COMPARATIVE VALUATION DYNAMICS IN MODELS WITH FINANCING FRICTIONS

II. MODELS WITH FRICTIONS

Today’s Lecture:
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1. Continuous-time recursive utility (Duffie-Epstein-Zin)

2. Model with production and adjustment costs

3. “Shock Elasticities” as model diagnostics

4. Illustration of how RRA and IES affect shock-exposure and shock-price elasticities, with and without production
1. Add heterogeneity and frictions to the frictionless continuous-time model
   • Heterogeneity in productivity, preferences, frictions
   • Theoretical solution method

2. Numerical solution method
   • PDEs solved using finite-differences
   • Computational considerations
Part I

Model
## Notation Differences from Markus

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**Preferences**

Recursive utility with small time-step $\epsilon$,

$$U_t = \left[ (1 - \exp(-\delta\epsilon))(C_t)^{1-\rho} + \exp(-\delta\epsilon)R_t(U_{t+\epsilon})^{1-\rho} \right]^{\frac{1}{1-\rho}}$$

where

$$R_t(U_{t+\epsilon}) = \mathbb{E} \left[ U_{t+\epsilon}^{1-\gamma} \mid \mathcal{F}_t \right]^{\frac{1}{1-\gamma}}$$

- $\delta$ – rate of time preference
- $1/\rho$ – intertemporal elasticity of substitution (IES)
- $\gamma$ – relative risk aversion (RRA)

Experts and households can have different preferences:

$$\delta_e \text{ vs } \delta_h \quad \rho_e \text{ vs } \rho_h \quad \gamma_e \text{ vs } \gamma_h$$
Agent \( j \in [0, 1] \) within agent group \( g \in \{e, h\} \) (experts versus households) holds capital \( K_{g,t}^{(j)} \).

Production with **differential productivity**:

\[
\text{a}_g K_{g,t}^{(j)} \quad \text{a}_e \geq \text{a}_h
\]

Capital evolution:

\[
\frac{dK_{g,t}^{(j)}}{K_{g,t}^{(j)}} = \left[ \Phi(I_{g,t}^{(j)}/K_{g,t}^{(j)}) + Z_t - \alpha_k \right] dt + \sqrt{V_t\sigma_k} \cdot dB_t + \sqrt{\tilde{V}_t\tilde{\sigma}_k} dB_{t}^{(j)}
\]

**endogenous growth**  
**exogenous growth**  
**aggregate shocks**  
**idiosyncratic shocks**

**note**: \( \int_0^1 K_{g,t}^{(j)} dB_{t}^{(j)} \, dj = 0 \)
Exogenous states

\[
\frac{dK_{g,t}^{(j)}}{K_{g,t}^{(j)}} = \left[ \Phi \left( \frac{I_{g,t}^{(j)}}{K_{g,t}^{(j)}} \right) + Z_t - \alpha_k \right] dt + \sqrt{V_t} \sigma_k \cdot dB_t + \sqrt{\tilde{V}_t} \tilde{\sigma}_k \cdot d\tilde{B}_t^{(j)}
\]

where

(exogenous growth) \[ dZ_t = -\lambda_z Z_t dt + \sqrt{V_t} \sigma_z \cdot dB_t \]

(aggregate variance) \[ dV_t = -\lambda_V (V_t - 1) dt + \sqrt{V_t} \sigma_V \cdot dB_t \]

(idiosyncratic variance) \[ d\tilde{V}_t = -\lambda_{\tilde{V}} (\tilde{V}_t - 1) dt + \sqrt{\tilde{V}_t} \tilde{\sigma}_V \cdot dB_t \]
Financial Markets and Constraints

- Frictionless capital market, with single price $Q_t$

- Frictionless short-term risk-free debt market, with return $r_t$

  \[
  \text{SDF drifts: } \frac{1}{dt} \mathbb{E}_t[\frac{dS_{e,t}^{(j)}}{S_{e,t}^{(j)}}] = \frac{1}{dt} \mathbb{E}_t[\frac{dS_{h,t}^{(j)}}{S_{h,t}^{(j)}}] = -r_t
  \]

- Expert equity market (when is this a restriction?), delivering market risk-price $\pi_t$

  \text{Skin-in-the-game constraint: Experts can issue equity, subject to retaining a fraction } \chi_t^{(j)} \geq \chi \in [0, 1] \text{ of their capital risk}

- Arrow-Debreu markets on the aggregate shocks $dB_t$, delivering market risk prices $\pi_t$

  \text{Restriction: Only households can trade in this market, so}

  \[
  \frac{1}{dt} \text{Cov}_t[\frac{dS_{h,t}^{(j)}}{S_{h,t}^{(j)}}, dB_t] := \pi_{h,t}^{(j)} = \pi_t \text{ but } \frac{1}{dt} \text{Cov}_t[\frac{dS_{e,t}^{(j)}}{S_{e,t}^{(j)}}, dB_t] := \pi_{e,t}^{(j)} \neq \pi_t
  \]
**Balance Sheets and Flows of Funds**

"Experts"

- **Assets**
  - Physical Capital
  - Risk Free Short Term Debt
  - Net Worth
  - External Equity

- **Liabilities**
  - Interest
  - Dividends
  - Derivatives

"Households"

