intermediaries ($\chi = \kappa$)

- Capital allocation must be such that

$$ \frac{(1 - \vartheta)\kappa \varphi \tilde{\sigma}_t}{\eta} = \frac{(1 - \vartheta)(1 - \kappa)\tilde{\sigma}_t}{1 - \eta} $$

$$ \Rightarrow \kappa = \frac{\eta}{\varphi^2(1 - \eta) + \eta} $$

- Policy marker may try to affect $\kappa$...
Welfare of Intermediaries \( I \) and HH \( h \)

- Intermediaries (Pareto weight \( \lambda \))

\[
E \left[ \int_0^\infty e^{-\rho t} \left( \log \eta_t + \log(a - \iota_t) + \frac{\Phi(t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \vartheta_t)^2 \kappa^2 \varphi^2 \sigma^2}{\eta^2} \right) dt \right]
\]

- Households (Pareto weight \( 1 - \lambda \))

\[
E \left[ \int_0^\infty e^{-\rho t} \left( \log(1 - \eta_t) + \log(a - \iota_t) + \frac{\Phi(t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \vartheta_t)^2 (1 - \kappa^2 \sigma^2)}{(1 - \eta)^2} \right) dt \right]
\]
Welfare

- Law of large numbers: switching risk does not matter. Everyone’s wealth growth averages out to $\Phi(\mu_t) - \delta$ and idiosyncratic risk exposure, to

$$
\eta(\bar{\sigma}^l)^2 + (1 - \eta)(\bar{\sigma}^h)^2 = (1 - \vartheta)^2 \bar{\sigma}^2 \left( \lambda \frac{\kappa^2 \varphi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \kappa)^2}{(1 - \eta)^2} \right)
$$

$$
(\bar{\sigma}^{Ave})^2 := \frac{(1 - \vartheta)\kappa \varphi \bar{\sigma}}{\eta}, \bar{\sigma}^l = \frac{(1 - \vartheta)\kappa \varphi \bar{\sigma}}{\eta}, \bar{\sigma}^h = \frac{(1 - \vartheta)(1 - \kappa)\bar{\sigma}}{1 - \eta}
$$

- Welfare

$$
E \left[ \int_0^\infty e^{-\rho t} \log(a - \mu_\vartheta) \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \frac{\Phi(\mu_\vartheta) - \delta}{\rho} \, dt \right] - E \left[ \int_0^\infty e^{-\rho t} \frac{(1 - \vartheta)^2 (\bar{\sigma}^{Ave})^2}{2\rho} \, dt \right]
$$

- Given $\bar{\sigma}^A$, optimal to set $\vartheta = \vartheta^* \left( (\bar{\sigma}^{Ave})^2 \right)$.

- Set $\lambda = \eta$ (Pareto weight is population share)
Money valuation

- Money valuation equation

\[ \rho - \left( \mu_t^g - \mu_t^M + \nu_t^M (\nu_t^M - \sigma_t^g) \right) = \eta (\tilde{\sigma}_t^l)^2 + (1 - \eta) (\tilde{\sigma}_t^h)^2 \]

\[ \begin{array}{c}
\mu_t^g \\
\mu_t^M \\
\mu_t^g/M \\
(1-\eta_t)^2 (\tilde{\sigma}_t^{Ave})^2
\end{array} \]
Macroprudential tools

- Average idiosyncratic risk of capital
  \[ \bar{\sigma}^2 \left( \frac{\kappa^2 \varphi^2}{\eta} + \frac{(1 - \kappa)^2}{1 - \eta} \right) \]
  is minimized when
  \[ \frac{\kappa \varphi^2}{\eta} = \frac{1 - \kappa}{1 - \eta} \Rightarrow \kappa = \frac{\eta}{\varphi^2 (1 - \eta) + \eta} \]
  - This is the equilibrium allocation!

