

On the Optimal Inflation Rate

By MARKUS K. BRUNNERMEIER AND YULIY SANNIKOV*

I. Introduction

How do financial frictions affect the optimal inflation rate? Can financial frictions alone annul the long-run super-neutrality of money? Should the inflation rate be higher in emerging market economies with less developed financial markets than in advanced economies, as is currently observed?¹

To answer these questions we set up an incomplete markets model in which households choose portfolios consisting of risky (physical) capital and money. Physical capital holdings are encumbered with idiosyncratic risk. Financial frictions prevent the diversification of the idiosyncratic risk. Our analysis in this paper can be seen as a simplified discrete-time version of the “I Theory of Money” (Brunnermeier and Sannikov, 2015) – but without the “I”, the intermediaries and inside money, and with an exclusive focus on the long-run steady state.

Like in Samuelson’s (1958) OLG and in Bewley’s (1980) uninsurable endowment risk model, money serves as store of value and can have positive value despite the fact that it never pays any dividends. Diamond (1965) introduces physical capital in Samuelson’s OLG model and Aiyagari (1994) in Bewley’s incomplete markets setting (but capital drives out money). In our setting, money and physical capital coexist and agents choose portfolios. Like in Diamond and Aiyagari, the market outcome is dynamically inefficient. In contrast, however, to Diamond and Aiyagari, in which

the interest rate is too low and savings and physical capital investment are excessive, in our setting the real risk-free interest rate is too high and the investment rate is inefficiently low.

The market outcome is constrained Pareto inefficient due to pecuniary externalities. Each individual agent takes the real interest rate as given, while in the aggregate it is driven by the economic growth rate, which in turn depends on individual portfolio decisions. Higher inflation due to higher money growth lowers the real interest rate (on money) and tilts the portfolio choice towards physical capital investment. This boosts the overall physical investment and endogenous growth rate.

We are able to solve the model and conduct the welfare analysis in closed form. We show that there is an optimal level of long-run inflation in a setting in which seigniorage is handed out in a wealth-distribution-neutral way. A government that faces the same constraints as markets can orchestrate a Pareto welfare improvement simply by printing the right amount of money. Second, we show that in countries with higher idiosyncratic risk, e.g. because the domestic financial sector is less developed, the optimal inflation rate is higher.

Most existing literature explores various rationales other than financial frictions to determine the optimal inflation and money growth rate.² The Friedman Rule advocates a policy that minimizes the cost of holding currency. Hence, currency should appreciate at the real risk-free rate of return, which implies deflation (unless currency, like short-term government debt, earns interest). Most New Keynesian models with price stickiness à la Calvo (1983) recommend a zero inflation rate in steady state. Zero inflation minimizes the price

* Brunnermeier: Princeton University, Princeton, NJ, 08540, email: markus@princeton.edu. Sannikov: Princeton University, Princeton, NJ, 08540, USA, email: Sannikov@princeton.edu. We thank Itamar Drechsler for helpful comments and Yann Koby, Christian Wolf and especially Sebastian Merkel for excellent research assistance.

¹The differences in inflation targets between emerging market and advanced economies are documented e.g. in Table 1 in Fraga, Goldfajn and Minella (2004) and Table 4.1 of International Monetary Fund (2005).

²See e.g. Schmitt-Grohé and Uribe (2010) and Woodford (1990).

dispersion between firms that accidentally had a chance to readjust their prices and those who did not have this opportunity. A higher inflation target can be justified if nominal interest rates are subject to the zero lower bound (ZLB). Hitting the ZLB may lead to an excessively high real interest rate causing output losses. In our setting the real interest rate is also too high, even without the ZLB, and capital investment is depressed. Tobin (1972) argues in favor of a positive inflation rate in order to overcome frictions resulting from downward wage rigidities. Phelps (1973) criticizes the Friedman rule and conjectures that the inflation tax should be part of an overall optimal tax scheme. Yet, the Friedman rule has turned out to be remarkably robust. A higher inflation tax may be optimal, only if it counteracts some monopolistic distortions or extends to an otherwise untaxable large shadow economy. In our setting inflation acts as a Pigouvian tax on money holding to overcome pecuniary externalities.