- **Assets**
  - Physical Capital
  - Risk Free Short Term Bonds
  - Equities

- **Liabilities**
  - Net Worth

"Experts" and "Households" exchange Interest and Dividends.
\[ \frac{dN_{g,t}^{(j)}}{N_{g,t}^{(j)}} = \left( \mu_{n,g,t}^{(j)} - C_{g,t}^{(j)} / N_{g,t}^{(j)} \right) dt + \sigma_{n,g,t}^{(j)} \cdot dB_t + \tilde{\sigma}_{n,g,t}^{(j)} d\tilde{B}_t, \]

where drifts and diffusions are

\[ \mu_{n,g,t}^{(j)} = r_t + \beta_{g,t}^{(j)} [\mu_{R,g,t} - r_t] + \theta_{g,t}^{(j)} \cdot \pi_{h,t} + \tilde{\theta}_{g,t}^{(j)} \cdot \mathbf{0} \]

expected excess ret-on-capital  
market compensation/payments

\[ \sigma_{n,g,t}^{(j)} = \beta_{g,t}^{(j)} \sigma_{R,t} + \theta_{g,t}^{(j)} \]

\[ \tilde{\sigma}_{n,g,t}^{(j)} = \beta_{g,t}^{(j)} \tilde{\sigma}_{R,t} + \tilde{\theta}_{g,t}^{(j)} \]

\[ \beta_{g,t}^{(j)} := \frac{Q_t K_{g,t}^{(j)}}{N_{g,t}^{(j)}} \geq \mathbf{0}, \text{ and trading constraints are given by} \]

\[ \theta_{h,t}^{(j)} \in \mathbb{R}^d \quad \text{and} \quad \theta_{e,t}^{(j)} \in \left\{ \theta \in \mathbb{R}^d : \theta = (\chi_t^{(j)} - 1) \beta_{e,t}^{(j)} \sigma_{R,t}; \chi_t^{(j)} \geq \chi \right\} \]

\[ \tilde{\theta}_{h,t}^{(j)} = \mathbf{0} \quad \text{and} \quad \tilde{\theta}_{e,t}^{(j)} \in \left\{ \theta \in \mathbb{R}^1 : \theta = (\chi_t^{(j)} - 1) \beta_{e,t}^{(j)} \tilde{\sigma}_{R,t}; \chi_t^{(j)} \geq \chi \right\} \]
Assumptions so far:

- Utility recursion is homogeneous of degree 1 in \((C_t, U_{t+\epsilon})\)
- Budget set is homogeneous of degree 1 in \(N_t\) (i.e., net worth evolutions are linear and trading constraints are homogeneous)

Common result:

- Utility separability:

\[
\log U_{g,t}^{(j)} = \log N_{g,t}^{(j)} + \xi_{g,t}
\]

continued utility

net worth

investment opportunities

- All appropriately-scaled choices \(I_{g,t}^{(j)}/K_{g,t}^{(j)}, K_{g,t}^{(j)}/N_{g,t}^{(j)}, C_{g,t}^{(j)}/N_{g,t}^{(j)}, \theta_{g,t}^{(j)}\) are independent of \(j\)
**Inhomogeneous Examples from Canonical Macro Models**

**Example 1.**

\[
\frac{dN_{g,t}^{(j)}}{N_{g,t}^{(j)}} = \left( \mu_{n,g,t}^{(j)} - \frac{C_{g,t}^{(j)}}{N_{g,t}^{(j)}} + \omega_t Y_{g,t}^{(j)} / N_{g,t}^{(j)} \right) dt + \sigma_{n,g,t}^{(j)} \cdot dB_t,
\]

where idiosyncratic labor productivity follows a (stationary) diffusion

\[
dY_{g,t}^{(j)} = \mu_{y,g} (Y_{g,t}^{(j)}) dt + \sigma_{y,g} (Y_{g,t}^{(j)}) \cdot dB_t + \tilde{\sigma}_{y,g} (Y_{g,t}^{(j)}) d\tilde{B}_t
\]

non-tradable piece

e.g., Aiyagari-Bewley-Huggett models, recently analyzed in continuous time by Achdou-Han-Lasry-Lions-Moll

**Example 2.**

Think about what happens if \( K_{g,t}^{(j)} \) is not tradable and production exhibits decreasing returns-to-scale.
Market clearing

• Goods market:
\[ a_e \int_0^1 K_{e,t}^{(j)}dj + a_h \int_0^1 K_{h,t}^{(j)}dj = \int_0^1 C_{e,t}^{(j)}dj + \int_0^1 C_{h,t}^{(j)}dj + \int_0^1 l_{e,t}^{(j)}dj + \int_0^1 l_{h,t}^{(j)}dj \]

• Capital market:
\[ K_t = \int_0^1 K_{e,t}^{(j)}dj + \int_0^1 K_{h,t}^{(j)}dj \]

• Aggregate risk markets:
\[ o = \int_0^1 \theta^{(j)}_{e,t}N_{e,t}^{(j)}dj + \int_0^1 \theta^{(j)}_{h,t}N_{h,t}^{(j)}dj \]

(recall: zero-net supply of equity and Arrow-Debreu securities)
Using the homogeneity properties, we can aggregate to representative expert and household.

- **Goods market:**
  \[ a_e K_{e,t} + a_h K_{h,t} = C_{e,t} + C_{h,t} + I_{e,t} + I_{h,t} \]

- **Capital market:**
  \[ K_t = K_{e,t} + K_{h,t} \]

- **Aggregate risk markets:**
  \[ O = \theta_{e,t} N_{e,t} + \theta_{h,t} N_{h,t} \]

(recall: zero-net supply of equity and Arrow-Debreu securities)
Last time, Lars showed that with recursive utility (limit as $\epsilon \to 0$):

$$
O = \sup \left\{ \frac{\delta (C_t/U_t)^{1-\rho} - 1}{1-\rho} + \frac{\mu_{u,t} - \gamma |\sigma_{u,t}|^2}{2} \right\}
$$

where

$$
dU_t = U_t[\mu_{u,t}dt + \sigma_{u,t} \cdot dB_t]
$$

**Digression:** sometimes people will instead write an equivalent “integral representation” for $\hat{U}_t := U_t^{1-\gamma}$, i.e.