- **Lemma:** Optimal not to use macroprudential tools.
  assuming \( \lambda = \eta \)
Remarks

- Same trade-off between insurance and investment
- Equilibrium allocation is efficient, minimizes the cost of risk exposure
- Policy space
  - (1) money growth and
  - (1) + (2) (money growth + macroprudential tools)
    leads to the same outcome
Roadmap

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  - I theory (with two technologies)
Endogenous law of motion of $\eta$

- Wealth distribution can change endogenously with
  - risk exposure of intermediaries and households
  - risk premia
  - consumption rates
- Consider the following model
Fixed types (no switching)

- **Model Setup**

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Types fixed (no switching)

You have already seen this model except here $\kappa = 1$

Two policy settings:
1. Money growth rate $\mu_t^M$ only
2. Also choose allocation (macroprudential) and transfer wealth between group (why/how?)
Welfare of Intermediaries $I$ and HH $h$

- Intermediaries (Pareto weight $\lambda$)

$$E \left[ \int_0^\infty e^{-\rho t} \left( \log \eta_t + \log(a - \iota_t) + \frac{\Phi(t_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \theta_t)^2}{2\rho} \frac{\kappa^2 \phi^2 \sigma^2}{\eta^2} \right) dt \right]$$

- Households (Pareto weight $1 - \lambda$)

$$E \left[ \int_0^\infty e^{-\rho t} \left( \log(1 - \eta_t) + \log(a - \iota_t) + \frac{\Phi(t_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \theta_t)^2}{2\rho} \frac{(1 - \kappa)^2 \sigma^2}{(1 - \eta)^2} \right) dt \right]$$
Optimal policy: (1) MoPo + (2) MacroPru

- Planner chooses $\theta$, $\kappa$ and $\eta$ to max discount integral of

$$\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \log(a - \iota(\vartheta_t)) + \frac{\Phi(\iota(\vartheta_t)) - \delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$- \frac{(1 - \vartheta_t)^2 \sigma^2}{2\rho} \left( \frac{\lambda \kappa^2 \varphi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \kappa)^2}{(1 - \eta)^2} \right)$$

$$\frac{\lambda(1 - \lambda)\varphi^2}{\lambda \varphi^2 (1 - \eta)^2 + (1 - \lambda)\eta^2}$$

given the optimal choice of $\kappa = \frac{(1 - \lambda)\eta^2}{\lambda \varphi^2 (1 - \eta)^2 + (1 - \lambda)\eta^2}$

not the competitive allocation (unless $\eta = \lambda$)
Optimal policy: (1) MoPo + (2) MacroPru

- **Step 1:** Solve optimal $\kappa$ (or $\chi$) for a given $\eta$ and $\lambda$

  Competitive $\kappa$ vs. minimizing cost of risk

  \[
  \kappa = \frac{(1 - \lambda)\eta^2}{\lambda\varphi^2(1 - \eta)^2 + (1 - \lambda)\eta^2}
  \]

  \[
  \kappa = \frac{\eta}{\varphi^2(1 - \eta) + \eta}
  \]

  Of course, here $\eta$ also can be chosen by the planner... but this is important because when $\eta$ can move freely, planner may want to push risk to the group whose wealth exceeds its welfare weight.
Optimal policy: (1) MoPo + (2) MacroPru

- Planner chooses $\vartheta, \kappa$ and $\eta$ to max discount integral of

$$\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \log(a - \iota(\vartheta_t)) + \frac{\Phi(\iota(\vartheta_t)) - \delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$\frac{(1 - \vartheta_t)^2 \tilde{\sigma}^2}{2 \rho} \left( \frac{\lambda}{\eta^2} + (1 - \lambda) \frac{(1 - \kappa)^2}{(1 - \eta)^2} \right)$$

- Step 2: Solve $\vartheta_t = \vartheta^* (\cdot)$ (having used optimal $\kappa_t$) for each given $\eta$

- Given $\kappa$ and $\eta$, optimal to set $\vartheta$ to

$$\vartheta = \vartheta^* (\frac{\lambda(1 - \lambda) \phi^2}{\lambda \phi^2 (1 - \eta)^2 + (1 - \lambda) \eta^2})$$

welfare weighted average risk exposure

given the optimal choice of $\kappa = \frac{(1 - \lambda) \eta^2}{\lambda \phi^2 (1 - \eta)^2 + (1 - \lambda) \eta^2}$

not the competitive allocation (unless $\eta = \lambda$)
Optimal policy: (1) MoPo + (2) MacroPru

- **Step 3: Optimal** $\eta$ *(given $\vartheta$)*
- let’s look at terms containing $\eta$
- Given $\kappa$ and $\eta$,

$$
\max_{\eta} \frac{\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t)}{\text{concave, max at $\eta=\lambda$, goes to $-\infty$ at 0 & 1}} - \frac{(1 - \vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \frac{\lambda(1 - \lambda)\varphi^2}{\lambda \varphi^2(1 - \eta)^2 + (1 - \lambda)\eta^2} \text{concave, max at } \frac{\lambda \varphi^2}{\lambda \varphi^2 + 1 - \lambda} < \lambda
$$

