LECTURE 07: MULTI-PERIOD MODEL
Overview

1. Generalization to a multi-period setting
   - Trees, modeling information and learning
     - Partitions, Algebra, Filtration
   - Security structure/trading strategy
     - Static vs. dynamic completeness

2. Pricing
   - Multi-period SDF and event prices
   - Martingale process – EMM
   - Forward measure

3. Ponzi scheme and Rational Bubbles
many one period models

how to model information?
Modeling information over time

• Partition

• Field/Algebra

• Filtration
Some probability theory

• **Measurability:** A random variable $y(s)$ is measure w.r.t. algebra $\mathcal{F}$ if
  - Pre-image of $y(s)$ are events (elements of $\mathcal{F}$)
    - for each $A \in \mathcal{F}, y(s) = y(s')$ for each $s \in A$ and $s' \in A$
    - $y(A) := y(s), s \in A$

• **Stochastic process:** A collection of random variables $y_t(s)$ for $t = 0, \ldots, T$

• Stochastic process is **adapted to filtration** $\mathcal{F} = \{\mathcal{F}_u\}_{u=t}^{T}$ if each $y_t(s)$ is measurable w.r.t. $\mathcal{F}_t$
  - Cannot see in the future
Multiple period Event Tree

- Last period events have prob., $\pi_{2,1}, \ldots, \pi_{2,4}$.
- To be consistent, the probability of an event is equal to the sum of the probabilities of its successor events.
  - E.g. $\pi_{1,1} = \pi_{2,1} + \pi_{2,2}$. 

$A_0$ $A_{1,1}$ $A_{2,1} = s_{2,1}$

$t=0$ $t=1$ $t=2$ $s_{2,2}$ $s_{2,3}$ $s_{2,4}$
2 Ways to reduce to One Period Model

\[ A_{1,1} = s_{2,1} \]

\[ A_0, A_{1,1}, A_{1,2} = s_{2,1}, s_{2,2}, s_{2,3}, s_{2,4} \]

\[ t=0, t=1, t=2 \]

Debreu
2 Ways to reduce to One Period Model

\[ A_{2,1} = s_{2,1} \]
Overview: from static to dynamic...

<table>
<thead>
<tr>
<th>Asset holdings</th>
<th>Dynamic strategy (adapted process)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset payoff $x$</td>
<td>Next period’s payoff $x_{t+1} + p_{t+1}$</td>
</tr>
<tr>
<td>Payoff of portfolio holding</td>
<td>Payoff of a strategy</td>
</tr>
<tr>
<td>span of assets</td>
<td>Marketed subspace of strategies</td>
</tr>
<tr>
<td>Market completeness</td>
<td>a) Static completeness (Debreu)</td>
</tr>
<tr>
<td></td>
<td>b) Dynamic completeness (Arrow)</td>
</tr>
<tr>
<td>No arbitrage w.r.t. holdings</td>
<td>No arbitrage w.r.t strategies</td>
</tr>
<tr>
<td>States $s = 1, \ldots, S$</td>
<td>Events $A_{t,i}$, states $s_{t,j}$</td>
</tr>
</tbody>
</table>
Overview: ...from static to dynamic

<table>
<thead>
<tr>
<th>State prices $q_s$</th>
<th>Event prices $q_{t,i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk free rate $R^f$</td>
<td>Risk free rate $R^f_t$ varies over time</td>
</tr>
<tr>
<td>DiscFactor: $\rho = 1/R^f$</td>
<td>Discount factor from $t$ to 0: $\rho_t$</td>
</tr>
<tr>
<td>Risk neutral prob.</td>
<td>Risk neutral prob.</td>
</tr>
<tr>
<td>$\pi^Q_s = q_s R^f$</td>
<td>$\pi^Q(\bar{A}<em>{t,i}) = \frac{q</em>{t,i}}{\rho_t}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pricing kernel</th>
<th>Pricing kernel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^j = E[m^* x^j]$</td>
<td>$M_t p^j_t = E_t[M_{t+1}(p^j_{t+1} + x^j_{t+1})]$</td>
</tr>
<tr>
<td>$1 = E[m^*] R^f$</td>
<td>$M_t = R^f_t E_t[M_{t+1}]$</td>
</tr>
</tbody>
</table>
Multiple period Event Tree