$$
\hat{U}_t = \mathbb{E}_t \left[ \int_t^\infty f(C_s, \hat{U}_s) ds \right], \quad \text{where} \quad f(c, \hat{u}) := \delta \frac{1-\gamma}{1-\rho} [c^{1-\rho} \hat{u}^{\frac{\rho-\gamma}{1-\gamma}} - \hat{u}].
$$
Using \( \log U_{g,t} = \log N_{g,t} + \xi_{g,t} \), and defining dynamics

\[
d\xi_{g,t} = \mu_{\xi,g,t} dt + \sigma_{\xi,g,t} \cdot dB_t,
\]

we have

\[
0 = \sup \left\{ \delta_g \frac{\exp(-\xi_{g,t}) C_{g,t}/N_{g,t})^{1-\rho_g - 1}}{1 - \rho_g} - C_{g,t}/N_{g,t}
+ \mu_{n,g,t} - \frac{\gamma_g}{2} |\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2} \tilde{\sigma}_{n,g,t}^2 - (\gamma_g - 1)\sigma_{n,g,t} \cdot \sigma_{\xi,g,t}
+ \mu_{\xi,g,t} - \frac{\gamma_g - 1}{2} |\sigma_{\xi,g,t}|^2 \right\}
\]
HJB EQUATIONS OF REPRESENTATIVE AGENTS

1. Consumption-savings

\[ O = \sup \left\{ \delta_g \frac{[\exp(-\xi_{g,t})C_{g,t}/N_{g,t}]^{1-\rho_g} - 1}{1 - \rho_g} - C_{g,t}/N_{g,t} \right\} \]

\[ + \mu_{n,g,t} - \frac{\gamma_g}{2} |\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2} \tilde{\sigma}^2_{n,g,t} - (\gamma_g - 1)\sigma_{n,g,t} \cdot \sigma_{\xi,g,t} \]

\[ + \mu_{\xi,g,t} - \frac{\gamma_g - 1}{2} |\sigma_{\xi,g,t}|^2 \right\} \]

so

\[ c_{g,t}^* := C_{g,t}/N_{g,t} = \delta_g^{1/\rho_g} \exp[(1 - 1/\rho_g)\xi_{g,t}] \]

• \((\rho_g = 1)\) \(c_g^* = \delta_g\)
• \((\rho_g > 1)\) \(c_g^*\) increasing in \(\xi_g\)
• \((\rho_g < 1)\) \(c_g^*\) decreasing in \(\xi_g\)
2. Portfolio-choice

\[ O = \sup \left\{ \delta_g \left[ \exp(-\xi_g,t) \frac{C_{g,t}}{N_g,t} \right]^{1-\rho_g} - 1 \right\} \]

\[ = \sup \left\{ \delta_g \left[ \exp(-\xi_g,t) \frac{C_{g,t}}{N_g,t} \right]^{1-\rho_g} - 1 - \frac{\mu_{n,g,t} - \gamma_g |\sigma_{n,g,t}|^2}{2} - \frac{\gamma_g \tilde{\sigma}_n^2,_{g,t} - \sigma_{n,g,t} \cdot \sigma_{\xi,g,t}}{2} \right\} \]

SO

\[ (\beta_{g,t}, \theta_{g,t}) \in \arg \max \left\{ \mu_{n,g,t} - \frac{\gamma_g}{2} |\sigma_{n,g,t}|^2 - \frac{\gamma_g \tilde{\sigma}_n^2,_{g,t} - \sigma_{n,g,t} \cdot \sigma_{\xi,g,t}}{2} \right\} \]

mean-variance

hedging-demand
2a. Expert portfolio-choice

\[(\beta_e, \theta_e) \in \arg \max \left\{ \mu_{n,e} - \frac{\gamma_e}{2} |\sigma_{n,e}|^2 - \frac{\gamma_e}{2} \tilde{\sigma}_{n,e}^2 - (\gamma_e - 1)\sigma_{n,e} \cdot \sigma_{\xi,e} \right\} \]

Define expert bonus risk premium:

\[\Delta_e := \chi^{-1}[\mu_{R,e} - r - \sigma_R \cdot \pi_h].\]

Optimality conditions:

\[[\theta_e, \tilde{\theta}_e, \chi] : \quad 0 = \min(\chi - \underline{\chi}, \Delta_e) \]

and

\[[\beta_e] : \quad \Delta_e + \sigma_R \cdot \pi_h = \gamma_e[\sigma_R \cdot \sigma_{n,e} + \tilde{\sigma}_R \tilde{\sigma}_{n,e}] + (\gamma_e - 1)\sigma_R \cdot \sigma_{\xi,e} \]
2b. Household portfolio-choice

\[(\beta_h, \theta_h) \in \arg\max \left\{ \mu_{n,h} - \frac{\gamma h}{2} |\sigma_{n,h}|^2 - \frac{\gamma h}{2} \tilde{\sigma}_{n,h}^2 - (\gamma h - 1)\sigma_{n,h} \cdot \sigma_{\xi,h} \right\} \]

Define household bonus risk premium:

\[\Delta_h := \mu_{R,h} - r - \sigma_R \cdot \pi_h.\]

Optimality conditions:

\[[\beta_h] : \quad 0 = \min(\beta_h, \gamma_h \tilde{\sigma}_R^2 \beta_h - \Delta_h)\]

and

\[[\theta_h] : \quad \pi_h = \gamma_h \sigma_{n,h} + (\gamma h - 1)\sigma_{\xi,h}\]
3. Continuation-utility dynamics

\[ O = \sup \left\{ \delta_g \frac{\exp(-\xi_{g,t})C_{g,t}/N_{g,t})^{1-\rho_g} - 1}{1 - \rho_g} - C_{g,t}/N_{g,t} \right\} \]
\[ + \mu_{n,g,t} - \frac{\gamma_g}{2} |\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2} \tilde{\sigma}_{n,g,t}^2 - (\gamma_g - 1)\sigma_{n,g,t} \cdot \sigma_{\xi,g,t} \]
\[ + \mu_{\xi,g,t} - \frac{\gamma_g - 1}{2} |\sigma_{\xi,g,t}|^2 \]

so we can iterate backward (like value-function-iteration) as follows:

(a) Given \( \xi_{g,t} = \xi_g(X_t) \) as a function of “state variables” \( X_t \), use Itô’s formula to get \( \mu_{\xi,g,t} = \mu_x(X_t) \partial_x \xi_g(X_t) + \frac{1}{2} \text{tr}[\sigma_x(X_t)\sigma_x(X_t)' \partial_{xx'} \xi_g(X_t)] \) and \( \sigma_{\xi,g,t} = \sigma_x(X_t) \partial_x \xi_g(X_t) \);

(b) Plug into the HJB equation above to obtain a PDE for \( \xi_g \).
**Markov Equilibrium: State Variables** $X_t$

**Exogenous states:**

$$\hat{X}_t := (Z_t, V_t, \tilde{V}_t)'$$

**Endogenous state:**

$$W_t := \frac{N_{e,t}}{N_{e,t} + N_{h,t}}$$

**Stack:**

$$X_t := (W_t, \hat{X}_t)'$$

$$dX_t = \mu_x(X_t)dt + \sigma_x(X_t)dB_t$$

where

$$\mu_x(X) := \begin{pmatrix} \mu_w(X) \\ \mu_{\hat{x}}(\hat{X}) \end{pmatrix} \quad \text{dim } 4 \times 1$$

$$\sigma_x(X) := \begin{pmatrix} \sigma_w(X) \\ \sigma_{\hat{x}}(\hat{X})' \end{pmatrix} \quad \text{dim } 4 \times d$$

**Next step: derive** $\mu_w, \sigma_w$
• Idiosyncratic Poisson birth/death at rate $\lambda_d$

• Fraction of newborns (population shares): $\nu$ experts; $1 - \nu$ households

• No bequest motive

• Preferences only altered by the discount rate, i.e., $\delta \mapsto \delta + \lambda_d$ [see Appendix D of Gârleanu-Panageas (2015)]

• Given absence of labor income, assume no “insurance company” offering life insurance [unlike Blanchard (1985) and Gârleanu-Panageas (2015)]

• Dying agents’ wealth redistributed equally to newborns
Wealth share dynamics:

Aggregate net worth dynamics:

\[
\frac{dN_{h,t}}{N_{h,t}} = \left[ r_t - c^*_h + \sigma_{n,h,t} \cdot \pi_{h,t} + \beta_{h,t} \Delta_{h,t} - \lambda_d + \frac{(1 - \nu)\lambda_d}{1 - W_t} \right] dt + \sigma_{n,h,t} \cdot dB_t
\]

\[
\frac{dN_{e,t}}{N_{e,t}} = \left[ r_t - c^*_e + \sigma_{n,e,t} \cdot \pi_{h,t} + \chi_t \beta_{e,t} \Delta_{e,t} - \lambda_d + \frac{\nu \lambda_d}{W_t} \right] dt + \sigma_{n,e,t} \cdot dB_t,
\]

where \( \kappa := K_e/K \) and

\[
\sigma_{n,h} = \frac{1 - \chi \kappa}{1 - W} \sigma_R
\]

\[
\sigma_{n,e} = \frac{\chi \kappa}{W} \sigma_R.
\]

Use Itô’s formula on \( W_t := N_{e,t}/(N_{e,t} + N_{h,t}) \) to get

\[
\mu_w = w(1 - w) \left[ c^*_h - c^*_e + \chi \beta_e \Delta_e - \beta_h \Delta_h \right] + \sigma_w \cdot (\pi_h - \sigma_R) + \lambda_d (\nu - w)
\]

\[
\sigma_w = (\chi \kappa - w) \sigma_R.
\]
In Markov equilibrium, \( Q_t = q(X_t) \), which solves the goods market clearing condition (given knowledge of \( \kappa \)):

\[
q[(1 - w)c^*_h + wc^*_e] + i^*(q) = (1 - \kappa)a_h + \kappa a_e.
\]

\( q \) can decrease for 3 reasons:

1. \( \kappa \downarrow \) [e.g., Brunnermeier-Sannikov 2014]
2. \( c^*_h, c^*_e \uparrow \) [e.g., Bansal-Yaron 2004]
3. \( w \downarrow \) and \( c^*_h > c^*_e \) [e.g., Gârleanu-Panageas 2015]

Plugging in \( \sigma_q = \sigma'_x \partial_x \log q \) and using the previous result for \( \sigma_w \):

\[
\sigma_R = \sqrt{v}\sigma_k + \sigma_q = \frac{\sqrt{v}\sigma_k + \sigma'_x \partial_x \log q}{1 - (\chi\kappa - w)\partial_w \log q}.
\]

\( \kappa \) still endogenous...
Solving for key equilibrium shares \((\chi, \kappa)\)

Recall FOCs for \(\chi\) and \(\beta_h\):

\[
o = \min(\chi - \underline{\chi}, \Delta_e)
\]

\[
o = \min(\beta_h, \gamma_h \tilde{\sigma}_{R}^2 \beta_h - \Delta_h)
\]
Substitute $\beta_h = (1 - \kappa)/(1 - w)$:

\[ o = \min(\chi - \chi, \Delta_e) \]

\[ o = \min(1 - \kappa, \gamma_h \tilde{\sigma}_R^2 \frac{1 - \kappa}{1 - w} - \Delta_h) \]
Solving for key equilibrium shares \((\chi, \kappa)\)

\[
o = \min(\chi - \underline{\chi}, \Delta e)
\]

\[
o = \min(1 - \kappa, \gamma h \tilde{\sigma}^2_R \frac{1 - \kappa}{1 - \omega} - \Delta h)
\]