II. The Economy

A. Model Setup

Our economy is populated by a continuum of households with identical preferences, but potentially different levels of wealth. Every household manages a private firm which operates a linear production technology with capital as the single input. Firms are subject to idiosyncratic, (real) cash-flow shocks of size proportional to the level of capital they manage. In addition, households can hold money, a bubble asset, which does not pay any dividends nor provides any other intrinsic service.

Time is divided into discrete intervals of length Δt , indexed by j .³ The timing within each period is as follows: Households enter period j with physical capital holdings k_j and nominal money holdings m_j . First, the physical capital produces

output $Ak_j\Delta t$, cash-flow shocks are realized and the household receives transfers from the government's seigniorage income. Second, households choose the investment rate ι_j . That is, they decide how many output units $\iota_j k_j \Delta t$ they use to produce new physical capital. Consequently, physical capital grows to $k_{j+1} = (1 + (\Phi(\iota_j) - \delta)\Delta t)k_j$, where δ is the depreciation rate, and the concave function $\Phi(\iota)$ reflects investment adjustment costs. Specifically, we assume the following functional form $\Phi(\iota) = \frac{1}{\kappa} \log(1 + \kappa\iota)$ with adjustment cost parameter κ . At the end of a period, households make their portfolio and consumption choices. That is, they trade physical capital, money and output goods to obtain the new capital holding k_{j+1} and nominal money holding m_{j+1} and consume the rest.

The consumption good serves as our numeraire. We restrict attention to equilibria with constant real price of physical capital q . The real value of aggregate physical capital, K_j , is qK_j . The real value of total money supply is pK_j . In other words, $p \geq 0$ is the real value of money normalized by the size of the economy, measured by the aggregate capital stock, K_j . The total wealth in the economy is $(p + q)K_j$. Given the quantity of money of M_j , the price level in the economy is $\mathcal{P}_j := \frac{M_j}{pK_j}$. The government chooses the money growth rate μ , i.e. $M_{j+1} = (1 + \mu\Delta t)M_j$, which impacts the real return on money R_j^m . The seigniorage revenues are redistributed in proportion to each household's wealth, w_j .⁴

We assume that all household maximize expected log utility with a time preference rate of $\rho\Delta t$. Given initial capital and nominal money holdings k_0 and m_0 , any household in the economy solves the problem

³As is common in discrete time models, we sum flows and ignore compounding effects within a Δt -period, while across periods we take compounding into account. For $\Delta t = 1$ our model resembles a standard discrete time model.

⁴In a world in which interest is paid on outside money (reserves) μ refers to the money growth beyond the interest payment while overall inflation is growing with the total money growth rate.

$$\begin{aligned} & \max E \left[\sum_{j=0}^{\infty} \left(\frac{1}{1 + \rho \Delta t} \right)^j \log c_j \cdot \Delta t \right] \\ \text{s.t. } & (c_j + \iota_j k_j) \Delta t + q k_{j+1} + \frac{m_{j+1}}{\mathcal{P}_j} = \\ & Ak_j \Delta t + z_j + q(1 + (\Phi(\iota_j) - \delta) \Delta t) k_j \\ & + R_j^m \frac{m_j}{\mathcal{P}_{j-1}} + \tau_j w_j \\ & w_j = q k_j + \frac{m_j}{\mathcal{P}_{j-1}}, \end{aligned}$$

where z_j are real cash flow shocks and $\tau_j w_j$ are transfers from the government.⁵ Households face idiosyncratic cash-flow shocks z_j which are proportional to the size of their business measured in the units of capital employed in production:

$$z_j = \sigma \varepsilon_j \sqrt{\Delta t} k_j$$

where ε_j is an iid shock, both over time and across households, with zero mean and unit variance,⁶ $\sigma > 0$ is a parameter. The shock is scaled by $\sqrt{\Delta t}$ instead of Δt to ensure that its impact does not become smaller with decreasing period length Δt . In sum, all flow variables contain a Δt -term, all shocks a $\sqrt{\Delta t}$ -term, while stocks do not depend on the length of the time period.