- Last period events have prob., $\pi_{2,1}, \ldots, \pi_{2,4}$.
- To be consistent, the probability of an event is equal to the sum of the probabilities of its successor events.
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Static Complete Markets

Debreu

All trading occurs at $t = 0$

6 independently traded assets needed
Dynamic Completion

Arrow (1953)

- Assets can be retraded
  - Conditional on event $A_{1,1}$ or $A_{1,2}$

- Completion with
  - Short-lived assets
    - Pays only next period
  - Long-lived assets
    - Payoff over many periods

- Trading strategy $h(A_{t,i})$
Completion with Short-lived Assets

• **Without uncertainty:**
  - No uncertainty and $T$ periods ($T$ can be infinite)
  - $T$ one period assets, from period 0 to period 1, from period 1 to 2, etc.
  - Let $p_t$ be the price of the short-term bond that begins in period $t$ and matures in period $t + 1$.

• Completeness requires
  Transfer of wealth between any two periods $t$ and $t'$, not just between consecutive periods.
  - Roll over short-term bonds
  - Cost of strategy: $p_t \cdot p_{t+1} \cdots p_{t'-1}$
Completion with Short-lived Assets

- With uncertainty

- \( p^{A_{t,i}} \) = price of an Arrow-Debreu asset that pays one unit in event \( A_{t,i} \). We want to transfer wealth from event \( A_0 \) to event-state \( s_{2,2} \).

- Go backwards:
  - in event \( A_{1,1} \), buy one event-state \( s_{2,2} \) asset for a price \( p^{A_{2,2}} \).
  - In event \( A_0 \), buy \( p^{A_{2,2}} \) shares of event \( A_{1,1} \) assets.

- Today’s cost \( p^{A_{2,2}} p^{A_{1,1}} \). The payoff is one unit in event \( s_{2,2} \) and nothing otherwise.
Completion with Long-lived Assets

• Without uncertainty:
  o $T$-period model ($T < \infty$).
  o Single asset
    • Discount bond maturing in $T$.
    • Tradable in each period for $p_t$.
  o $T$ prices (not simultaneously, but sequentially)
  o Payoff can be transferred from period $t$ to period $t' > t$ by purchasing the bond in period $t$ and selling it in period $t'$. 
Completion with Long-lived Assets

- With uncertainty

- At $t = 1$ it is as if one has 2 Arrow-Debreu securities (in each event $A_{1,i}$).
  - From perspective of $t = 0$ it is as if one has 4 Arrow-Debreu assets at $t = 1$. 
Completion with Long-lived Assets

- With uncertainty
  - 2 long-lived assets
  - 6 prices
    - Each asset is traded in 3 events
  - Payoff
    - In $t = 1$ is endogenous price $p_1$
One-period holding

- Call “trading strategy \([j, A_{t,i}]\)” the cash flow of asset \(j\) that is purchased in event \(A_{t,i}\) and is sold one period later.

6 trading strategies:
\([1, A_0], [1, A_{1,1}], [1, A_{1,2}], [2, A_0], [2, A_{1,1}], [2, A_{1,1}]\)

(Note that this is potentially sufficient to span the complete space.)
## Extended Payoff Matrix

6x6 payoff matrix.

<table>
<thead>
<tr>
<th>Asset</th>
<th>[1, A_0]</th>
<th>[2, A_0]</th>
<th>[1, A_{1,1}]</th>
<th>[2, A_{1,1}]</th>
<th>[1, A_{1,2}]</th>
<th>[2, A_{1,2}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>event $A_0$</td>
<td>$-p_0^1$</td>
<td>$-p_0^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>event $A_{1,1}$</td>
<td>$p_{1,1}$</td>
<td>$p_{1,1}$</td>
<td>$-p_{1,1}$</td>
<td>$-p_{1,1}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>event $A_{1,2}$</td>
<td>$p_{1,2}$</td>
<td>$p_{1,2}$</td>
<td>0</td>
<td>0</td>
<td>$-p_{1,2}$</td>
<td>$-p_{1,2}$</td>
</tr>
<tr>
<td>state $s_{2,1}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>state $s_{2,2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>state $s_{2,3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>state $s_{2,4}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

This matrix is full rank/regular (and hence the market complete) if the red framed submatrix is regular (of rank 2).
When Dynamically Complete?