Recall

\[
\Delta h := \mu_{R,h} - r - \sigma_R \cdot \pi_h
\]

\[
= \mu_{R,e} - r - \sigma_R \cdot \pi_h - (\mu_{R,e} - \mu_{R,h})
\]

\[
= \underline{\chi} \Delta e - q^{-1}(a_e - a_h)
\]
SOLVING FOR KEY EQUILIBRIUM SHARES $(\chi, \kappa)$

\[
0 = \min(\chi - \underline{\chi}, \Delta_e)
\]

\[
0 = \min(1 - \kappa, \gamma_h \tilde{\sigma}_R^2 \frac{1 - \kappa}{1 - \nu} - \Delta_h)
\]

In addition, we have equations for $(\Delta_e, \pi_h)$ from the other portfolio FOCs:

\[
\Delta_h = \chi \Delta_e - q^{-1}(a_e - a_h)
\]

\[
\Delta_e = -\sigma_R \cdot \pi_h + \gamma_e [\sigma_R \cdot \sigma_{n,e} + \tilde{\sigma}_R \tilde{\sigma}_{n,e}] + (\gamma_e - 1)\sigma_R \cdot \sigma_{\xi,e}
\]

\[
\pi_h = \gamma_h \sigma_{n,h} + (\gamma_h - 1)\sigma_{\xi,h}
\]
SOLVING FOR KEY EQUILIBRIUM SHARES $(\chi, \kappa)$

\[
o = \min(\chi - \chi, \Delta_e)
\]

\[
o = \min(1 - \kappa, \gamma_h \tilde{\sigma}_R^2 \frac{1 - \kappa}{1 - w} - \Delta_h)
\]

Plug $\pi_h$ into $\Delta_e$ and plug $\Delta_e$ into $\Delta_h$:

\[
\Delta_h = -q^{-1}(a_e - a_h)
\]

\[
+ \chi \left\{ \sigma_R \cdot \left[ \gamma_e \sigma_{n,e} - \gamma_h \sigma_{n,h} \right] + \gamma_e \tilde{\sigma}_R \tilde{\sigma}_{n,e} + \sigma_R \cdot \left[ (\gamma_e - 1) \sigma_{\xi,e} - (\gamma_h - 1) \sigma_{\xi,h} \right] \right\}
\]

\[
\Delta_e = \sigma_R \cdot \left[ \gamma_e \sigma_{n,e} - \gamma_h \sigma_{n,h} \right] + \gamma_e \tilde{\sigma}_R \tilde{\sigma}_{n,e} + \sigma_R \cdot \left[ (\gamma_e - 1) \sigma_{\xi,e} - (\gamma_h - 1) \sigma_{\xi,h} \right]
\]
SOLVING FOR KEY EQUILIBRIUM SHARES \((\chi, \kappa)\)

\[
0 = \min(\chi - \chi, \Delta_e)
\]

\[
0 = \min(1 - \kappa, \gamma_h \tilde{\sigma}_R^2 \frac{1 - \kappa}{1 - W} - \Delta_h)
\]

If \(\chi > \chi\), then \(\Delta_h < \Delta_e = 0\), implying \(\kappa = 1\). Thus, we may substitute

- \(\chi = \chi\) into the expression for \(\Delta_h\)
- \(\kappa = 1\) into the expression for \(\Delta_e\)

These equations become decoupled.
SOLVING FOR KEY EQUILIBRIUM SHARES \((\chi, \kappa)\)

\[
O = \min(\chi - \chi, \Delta_e^{\kappa=1})
\]

\[
O = \min(1 - \kappa, \gamma_h \tilde{\sigma}_R^2 \frac{1 - \kappa}{1 - W} - \Delta_h^{\chi=\chi})
\]

\[
\Delta_h^{\chi=\chi} = -q^{-1}(a_e - a_h)
\]

\[
+ \chi \left\{ |\sigma_R^{\chi=\chi}|^2 \left[ \gamma_e \frac{\chi \kappa}{W} - \gamma_h \frac{1 - \chi \kappa}{1 - W} \right] + \gamma e \tilde{\sigma}_R^2 \frac{\chi \kappa}{W} 
+ \sigma_R^{\chi=\chi} \cdot \left[ (\gamma_e - 1)\sigma_{\xi,e}^{\chi=\chi} - (\gamma_h - 1)\sigma_{\xi,h}^{\chi=\chi} \right] \right\}
\]

\[
\Delta_e^{\kappa=1} = |\sigma_R^{\kappa=1}|^2 \left[ \gamma_e \frac{\chi}{W} - \gamma_h \frac{1 - \chi}{1 - W} \right] + \gamma e \tilde{\sigma}_R^2 \frac{\chi}{W}
\]

\[
+ \sigma_R^{\kappa=1} \cdot \left[ (\gamma_e - 1)\sigma_{\xi,e}^{\kappa=1} - (\gamma_h - 1)\sigma_{\xi,h}^{\kappa=1} \right]
\]
Solving for key equilibrium shares \((\chi, \kappa)\)

\[
O = \min(\chi - \chi, \Delta_e^{\kappa=1})
\]

\[
O = \min(1 - \kappa, \gamma h \bar{\sigma}_R^2 \frac{1 - \kappa}{1 - w} - \Delta_h^{\chi=\chi})
\]

Finally, recall:

\[
q[(1 - w)c_h^* + wc_e^*] + i^*(q) = (1 - \kappa)a_h + \kappa a_e
\]

\[
\sigma_R = \frac{\sqrt{v} \sigma_k + \sigma'_x \partial_x \log q}{1 - (\chi \kappa - w) \partial_w \log q}
\]

\[
\sigma_{\xi,g} = \sigma_x \cdot \partial_x \xi_g, \quad g \in \{e, h\}.
\]