- hence it is **optimal to set** $\eta > \lambda$
  (unfortunately no closed-form expression for the optimal $\eta$)
- push more risk onto intermediaries than they’d take under competitive outcome
- relative to previous infinite switching model
  - it is optimal to give intermediaries more wealth, because they are more efficient at absorbing risk
  - overall risk is reduced and the value of money is lower (more intermediation)
Optimizing over $\eta$

$\rho = 0.05, \phi = 2, \bar{\sigma} = 0.3, \varphi = 0.5, \lambda = 0.2$
Optimal policy: (1) MoPo only

- Planner cannot alter competitive alloc. \( \kappa_t = \frac{\eta_t}{\varphi^2(1-\eta_t)+\eta_t} \)

- Welfare is the discount integral of

\[
\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \log(a - \varrho(\vartheta_t)) + \frac{\Phi(\varrho(\vartheta_t)) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \vartheta_t)^2 \bar{\sigma}^2}{2\rho} \left( \frac{\lambda \kappa_t^2 \varphi^2}{\eta_t^2} + (1 - \lambda) \frac{(1 - \kappa_t)^2}{(1 - \eta_t)^2} \right) - \frac{\lambda \varphi^2 + (1 - \lambda) \varphi^4}{(\varphi^2(1-\eta_t) + \eta_t)^2}
\]

s.t.

\[
\frac{d\eta_t}{\eta_t} = (1 - \eta_t) \left( (\bar{\sigma}_t')^2 - (\bar{\sigma}_t^h)^2 \right) dt = (1 - \eta_t) \frac{(1 - \vartheta_t)^2 \bar{\sigma}^2 \varphi^2 (1 - \varphi^2)}{(\varphi^2(1 - \eta_t) + \eta_t)^2} dt
\]

- Planner can not choose \( \kappa_t \) or \( \eta_t \) but has some control over \( \mu_t^\eta \)

- Now, fully dynamic problem!
Optimal policy: (1) MoPo only

- Payoff flow: 
  \[ f(\eta_t, \vartheta_t) = \lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \frac{\log(1 - \vartheta_t)}{\rho \phi} \]
  \[ -\frac{\rho \phi + 1}{\rho \phi} \log(\rho \phi + 1 - \vartheta_t) - \frac{(1 - \vartheta_t)^2 \bar{\sigma}^2}{2 \rho} \left( \lambda \frac{\kappa_t^2 \varphi^2}{\eta_t^2} + (1 - \lambda) \frac{(1 - \kappa_t^2)}{(1 - \eta_t)^2} \right), \]

  with \( \kappa = \frac{\eta}{\varphi^2(1 - \eta) + \eta} \)

- HJB equation

  \[ \rho V(\eta) = \max_{\vartheta} f(\eta, \vartheta) + V'(\eta)\mu\eta\eta + \frac{1}{2} V''(\eta)(\sigma^2 \eta^2) \]

- Law of motion of \( \eta \)

  \[ \frac{d\eta}{\eta} = (1 - \eta) \frac{(1 - \vartheta)^2 \bar{\sigma}^2 \varphi^2 (1 - \varphi^2)}{(\varphi^2(1 - \eta) + \eta)^2} dt + 0dZ \]
Optimal policy: (1) MoPo only

- Payoff flow: $f(\eta_t, \vartheta_t) = \lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \frac{\log(1 - \vartheta_t)}{\rho \phi}$

$$- \frac{\rho \phi + 1}{\rho \phi} \log(\rho \phi + 1 - \vartheta_t) - \frac{(1 - \vartheta_t)^2 \tilde{\sigma}^2}{2 \rho} \left( \lambda \frac{\kappa_t^2 \varphi^2}{\eta_t^2} + (1 - \lambda) \frac{(1 - \kappa_t)^2}{(1 - \eta_t)^2} \right),$$

- with $\kappa = \frac{\eta}{\varphi^2 (1 - \eta) + \eta}$

- HJB equation

$$\rho V(\eta) = \max_{\vartheta} f(\eta, \vartheta) + V'(\eta) \mu \eta \eta + \frac{1}{2} V''(\eta) (\sigma'' \eta)^2$$