- Is the red-framed submatrix of rank 2?
- Payoffs are endogenous future prices
- There are cases in which \((p_{1,1}^{1}, p_{1,1}^{2})\) and \((p_{1,2}^{1}, p_{1,2}^{2})\) are collinear in equilibrium.
  - Example: If per capita endowment is the same in event \(A_{1,1}\) and \(A_{1,2}\), in state \(s_{2,1}\) and \(s_{2,3}\), and in state \(s_{2,2}\) and \(s_{2,4}\), respectively, and if the probability of reaching state \(s_{1,1}\) after event \(A_{1,1}\) is the same as the probability of reaching state \(s_{2,3}\) after event \(A_{1,2}\)
    \[ \text{submatrix is singular (only of rank 1).} \]
  - then events \(A_{1,1}\) and \(A_{1,2}\) are effectively identical, and we may collapse them into a single event.
Accidental Incompleteness

- A *random* square matrix is of full rank (regular). So outside of special cases, the red-framed submatrix is of full ("almost surely").
- The 2x2 submatrix may still be singular *by accident*.
- In that case it can be made regular again by applying a small perturbation of the returns of the long-lived assets, by perturbing aggregate endowment, the probabilities, or the utility function.
- *Generically*, the market is dynamically complete.
Dynamic Completeness in General

- *branching number* = The maximum number of branches fanning out from any event.
- = number of assets necessary for dynamic completion.
- Generalization by Duffie and Huang (1985): continuous time $\rightarrow$ continuity of events $\rightarrow$ but a small number of assets is sufficient.
- The large power of the event space is matched by continuously trading few assets, thereby generating a continuity of trading strategies and of prices.
Example: Black-Scholes Formula

- Cox, Ross, Rubinstein binominal tree model of B-S
- Stock price goes up or down (follows binominal tree) interest rate is constant
- Market is dynamically complete with 2 assets
  - Stock
  - Risk-free asset (bond)
- Replicate payoff of a call option with (dynamic Δ-hedging)
- (later more)
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specify Preferences & Technology

- evolution of states
- risk preferences
- aggregation

observe/specify existing Asset Prices

State Prices $q$
(or stochastic discount factor/Martingale measure)

derive Price for (new) asset

Only works as long as market completeness doesn’t change

derive Asset Prices

Absolute asset pricing

Relative asset pricing

NAC/LOOP

LOOP

Only works as long as market completeness doesn’t change
No Arbitrage

- No dynamic trading strategy
  - No cost today about some positive payoff along the tree
  - Negative cost today and no negative payoff along the tree

- No dynamic trading strategy
  = no (static) arbitrage in each subperiod
Existence of Multi-period SDF $M$

- No Arbitrage $\iff$ there exists $m_{t+1} \gg 0$ for each one period subproblem
  - such that $p_t = E_t[m_{t+1}(p_{t+1} + x_{t+1})]$

- Define multi-period SDF (discounts back to $t = 0$)
  $M_{t+1} = m_1 \cdot m_2 \cdot ... \cdot m_{t+1}$
  (waking along the event tree)
  adapted process that is measurable w.r.t. filtration $\{F_{t+1}\}^T_t$
  
  $M_t p_t = E_t[M_{t+1}(p_{t+1} + x_{t+1})]$
The Fundamental Pricing Formula

• To price an arbitrary asset \( x \), portfolio of STRIPped cash flows, \( x_1^j, x_2^j, \ldots x_\infty^j \), where \( x_t^j \) denotes the cash-flows in event \( A_{t,s} \)

• The price of asset \( x^j \) is simply the sum of the prices of its STRIPed payoffs, so

\[
p_0^j = \sum_t E_0[M_t x_t^j]
\]
Pricing Kernel $M_t^*$

- Recall $m_{t+1}^* = proj(m_{t+1} \mid < X_{t+1} >)$
  - That is, there exists $h_t^*$ s.t. $m_{t+1}^* = X_{t+1} h_t^*$ and $p_t = E_t[X'_{t+1} m_{t+1}^*] = E_t[X'_{t+1} X_{t+1} h_t^*] = E_t[X'_{t+1} X_{t+1}] h_t^*$
  - $h_t^* = (E_t[X'_{t+1} X_{t+1}])^{-1} p_t$
  - Hence, $m_{t+1}^* = X_{t+1} (E_t[X'_{t+1} X_{t+1}])^{-1} p_t$