B. Optimality Conditions

LEMMA 1: *The optimal internal investment rate ι^* is an intra-period problem, solves*

$$(1) \quad q = \frac{1}{\Phi'(\iota^*)} = 1 + \kappa \iota^*$$

and is identical across all households and constant over time.

Note that if all households choose the same investment rate ι^* , the law of motion of the aggregate capital stock is

$$K_{j+1} = (1 + \underbrace{(\Phi(\iota^*) - \delta)}_{g:=}) \Delta t K_j.$$

The (gross) returns on physical capital and on money are then given by

⁵Money holdings m_j in the beginning of period j are divided by \mathcal{P}_{j-1} , not \mathcal{P}_j , because R_j^m is already the *real* rate of return.

⁶Note that the distribution needs to have bounded support to avoid the possibility of wealth becoming negative. Unbounded normally distributed shocks only work in the continuous-time limit.

$$\begin{aligned} R_j^k &= 1 + \left(\frac{A - \iota^*}{q} + g \right) \Delta t + \frac{\sigma}{q} \varepsilon_j \sqrt{\Delta t}, \\ R_j^m &= \frac{1 + g \Delta t}{1 + \mu \Delta t}. \end{aligned}$$

Let the “portfolio return” if the household holds a fraction x^k in physical capital and fraction $(1 - x^k)$ in money be

$$R_j^p(x_j^k) := x_j^k R_j^k + (1 - x_j^k) R_j^m + \tau_j$$

We include seigniorage transfers τ_j as they are also proportional to household wealth.

Denote by $w'_j = q k_j R_j^k + \frac{m_j}{\mathcal{P}_{j-1}} R_j^m + \tau_j w_j$ household wealth immediately before consumption (sometimes referred to as “cash at hand”), i.e. for period j , $w'_j - c_j \Delta t = w_{j+1}$. Given the optimal investment rate ι^* one can then rewrite the household’s problem as a Bellman equation in terms of the single state variable w' . Conjecturing value function of the form $V(w') = \alpha_0 + \alpha_1 \log w'$, where α_0, α_1 are undetermined coefficients, conveniently separates the optimization problem in the Bellman equation into the sum of two: (i) the optimal intertemporal consumption-savings decision and (ii) the static optimal portfolio choice between money and capital.⁷

LEMMA 2: *The optimal consumption level is $c^* = \frac{\rho}{1 + \rho \Delta t} w'$.*

Up to a positive scaling factor, the portfolio allocation problem is given by

$$\max_{x^k} E[\log R^p(x^k)].$$

We solve for an approximate solution which is exact in the continuous-time limit. This solution is obtained by evaluating $E[\log R^p(x^k)]$ using Taylor expansion around $R^p(x^k) = 1$ up to a term of $o(\Delta t)$. Since $R^p(x^k) = 1 + O(\sqrt{\Delta t})$, we need to include only terms up to degree 2. Notice also that $(R^p(x^k) - 1)^2 = x^k \frac{\sigma^2}{q^2} \varepsilon^2 \Delta t + o(\Delta t)$. Hence $E[\log R^p(x^k)]$ can be written as

$$\begin{aligned} & E\left[(R^p(x^k) - 1) - \frac{1}{2} (R^p(x^k) - 1)^2 \right] + o(\Delta t) \\ & \approx \left(g - \tau + x^k \left(\frac{A - \iota^*}{q} + \mu \right) - \frac{1}{2} (x^k)^2 \frac{\sigma^2}{q^2} \right) \Delta t \end{aligned}$$

where in the last equation we use the fact that ε has zero mean and unit variance. This approximated portfolio problem

⁷We refer to the working paper version of this article (Brunnermeier and Samikov, 2016) for further details.

is now quadratic in x^k and straightforward to solve.