- If \(\kappa = 1\), then \(q(x; \xi_e, \xi_h)\) is known, so the equation for \(\chi\) is algebraic.
- The equation for \(\kappa\) is differential.
Part II

Numerical Solution Method
**Value function iteration**

**Statement of the problem.** Scaled value functions \( \{ \xi_g \}_{g=e,h} \) solve PDEs like

\[
0 = F_g + A_g \xi_g + B_g \cdot \partial_x \xi_g + \frac{1}{2} \text{tr}[C_g C'_g \partial_{xx} \xi_g], \quad x = (w, z, v, \tilde{v}),
\]

where the coefficients are:

\[
\begin{align*}
F_g &= F_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
A_g &= A_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
B_g &= B_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
C_g &= C_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)
\end{align*}
\]

The dependence of \( A, B, C \) on \((\xi_e, \xi_h)\) arises due to the preferences and general equilibrium. Solve this PDE system with a back-and-forth iteration:

1. given coefficients, solve the linear PDE system and obtain \( \{ \xi_g \}_{g=e,h} \)
2. given PDE solution \( \{ \xi_g \}_{g=e,h} \), update coefficients
**Value function iteration**

**Step 1.** Augment the PDE with a “false transient,” which is an artificial time-derivative \( \partial_t \xi_g \) (Itô with time \( t \)), \( \mu_{\xi,g} = \partial_t \xi_g + \mu'_x \partial_x \xi_g + \frac{1}{2} \text{tr}[\sigma_x \sigma'_x \partial_{xx'} \xi_g] \):

\[
o = F_g + \partial_t \xi_g + A_g \xi_g + B_g \cdot \partial_x \xi_g + \frac{1}{2} \text{tr}[C_g C'_g \partial_{xx'} \xi_g],
\]

where

\[
F_g = F_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
A_g = A_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
B_g = B_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
C_g = C_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)
\]

We will use this to “work backward” from the distant future (\( T \)), just as in discrete-time value function iteration (may set terminal condition \( \xi_g(T) \) to anything in a stationary environment).

Stop iterating when reaching fixed point, i.e., \( \partial_t \xi_g \approx 0 \).
Value function iteration

Step 2. Given an iterant or guess \((\xi^{(t)}_e, \xi^{(t)}_h)\), we substitute the coefficients \((F^{(t)}_g, A^{(t)}_g, B^{(t)}_g, C^{(t)}_g)\).

\[
o = F^{(t)}_g + \partial_t \xi_g + A^{(t)}_g \xi_g + B^{(t)}_g \cdot \partial_x \xi_g + \frac{1}{2} \text{tr}[C^{(t)}_g C^{(t)}_g', \partial_{xx'} \xi_g],
\]

where

\[
F^{(t)}_g := F_g(x, \xi^{(t)}_e, \xi^{(t)}_h, \partial_x \xi^{(t)}_e, \partial_x \xi^{(t)}_h)
\]
\[
A^{(t)}_g := A_g(x, \xi^{(t)}_e, \xi^{(t)}_h, \partial_x \xi^{(t)}_e, \partial_x \xi^{(t)}_h)
\]
\[
B^{(t)}_g := B_g(x, \xi^{(t)}_e, \xi^{(t)}_h, \partial_x \xi^{(t)}_e, \partial_x \xi^{(t)}_h)
\]
\[
C^{(t)}_g := C_g(x, \xi^{(t)}_e, \xi^{(t)}_h, \partial_x \xi^{(t)}_e, \partial_x \xi^{(t)}_h)
\]
Step 3. Discretize the time derivatives and write all spatial derivatives in terms of $\xi_g^{(t−Δ)}$ (“implicit” finite differences, as opposed to “explicit”), i.e.,

$$\frac{\xi_g^{(t−Δ)} − \xi_g^{(t)}}{Δ} = F_g^{(t)} + A_g^{(t)} \xi_g^{(t−Δ)} + B_g^{(t)} \cdot \partial_x \xi_g^{(t−Δ)} + \frac{1}{2} \text{tr}[C_g^{(t)} C_g^{(t)′} \partial_{xx} \xi_g^{(t−Δ)}],$$

where

$$F_g^{(t)} = F_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})$$
$$A_g^{(t)} = A_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})$$
$$B_g^{(t)} = B_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})$$
$$C_g^{(t)} = C_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})$$

To hope for scheme “monotonicity” [i.e., Barles-Souganidis (1991)]:

- “Upwinding” for discretization of $\partial_x \xi_g^{(t−Δ)}$;
- Cross-partial derivatives computed using $\xi_g^{(t)}$ and added into $F_g^{(t)}$
**Value function iteration**

**Step 3-alt.** Discretize the time derivatives and write all spatial derivatives in terms of $\xi_g^{(t)}$ (“explicit” finite differences, as opposed to “implicit”), i.e.,

$$ \frac{\xi_g^{(t-\Delta)} - \xi_g^{(t)}}{\Delta} = F_g^{(t)} + A_g^{(t)} \xi_g^{(t)} + B_g^{(t)} \cdot \partial_x \xi_g^{(t)} + \frac{1}{2} \text{tr}[C_g^{(t)} C_g^{(t)'} \partial_{xx'} \xi_g^{(t)}], $$

where

$$ F_g^{(t)} = F_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) $$
$$ A_g^{(t)} = A_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) $$
$$ B_g^{(t)} = B_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) $$
$$ C_g^{(t)} = C_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) $$

With explicit schemes, often a smaller $\Delta$ is required (e.g., CFL condition). We use implicit schemes.
Step 4. By discretizing the spatial derivatives $\partial_x \xi_g^{(t-\Delta)}$ and $\partial_{xx'} \xi_g^{(t-\Delta)}$, the PDE becomes a system of linear equations in the unknown value function at the discretization points:

$$\xi_g^{(t-\Delta)} = \xi_g^{(t)} + \Delta F_g^{(t)} + \Delta L_g^{(t)} \xi_g^{(t-\Delta)},$$

where $L_g^{(t)} \xi_g^{(t-\Delta)}$ is the discretized version of

$$A_g^{(t)} \xi_g^{(t-\Delta)} + B_g^{(t)} \cdot \partial_x \xi_g^{(t-\Delta)} + \frac{1}{2} \text{tr}[C_g^{(t)} C_g^{(t)'} \partial_{xx'} \xi_g^{(t-\Delta)}].$$

Solve this system for $(\xi_e^{(t-\Delta)}, \xi_h^{(t-\Delta)})$. 