- Define $M_{t+1}^* = m_1^* \cdot m_2^* \cdot \ldots \cdot m_{t+1}^*$
- Part of asset span
Aside: Alternative Formula for $m^*$

- \( \frac{M_{t+1}^*}{M_t^*} = m_{t+1} = X_{t+1}(E_t[X'_{t+1}X_{t+1}])^{-1}p_t \)
  - where \( E_t[X'_{t+1}X_{t+1}] \) is a second moment \((J \times J)\) matrix

- Expressed in covariance-matrix, \( \Sigma_t \)
  \[
  m_{t+1}^* = E[m_{t+1}^*] + [p_t - E[m_{t+1}^*]E[X_{t+1}]]' \Sigma_t^{-1}(X_{t+1} - E[X_{t+1}])
  \]

- In excess returns, \( R^e \) and now return \( \Sigma_t \equiv \text{Cov}_t[R^e_{t+1}] \)
  \[
  m_{t+1}^* = \frac{1}{R_t^F} - \frac{1}{R_t^F}E[R^e_{t+1}]' \Sigma_t^{-1}(R^e_{t+1} - E[R^e_{t+1}])
  \]

- Continuous time analogous
  \[
  \frac{dM^*}{M} = -r^F dt - \left( \mu + \frac{D}{P} - r^F \right)' \Sigma^{-1} dz
  \]
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Martingales

• Let $X_1$ be a random variable and let $x_1$ be the realization of this random variable.
• Let $X_2$ be another random variable and assume that the distribution of $X_2$ depends on $x_1$.
• Let $X_3$ be a third random variable and assume that the distribution of $X_3$ depends on $x_1, x_2$.

• Such a sequence of random variables, $(X_1, X_2, X_3, ...)$ is called a stochastic process.

• A stochastic process is a martingale if $E[X_{t+1} | x_t, ...] = x_t$
History of the Word Martingale

• “martingale” originally refers to a sort of pants worn by “Martigaux” people living in Martigues located in Provence in the south of France. By analogy, it is used to refer to a strap in equestrian. This strap is tied at one end to the girth of the saddle and at the other end to the head of the horse. It has the shape of a fork and divides in two.

• In comparison to this division, martingale refers to a strategy which consists in playing twice the amount you lost at the previous round. Now, it refers to any strategy used to increase one's probability to win by respecting the rules.

• The notion of martingale appears in 1718 (The Doctrine of Chance by Abraham de Moivre) referring to a strategy that makes you sure to win in a fair game.

• See also www.math.harvard.edu/~ctm/sem/martingales.pdf
$M_t p_t$ is Martingale

- $M_t p_t = E_t[M_{t+1} p_{t+1}] + E_t[M_{t+1} x_{t+1}]$

- ... but consider
  1. Dividend payments $x_{t+1}$ fund that reinvests
Prices are Martingales...

• Samuelson (1965) has argued that prices have to be martingales in equilibrium.

\[ p_t = \frac{1}{1+r_{t,t+1}^f} E_t^Q [p_{t+1} + x_{t+1}] \]

• ... 3 “butts” consider
  
  1. Dividend payments fund that reinvests dividends
  2. Discounting discounted process
  3. Risk aversion risk-neutral measure \( \pi_t^Q \)
Equivalent Martingale Measure

- risk-neutral probabilities

\[ \pi_{A_t}^* = \frac{\pi_{A_t} M_{A_t}}{\rho_{A_t}} \]

where \( \rho_{A_t} \) is the discount-factor from event \( A_t \) to 0.

(state dependent)

- Discount everything back to \( t = 0 \).

- Why not “upcount”/compound to \( t = T \)?
Risk-Forward Pricing Measure

• \( P_s(t, T) \) be the time- \( s \) price of a bond purchased at time \( t \) with maturity \( T \), with \( s < t < T \).