LEMMA 3: *The optimal portfolio share of capital is $x^{k*} = \frac{E[R^k - R^m]}{\text{Var}[R^k - R^m]} = \frac{q(A - \iota^*)}{\sigma^2} + \frac{q^2 \mu}{\sigma^2}$.*

C. Market Clearing Conditions

The *goods market* clears if total output equals the sum of investment and consumption. Since individual cash-flow shocks cancel out in the aggregate and every household chooses the same ι^* ,⁸

$$AK\Delta t = \iota^* K\Delta t + C\Delta t.$$

By Lemma 2 individual consumption is a constant fraction of end of period wealth before consumption w' , which easily aggregates to $C = \frac{\rho}{1+\rho\Delta t}W'$. To obtain a closed form solution we approximate the market clearing condition by its continuous time limit in which $W' = W$ and hence $C = \rho W$. Noting that $W = (p+q)K$ and dividing by K ,

$$(2) \quad A = \iota^* + \rho(p+q).$$

The *capital market* clears if aggregate capital demand equals capital supply, $\frac{x^{k*}W}{q} = K$. Using $W = (p+q)K$ and the optimal portfolio share from Lemma 3 yields

$$(3) \quad \frac{1}{p+q} = \frac{A - \iota^*}{\sigma^2} + \frac{q\mu}{\sigma^2}.$$

The *money market* clears by Walras law.

D. Equilibrium

The optimal investment decision (1) and the market clearing equations (2) and (3) fully describe the (approximated) equilibrium in our model economy.

While these three equations can be solved in closed form, the solution is significantly simplified if it is expressed in terms of “transformed money growth” $\hat{\mu} := x^{k*}\mu$ instead of μ itself.⁹

⁸Capital letters are the aggregate counterparts of the lower-case letters in the individual decision problem.

⁹We show in Proposition 1 that $\hat{\mu}$ is strictly increasing in μ . Any qualitative statement in terms of $\hat{\mu}$ holds thus also in terms of μ .

PROPOSITION 1: *In the equilibrium with money and capital*¹⁰

$$(4) \quad p = \frac{\sigma(1 + \kappa\rho)}{\sqrt{\rho + \hat{\mu}}} - (1 + \kappa A),$$

$$(5) \quad q = 1 + \kappa A - \frac{\kappa\rho\sigma}{\sqrt{\rho + \hat{\mu}}},$$

$$(6) \quad \iota^* = A - \rho \frac{\sigma}{\sqrt{\rho + \hat{\mu}}},$$

where the transformed money growth rate $\hat{\mu} = x^k\mu$ is strictly increasing in μ , and this equilibrium exists if

$$\sigma / [(1 + \kappa A)\sqrt{\rho + \hat{\mu}}] \in \left(\frac{1}{\kappa\rho}, \frac{1}{1 + \kappa\rho} \right).$$

There always exists a moneyless equilibrium with $p = 0$, $q = \frac{1 + \kappa A}{1 + \kappa\rho}$, $\iota^* = \frac{A - \rho}{1 + \kappa\rho}$.

Proposition 1 reveals that in economies with high idiosyncratic risk, e.g. with poorly developed (internal) financial markets, money is more valuable. Indeed, for money to have positive value some minimum amount of idiosyncratic risk is necessary. Note that capital investment yield positive output Ak_j , while money does not. For sufficiently low σ or sufficiently high A capital investment is too attractive and we are in a moneyless economy, i.e. $p = 0$. Another interesting fact is that the capital depreciation rate δ does not affect prices nor the investment rate. However, it does affect the evolutions of individual and aggregate capital and through it also households' overall utility level.

Note that in the moneyless equilibrium $x^{k*} = 1$ and hence equilibrium is determined by (1) and (2) with $p = 0$.