---

**Value function iteration**

Step 4. By discretizing the spatial derivatives $\partial_x \xi_g^{(t-\Delta)}$ and $\partial_{xx'} \xi_g^{(t-\Delta)}$, the PDE becomes a system of linear equations in the unknown value function at the discretization points:

$$\xi_g^{(t-\Delta)} = \xi_g^{(t)} + \Delta F_g^{(t)} + \Delta L_g^{(t)} \xi_g^{(t-\Delta)},$$

where $L_g^{(t)} \xi_g^{(t-\Delta)}$ is the discretized version of

$$A_g^{(t)} \xi_g^{(t-\Delta)} + B_g^{(t)} \cdot \partial_x \xi_g^{(t-\Delta)} + \frac{1}{2} \text{tr}[C_g^{(t)} C_g^{(t)'} \partial_{xx'} \xi_g^{(t-\Delta)}].$$

Solve this system for $(\xi_e^{(t-\Delta)}, \xi_h^{(t-\Delta)})$. 

---
**Implicit FD example.** Suppose spatial variable $x$ is one-dimensional:

$$0 = F + \partial_t \xi + A\xi + B\partial_x \xi + \frac{1}{2}C^2\partial_{xx} \xi.$$ 

Discretization with space step “$dx$”:

$$\frac{\xi^{(t-\Delta)}(x) - \xi^{(t)}(x)}{\Delta} = F^{(t)}(x) + A^{(t)}(x)\xi^{(t-\Delta)}(x)$$

$$+ B^{(t)}(x) \left[ 1_{\{B^{(t)}(x) > 0\}} \frac{\xi^{(t-\Delta)}(x + dx) - \xi^{(t-\Delta)}(x)}{dx} + 1_{\{B^{(t)}(x) < 0\}} \frac{\xi^{(t-\Delta)}(x) - \xi^{(t-\Delta)}(x - dx)}{dx} \right]$$

“upwinding” for first derivative

$$+ \frac{1}{2} \left( C^{(t)}(x) \right)^2 \frac{\xi^{(t-\Delta)}(x + dx) - 2\xi^{(t-\Delta)}(x) + \xi^{(t-\Delta)}(x - dx)}{dx^2}$$

second derivative approximation
**Implicit FD example continued.** Write the system as

\[ \frac{\xi(t - \Delta) - \xi(t)}{\Delta} = F(t) + L(t)\xi(t - \Delta) \Rightarrow [I - \Delta L(t)]\xi(t - \Delta) = \xi(t) + \Delta F(t). \]

The row for \( x \) has \( L(t)(x, :) \) constructed as...

\[
\begin{align*}
L(t)(x, x) &= A(t)(x) - \frac{|B(t)(x)|}{dx} - \frac{(C(t)(x))^2}{dx^2} < 0 \quad \text{if} \quad A(t)(x) < 0 \\
L(t)(x, x + dx) &= \frac{\max[0, B(t)(x)]}{dx} + \frac{1}{2} \frac{(C(t)(x))^2}{dx^2} > 0 \\
L(t)(x, x - dx) &= \frac{\min[0, B(t)(x)]}{dx} + \frac{1}{2} \frac{(C(t)(x))^2}{dx^2} > 0 \\
L(t)(x, y) &= 0 \quad \text{for} \quad y \notin \{x - dx, x, x + dx\}
\end{align*}
\]

**Sparsity:** \( I - \Delta L(t) \) is a highly-sparse (tri-diagonal) matrix.

**Monotonicity:** Opposing signs of diagonal \( I - \Delta L(t)(x, x) > 0 \) and off-diagonal elements \( I - \Delta L(t)(x, y) < 0 \) for \( y \neq x \).
Computational considerations. Solving \[ 1 - \Delta \mathbf{L}_g^{(t)} \] \( \xi_g^{(t-\Delta)} = \xi_g^{(t)} + \Delta F_g^{(t)} \).

**Direct approach:** essentially “invert” \( 1 - \Delta \mathbf{L}_g^{(t)} \) to the other side (technically, solve system using LU decomposition)

- **Upside:** generates exact solution for \( \xi_g^{(t-\Delta)} \)
- **Downside:** each iteration \( (t) \), the problem of inverting \( 1 - \Delta \mathbf{L}_g^{(t)} \) changes

**Iterative approach:** solve (using “conjugate gradient” algorithm)

\[
\xi_g^{(t-\Delta)} = \arg \min_{\mathbf{v}} \left\{ \frac{1}{2} \mathbf{v}' \left[ 1 - \Delta \mathbf{L}_g^{(t)} \right]' \left[ 1 - \Delta \mathbf{L}_g^{(t)} \right] \mathbf{v} - \mathbf{v}' \left[ 1 - \Delta \mathbf{L}_g^{(t)} \right]' \left[ \xi_g^{(t)} + \Delta F_g^{(t)} \right] \right\}.
\]