• The fundamental pricing equation is
  \[
P_t(t, T) = E_t[m_{t+1}P_{t+1}(t + 1, T)]
  \]

• Dividing the pricing equation for a generic asset \( j \) by this relation and rearranging we get
  \[
  \frac{p^j_t}{P_t(t, T)} = E_t \left[ \frac{P_{t+1}(t + 1, T)}{E_t[m_{t+1}P_{t+1}(t + 1, T)]} \frac{m_{t+1}(x_{t+1}^j + p_{t+1}^j)}{P_{t+1}(t + 1, T)} \right] = E_T^F \left[ \frac{x_{t+1}^j + p_{t+1}^j}{P_{t+1}(t + 1, T)} \right]
  \]

• Where \( \pi_S^{FT} = \frac{\pi_s P_{t+1,s}(t+1,T)m_{t+1,s}}{E_t[m_{t+1}P_{t+1}(t+1,T)]} \).

• Useful for pricing of bond options and
• coincides with the risk-neutral measure
  o for \( t + 1 = T \).
  o ... (connection with expectations hypothesis?)
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Ponzi Schemes: Infinite Horizon Max.-problem

• Infinite horizon allows agents to borrow an arbitrarily large amount without effectively ever repaying, by rolling over debt forever.
  o Ponzi scheme - allows infinite consumption.

• Example
  o Consider an infinite horizon model, no uncertainty, and a complete set of short-lived bonds.
  o $z_t$ is the amount of bonds maturing in period $t$. 
Ponzi Schemes: Rolling over Debt Forever

• The following consumption path is possible:
  \[ c_t = y_t + 1 \]

• Note that agent consumes *more than his endowment, \( y_t \), in each period, forever* financed with *ever increasing debt*

• Ponzi schemes
  o can never be part of an equilibrium.
  o destroys the existence of a utility maximum because the choice set of an agent is unbounded above.
  o additional constraint is needed.
Ponzi Schemes: Transversality

- The constraint that is typically imposed on top of the budget constraint is the transversality condition,

\[
\lim_{t \to \infty} p_t^{bond} z_t \geq 0
\]

- This constraint implies that the value of debt cannot diverge to infinity.
  - More precisely, it requires that all debt must be redeemed eventually (i.e. in the limit).
Fundamental and Bubble Component

- Our formula
  \[ M_t p_t = E_t[M_{t+1}(p_{t+1} + x_{t+1})] \]
  or
  \[ M_t p_t = E_t M_{t+1} p_{t+1} + E_t M_{t+1} x_{t+1} \]

- Solve forward – (many solutions)

  \[ \Rightarrow p_0 = \sum_{t=1}^{\infty} E_0[M_t x_t] + \lim_{T \to \infty} E_0 M_T p_T \]
  \[ \text{fund. value} \quad \text{bubble comp.} \]
Money as a Bubble

\[ p_0 = \sum_{t=1}^{\infty} M_t + \lim_{T \to \infty} M_T p_T \]

- The fundamental value = price in the static-dynamic model.
- Repeated trading gives rise to the possibility of a bubble.
- Fiat money as a store of value can be understood as an asset with no dividends. The fundamental value of such an asset would be zero. But in a world of frictions fiat money can have positive value (a bubble) (e.g. in Samuelson 195X, Bewley, 1980).
- In asset pricing theory, we often rule out bubbles simply by imposing \( \lim_{T \to \infty} M_T p_T = 0 \).
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Time-varying $R_t^*$ (SDF)

- If one-period SDF $m_{t+1}^*$ is not time-varying (i.e. distribution of $m_{t+1}^*$ is i.i.d., then
  - Expectations hypothesis holds
  - Investment opportunity set does not vary
  - Corresponding $R_t^{*+1}$ of single factor state-price beta model can be easily estimate (because over time one more and more observations about $R_t^{*+1}$)

- If not, then $m_t^*$ (or corresponding $R_t^*$)
  - depends on state variable
  - multiple factor model
$R_t^*$ depends on State Variable

- $R_t^* = R^*(z_t)$, with state variable $z_t$
- Example:
  - $z_t = 1$ or $2$ with equal probability
  - Idea:
    - Take all periods with $z_t = 1$ and figure out $R^*(1)$
    - Take all periods with $z_t = 2$ and figure out $R^*(2)$
  - Can one do that?
    - No – hedge across state variables
- Potential state-variables: predict future return
Empirical: Single Factor (CAPM) fails

$$E[r] = \alpha + \beta E[R_{MRF}]$$

Average $|\alpha| = 0.296$
Three Factor Model works

Fama-French Model

\[ E[r] = \alpha + \beta \ E[RMRF] + \beta_s \ E[SMB] + \beta_h \ E[HML] \]