- Law of motion of $\eta$

$$\frac{d\eta}{\eta} = (1 - \eta) \frac{(1 - \vartheta)^2 \tilde{\sigma}^2 \varphi^2 (1 - \varphi^2)}{(\varphi^2 (1 - \eta) + \eta)^2} dt + 0dZ$$
Optimal policy: (1) MoPo only

- Optimal $\theta^*$
- HJB equation

$$
\max_{\theta} \frac{\log(1 - \theta)}{\rho \phi} - \frac{\rho \phi + 1}{\rho \phi} \log(\rho \phi + 1 - \theta) - \frac{(1 - \theta_t)^2 \bar{\sigma}^2}{2 \rho} \left( \frac{\lambda \kappa^2 \varphi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \kappa)^2}{(1 - \eta)^2} \right) \\
+ V' (\eta) (1 - \theta_t)^2 \frac{\eta (1 - \eta) \bar{\sigma}^2 \varphi^2 (1 - \varphi^2)}{(\varphi^2 (1 - \eta) + \eta)^2}
$$

- $\theta$ affects the drift of $\eta$, it is optimal to choose

$$
\theta^* \left( \bar{\sigma}^2 \left( \frac{\lambda \kappa^2 \varphi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \kappa)^2}{(1 - \eta)^2} \right) - 2 \rho V' (\eta) \frac{\eta (1 - \eta) \bar{\sigma}^2 \varphi^2 (1 - \varphi^2)}{(\varphi^2 (1 - \eta) + \eta)^2} \right)
$$

- Speed up $\eta$ when $V' > 0$, slow down when $V' < 0$. 
Example: using $\theta$ to push $\eta$

$\rho = .05, \phi = 2, \sigma = .3, \varphi = .5, \lambda = .2$
Optimal policy: (1) MoPo only

- Using MoPo $\psi$ to push $\eta$ (to recapitalize banks via risk premia)
- Using screwdriver as hammer
Roadmap

- Expected Utility/Value function with log-utility

- One sector model with stochastic idiosyncratic volatility

- Two sector model
  - With exogenously fixed net worth share $\eta$
  - With endogenous wealth share $\eta$
  - I theory (with two technologies)
I Theory of Money

- Aim: intermediary sector is not perfectly hedged
- Idiosyncratic risk that HH have to bear is time-varying
- Needed: Intermediaries’ aggregate risk ≠ aggregate risk of economy
  - One way to model: 2 technologies $a$ and $b$

<table>
<thead>
<tr>
<th>Technology</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital share (Leontieff)</td>
<td>$1 - \bar{\kappa}$</td>
<td>$\bar{\kappa}$</td>
</tr>
<tr>
<td>Risk</td>
<td>$\frac{dk_t}{k_t} = (\cdot)dt + \sigma^a dZ_t + \tilde{\sigma} d\tilde{Z}_t$</td>
<td>$\frac{dk_t}{k_t} = (\cdot)dt + \sigma^b dZ_t + \tilde{\sigma} d\tilde{Z}_t$</td>
</tr>
<tr>
<td>Intermediaries</td>
<td>No</td>
<td>Yes, reduce to $\varphi \tilde{\sigma}$</td>
</tr>
<tr>
<td>Excess risk</td>
<td>$-\bar{\kappa} \sigma - \frac{\sigma^\vartheta - \sigma^M}{1 - \vartheta}$</td>
<td>$(1 - \bar{\kappa}) \sigma - \frac{\sigma^\vartheta - \sigma^M}{1 - \vartheta}$</td>
</tr>
</tbody>
</table>
I Theory: Balance Sheets

- **Frictions:**
  - Household cannot diversify idio risk
  - Limited risky claims issuance
  - Only nominal deposits
Model with Intermediaries – new policy

- **Model Setup**

  \[
  \frac{d k_t}{k_t} = (\Phi(\omega_t) - \delta) dt + \sigma d Z_t + \tilde{\sigma} d \tilde{Z}_t
  \]

  - Aggregate
  - Idiosyncratic

- Intermediaries can hold equality share up to \( \tilde{\kappa} \)
- Can diversify some idiosyncratic risk, reduce it to \( \varphi \tilde{\sigma} \)
- Intermediaries’ wealth share \( \eta_t = N_t / (p_t + q_t)K_t \)
- Welfare weights \( \lambda \) on intermediaries, \( 1 - \lambda \) on HH