III. Welfare

In this section we derive households' overall expected utility as a function of exogenous parameters and transformed money growth rate $\hat{\mu}$. The tractability of our framework allows us to characterize welfare as a function of portfolio return and asset prices in closed form without approximation. Going beyond that and in order to characterize welfare as a function of exogenous parameters and the government policy

¹⁰These equations also hold for the special case of no capital adjustment costs, $\kappa = 0$. In this case our model is a version of Angeletos (2007) but with money.

variable, the (transformed) money growth $\hat{\mu}$, we use our approximated equilibrium prices and returns. As our equilibrium results are exact in the continuous time limit, so are our welfare results.

An individual household's expected utility can be calculated by solving for the undetermined coefficients α_0, α_1 in the household's value function $V(w')$ and writing the remaining expressions in terms of model parameters and transformed money growth.

PROPOSITION 2: *The expected utility of a household with initial capital stock $k_0 = 1$ multiplied by ρ^2 is given by a constant plus $\frac{1}{\kappa} \log \left(1 + \kappa A - \frac{\kappa \rho \sigma}{\sqrt{\rho + \hat{\mu}}} \right) - \delta - \frac{\hat{\mu}}{2} + \rho \log \left(\frac{\sigma}{\sqrt{\rho + \hat{\mu}}} \right)$.*

If we assume that all households are equally wealthy in the beginning we can simply integrate over all individual households' utility levels. That is, we can simply take an individual household's utility level as our economy wide welfare measure. In this case any welfare improvement is also a Pareto improvement.

IV. Optimal Money Growth and Inflation Rate

Increasing the money growth rate increases inflation and lowers the real return on money. This encourages households to tilt their portfolio towards real assets at the expense of money holdings. The higher investment rate increases the real growth rate in the economy – a point originally made by Tobin (1965). Sidrauski (1967) showed that this is not welfare improving within a representative agent model, i.e. absent financial frictions. Our analysis revives Tobin's intuition by showing that the welfare-maximizing money growth rate is not zero in a setting with incomplete markets. In particular, if the (uninsurable) idiosyncratic risk is sufficiently large, the optimal money growth rate is positive.

PROPOSITION 3 (Optimal money growth): *There always exists a unique optimal growth rate of money μ^* , which is positive (negative), if*

$$\sigma > (<) \frac{2\sqrt{\rho}(A\kappa + 1)}{1 + 2\kappa\rho}.$$

*The competitive equilibrium outcome with $\mu = 0$ is constrained Pareto inefficient except for the knife-edge case in which this condition holds with equality.*¹¹

The steady state (long-run) money growth rate affects equilibrium allocation and economic growth. In short, money is not superneutral, despite the absence of any nominal rigidities and of the transaction role of money.

COROLLARY 1 (No Superneutrality): *Money is not superneutral in our flexible price (steady state) economy since a steady state increase in money supply growth affects the steady state economic growth rate $\Phi(\iota^*) - \delta$.*

As one increases money growth, output also increases. However, output maximizing money growth is excessive since it ignores the utility costs from bearing idiosyncratic risk. Indeed, it would make money so unattractive that it loses its value altogether, leading to a suboptimal welfare outcome.

Zero money growth is also constrained Pareto inefficient, despite perfect competition and flexible prices. A government that faces the same constraints as markets can orchestrate a Pareto welfare improvement simply by printing the right amount of money. Competitive equilibrium prices are distorted due to pecuniary externalities. Each individual household does not internalize that, by tilting its portfolio towards real assets, it boosts real growth in the economy and with it also the real interest rate on money holdings. The social planner internalizes this pecuniary externality and an inflation tax works like a Pigouvian tax in this environment.

Finally, the optimal money growth rate is higher for economies with higher idiosyncratic risks.