Average \( |\alpha| = 0.094 \)
International data: Out of Sample Test

E[r] = α + β E[RMRF]

Average |α| = 0.229
International Data: Out of Sample Test

Fama-French Model

\[ E[r] = \alpha + \beta \ E[R_{MRF}] + \beta_h \ E[HML] \]

Average \(|\alpha| = 0.176\)
Fama-MacBeth 2 Stage Method

- **Stage 1:** Use *time series* data to obtain estimates for each individual stock’s $\beta^j$
  
  $$R_t^j - R_f = \alpha + \beta^j (R_t^m - R_t^f) + \epsilon_t^j$$
  
  (e.g. use monthly data for last 5 years)
  
  Note: $\hat{\beta}^j$ is just an estimate [around true $\beta^j$]

- **Stage 2:** Use *cross sectional* data and estimated $\beta^j$s to estimate SML
  
  $$R_{next \ month}^j = \alpha + b \hat{\beta}^j + e^j$$
  
  $b=$ market risk premium
CAPM $\beta$–Testing Fama French (1992)

- Using newer data slope of SML $b$ is not significant (adding size and B/M)
- Dealing with econometrics problem:
  - $\hat{\beta}_j$'s are only noisy estimates, hence estimate of $b$ is biased
  - Solution:
    - Standard Answer: Find instrumental variable
    - Answer in Finance: Derive $\hat{\beta}$ estimates for portfolios
      - Group stocks in 10 x 10 groups sorted to size and estimated $\hat{\beta}_j$
      - Conduct Stage 1 of Fama-MacBeth for portfolios
      - Assign all stocks in same portfolio same $\beta$
      - Problem: Does not resolve insignificance
- CAPM predictions: $b$ is significant, all other variables insignificant
- Regressions: size and B/M are significant, $b$ becomes insignificant
  - Rejects CAPM
Book to Market and Size

Small "value" companies have higher returns

AVERAGE RETURNS ON U.S. STOCKS DEPENDING ON SIZE AND B/M
Percent per month

High B/M is similar to low P/E, it means "value". The opposite is "growth".

Source: Mertens, Data from Fama and French (1992)
Fama French Three Factor Model

- Form 2x3 portfolios
  - Size factor (SMB)
    - Return of small minus big
  - Book/Market factor (HML)
    - Return of high minus low
- For $R_t^j - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f)$
  - $\alpha$s are big and $\beta$s do not vary much
- For $R_t^p - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f) + \gamma^p \text{SMB}_t + \delta^p \text{HML}_t$
  - (for each portfolio $p$ using time series data)
  - $\alpha$s are zero, coefficients significant, high $R^2$. 
Fama French Three Factor Model

• Form 2x3 portfolios
  o Size factor (SMB)
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• For \( R_t^p - R_t^f = \alpha_p + \beta_p (R_t^m - R_t^f) + \gamma_p SMB_t^p + \delta_p HML_t^p \)
  (for each portfolio \( p \) using time series data)
  \( \alpha_p \)'s are zero, coefficients significant, high \( R^2 \).
Book to Market as a Predictor of Return

Annualized Rate of Return

<table>
<thead>
<tr>
<th>High Book/Market</th>
<th>Low Book/Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>Growth</td>
</tr>
</tbody>
</table>

The chart shows the annualized rate of return for different book-to-market ratios. The x-axis represents the book-to-market ratio, with categories from 1 to 10. The y-axis represents the annualized rate of return, ranging from 0% to 25%. The chart indicates that assets with a high book-to-market ratio tend to have a higher annualized rate of return compared to those with a low book-to-market ratio, supporting the concept that Value stocks may outperform Growth stocks.
Book to Market Equity of Portfolios Ranked by Beta
Adding Momentum Factor

- 5x5x5 portfolios
- Jegadeesh & Titman 1993 JF rank stocks according to performance to past 6 months
  - Momentum Factor
    Top Winner minus Bottom Losers Portfolios
MONTHLY DIFFERENCE BETWEEN WINNER AND LOSER PORTFOLIOS AT ANNOUNCEMENT DATES

MONTHS FOLLOWING 6 MONTH PERFORMANCE PERIOD
Cumulative Difference Between Winner and Loser Portfolios at Announcement Dates

Months Following 6 Month Performance Period

Cumulative Difference Between Winner and Loser Portfolios at Announcement Dates