Two policy settings:
(1) money growth rate \( \mu_t^M \) only
(1) + (2) also choose allocation (macroprudential) and transfer wealth between group (why/how?)
Optimal policy: (1) MoPo + (2) MacroPru

- Same steps as above

- **Step 1**: Optimal $\kappa = \min\left(\frac{(1-\lambda)\eta^2}{\lambda\varphi^2(1-\eta)^2+(1-\lambda)\eta^2}, \bar{\kappa}\right)$ given $\eta$

- **Step 2**: Optimal $\vartheta = \vartheta^*\left(\frac{\bar{\sigma}^2}{\lambda\varphi^2(1-\eta)^2+(1-\lambda)\eta^2} \left(\frac{\lambda(1-\lambda)\varphi^2}{\lambda\varphi^2(1-\eta)^2+(1-\lambda)\eta^2}\right)\right)$ welfare weighted average risk exposure

- **Step 3**: Optimal $\eta$ (given $\vartheta$) as a function of Pareto weight $\lambda$
Optimal policy: (1) MoPo + (2) MacroPru

- **Step 3:** Optimal $\eta$ (given $\vartheta$) - let’s look at terms containing $\eta$

$$\max_{\eta} \frac{\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t)}{\text{concave, max at } \eta = \lambda, \text{ goes to } -\infty \text{ at 0 & 1}}$$

$$= \frac{-(1 - \vartheta_t)^2 \bar{\sigma}^2}{2\rho} - \frac{\lambda(1 - \lambda)}{\lambda \varphi^2 (1 - \eta)^2 + (1 - \lambda)\eta^2}$$

For $\varphi = 1$, the optimal policy as a function of $\lambda$ is
Optimal policy: (1) MoPo + (2) MacroPru

- For $\varphi = 1$, and $\bar{\kappa} = 0.6$ (intermediaries’ risk taking is constrained)
Optimal policy: (1) MoPo + (2) MacroPru

- For $\varphi = 0.8$, $\bar{\kappa} = 1$, and $\varphi = 0.8$, $\bar{\kappa} = 0.8$
- Intermediaries given a lot more risk when they can diversify it
Optimal policy: (1) MoPo + (2) MacroPru

- **Step 3**: Optimal $\eta$ (given $\theta$) - let’s look at terms containing $\eta$

- Same as above

- Given $\kappa$ and $\eta$,
  \[
  \max_{\eta} \frac{\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t)}{\text{concave, max at } \eta = \lambda, \text{ goes to } -\infty \text{ at } 0 \text{ & } 1} - \frac{(1 - \theta_t)^2 \sigma^2}{2 \rho} \quad \frac{\lambda(1 - \lambda) \phi^2}{\lambda \phi^2 (1 - \eta)^2 + (1 - \lambda) \eta^2} \text{ concave also, max at } \frac{\lambda \phi^2}{\lambda \phi^2 + 1 - \lambda} < \lambda
  \]

- Assuming FOC holds uniquely, it is optimal to set $\eta > \lambda$

- push more risk to intermediaries and they’d take under competitive outcome

- relative to previous infinite switching model
  - it is optimal to give intermediaries more wealth, because they are more efficient at absorbing risk
  - overall risk is reduced and the value of money is lower (more intermediation)
Optimizing over $\eta$

\[ \rho = .05, \phi = 2, \tilde{\sigma} = .3, \varphi = .5, \lambda = .2 \]
Optimal policy, (1) MoPo only

- Using $\vartheta$ to push $\eta$ - Same analytical steps as before

\[ \rho = 0.05, \phi = 2, \sigma = 0.3, \varphi = 0.5, \lambda = 0.2 \]
Take-aways of Optimal Policy

- Baseline (one-sector) model
  - Trade-off: insurance vs. investment (growth)

- Multi-sector model
  - Allocation of risk/assets

- Money is not super-neutral
  - since it affect portfolio choice, risk allocation
  - Price of risk (risk premia), $\eta$-drift

- (1) MoPo + (2) MacroPru
  - Static problem – 3 steps maximization
    - Always $\theta^*(\cdot)$-function

- (1) MoPo only
  - Using screwdriver as hammer to push $\eta$
Thank you!

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