¹¹The optimal transformed money growth rate $\hat{\mu}^*$ is characterized by

$$(1 + \kappa A) \sqrt{\rho + \hat{\mu}^*} \left(2 + \frac{\hat{\mu}^*}{\rho} \right) = \sigma (1 + \kappa (2\rho + \hat{\mu}^*)).$$

PROPOSITION 4 (Comparative Statics): *The optimal money growth rate μ^* is strictly increasing in idiosyncratic risk σ .*¹²

Proposition 4 provides an explanation for why emerging market economies have higher inflation targets than advanced economies in which financial markets enable better risk sharing. Inflation

$$\pi = \mu - (\Phi(\iota^*(\mu)) - \delta)$$

increases in μ , but less than one-to-one since a higher μ also boosts the growth rate of the economy through a higher investment rate $\iota^*(\mu)$. Note that δ affects the optimal inflation target but not the optimal money growth rate.

REFERENCES

- Aiyagari, S. Rao.** 1994. “Uninsured Idiosyncratic Risk and Aggregate Saving.” *Quarterly Journal of Economics*, 109(3): 659–684.
- Angeletos, George-Marios.** 2007. “Uninsured Idiosyncratic Investment Risk and Aggregate Saving.” *Review of Economic Dynamics*, 10(1): 1–30.
- Bewley, Truman F.** 1980. “The Optimum Quantity of Money.” In *Models of Monetary Economies*, ed. John H. Kareken and Neil Wallace. Federal Reserve Bank of Minneapolis.
- Brunnermeier, Markus K., and Yuliy Sannikov.** 2015. “The I Theory of Money.” Working Paper.
- Brunnermeier, Markus K., and Yuliy Sannikov.** 2016. “On the Optimal Inflation Rate.” Working Paper.
- ¹²While the full proof of this result can be found in Brunnermeier and Sannikov (2016), we want to illustrate its main idea for the special case of no capital adjustment costs, $\kappa = 0$. In this case, the characterization of $\hat{\mu}^*$ from footnote 11 simplifies to
- $$\sqrt{\rho + \hat{\mu}^*}(2 + \frac{\hat{\mu}^*}{\rho}) = \sigma.$$
- Since the left-hand side is increasing in $\hat{\mu}^*$, a higher σ on the right must be accommodated by a higher $\hat{\mu}^*$ for the equation to hold.
- Calvo, Guillermo A.** 1983. “Staggered prices in a utility-maximizing framework.” *Journal of Monetary Economics*, 12(3): 383–398.
- Diamond, Peter A.** 1965. “National Debt in a Neoclassical Growth Model.” *American Economic Review*, 55(5): 1126–1150.
- Fraga, Arminio, Ilan Goldfajn, and André Minella.** 2004. “Inflation Targeting in Emerging Market Economies.” In *NBER Macroeconomics Annual 2003*. Vol. 18, 365–416. The MIT Press.
- International Monetary Fund.** 2005. “Does Inflation Targeting Work in Emerging Markets?” In *World Economic Outlook, September*. Chapter 4.
- Phelps, Edmund S.** 1973. “Inflation in the Theory of Public Finance.” *The Swedish Journal of Economics*, 75(1): 67–82.
- Samuelson, Paul A.** 1958. “An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money.” *Journal of Political Economy*, 66(6): 467–482.
- Schmitt-Grohé, Stephanie, and Martín Uribe.** 2010. “The Optimal Rate of Inflation.” In . Vol. 3 of *Handbook of Monetary Economics*, ed. Benjamin M. Friedman and Michael Woodford, Chapter 13, 653 – 722. Elsevier.
- Sidrauski, Miguel.** 1967. “Rational Choice and Patterns of Growth in a Monetary Economy.” *American Economic Review*, 57(2): 534–544.
- Tobin, James.** 1965. “Money and Economic Growth.” *Econometrica*, 33(4): 671–684.
- Tobin, James.** 1972. “Inflation and Unemployment.” *American Economic Review*, 62(1): 1–18.
- Woodford, Michael.** 1990. “The Optimum Quantity of Money.” In . Vol. 2 of *Handbook of Monetary Economics*, 1067 – 1152. Elsevier.