Any equation \( \mathbf{A} \mathbf{x} = \mathbf{b} \) can be solved for \( \mathbf{x} \) using \( \min_{\mathbf{x}} \frac{1}{2} \mathbf{x}' \mathbf{A}' \mathbf{A} \mathbf{x} - \mathbf{x}' \mathbf{b} \) as long as \( \mathbf{A}' \mathbf{A} \) is positive definite

- **Upside:** can provide “smart guess” based on \( \xi_g^{(t)} \)
- **Downside:** only an approximate solution for \( \xi_g^{(t-\Delta)} \)
LU versus CG. Solving \[ \left[ I - \Delta L_g^{(t)} \right] \xi_g^{(t-\Delta)} = \xi_g^{(t)} + \Delta F_g^{(t)} \].
So far, I said nothing about **boundary conditions**! These models usually have sensitive boundaries (example: \(\pi_e(o+) = +\infty\) is possible)

But the boundaries are **unattainable** in the sense of zero-probability events (example: \(\pi_e(o+) = +\infty\) implies \(\mu_w(o+) = +\infty\))

Therefore, we **need not provide any special boundary conditions**!

Heuristic idea: if \((F, A, B, C)\) are known functions in the PDE

\[
0 = F + \partial_t \xi + A \xi + B \partial_x \xi + \frac{1}{2} \text{tr}[CC' \partial_{xx'} \xi],
\]

then the solution can be written (Feynman-Kac theorem)

\[
\xi(x) = \mathbb{E}\left[ \int_0^\infty e^{\int_0^t A(s, X_s)ds} F(t, X_t) dt \mid X_0 = x \right]
\]

subject to \(dX_t = B(t, X_t)dt + C(t, X_t) \cdot dZ_t\). Brownian motion
Constraints and \((\chi, \kappa)\)

**Statement of the problem.** Capital distribution \(\kappa \in [0, 1]\) and expert equity-retention \(\chi \in [\chi, 1]\) determine occasionally-binding constraints

\[
\begin{align*}
o &= \min(1 - \kappa, G_h) \\
o &= \min(\chi - \chi, G_e)
\end{align*}
\]

where we showed theoretically that

\[
\begin{align*}
G_h &= G_h(x, \kappa, \partial_x \kappa; \xi_e, \xi_h) \\
G_e &= G_e(x, \chi; \xi_e, \xi_h).
\end{align*}
\]

These are sometimes called **variational inequalities.**
**Solution method.**

1. Given an iterant \( \xi_e^{(t)}, \xi_h^{(t)} \), construct

\[
G_h^{(t)}(x, \kappa, \partial_x \kappa) := G_h(x, \kappa, \partial_x \kappa; \xi_e^{(t)}, \xi_h^{(t)})
\]
\[
G_e^{(t)}(x, \chi) := G_e(x, \chi; \xi_e^{(t)}, \xi_h^{(t)})
\]

2. Since \( O = \min(\chi - \chi, G_e^{(t)}) \) is a univariate **algebraic** equation in \( \chi \), simply use nonlinear solver to obtain solution \( \chi^{(t)} \)

3. Since \( O = \min(1 - \kappa, G_h^{(t)}) \) is a univariate **differential** equation in \( \kappa \), use explicit finite-difference scheme with false transient, i.e.,

\[
\frac{\tilde{\kappa}(\tau + \tilde{\Delta}) - \tilde{\kappa}(\tau)}{\tilde{\Delta}} = \min \left( 1 - \tilde{\kappa}(\tau), G_h^{(t)}(x, \tilde{\kappa}(\tau), \partial_x \tilde{\kappa}(\tau)) \right), \quad \tilde{\kappa}^{(o)} = \kappa^{(t+\Delta)}.
\]

If LHS becomes small at \( \tau \), put \( \kappa^{(t)} := \tilde{\kappa}(\tau) \).

[See Oberman (2006) for nonlinear first-order PDE schemes of this type; small enough \( \tilde{\Delta} \) is required.]
Stationary Density

**Step 1.** After solving for all value functions and equilibrium objects, we have the equilibrium state dynamics $\mu_x$ and $\sigma_x$.

Recall the “transition operator” associated with the Kolmogorov Backward Equation (also called the “generator” of a diffusion):

$$Pf := \mu'_x \partial_x f + \frac{1}{2} \text{tr}[\sigma_x \sigma'_x \partial_{xx'} f]$$

Discretize this linear operator with a matrix $P$, e.g. in 1D example:

$$P(x, x) = -\frac{|\mu_x(x)|}{dx} - \frac{(\sigma_x(x))^2}{dx^2}$$

$$P(x, x + dx) = \frac{|\max[0, \mu_x(x)]|}{dx} + \frac{1}{2} \frac{(\sigma_x(x))^2}{dx^2}$$

$$P(x, x - dx) = \frac{|\min[0, \mu_x(x)]|}{dx} + \frac{1}{2} \frac{(\sigma_x(x))^2}{dx^2}$$

Notice that $P$ is a transition matrix for a continuous-time Markov chain (e.g., row-sums are 0).
**Step 2.** Can obtain stationary density approximation $\omega$ by solving (as in CTMC theory)

$$P'\omega = 0.$$

- **Alternative 1:** Recall that the Kolmogorov Forward Equation is the adjoint equation to the backward equation, and since adjoints in finite-dimensional space are matrix transposes, $P'\omega = 0$ is the discretized adjoint equation to $Pf = 0$.

- **Alternative 2:** $I + \Delta P$ is a discrete-time Markov matrix, for small time-step $\Delta$, so just solve $\omega'(I + \Delta P) = \omega'$.

Just an eigenvector-eigenvalue problem.
Fabrice will talk about:

- Evaluating this class of models

- Comparing different models

**Example:** what is similar and different about models in which the “experts” are more productive (i.e., $a_e > a_h$) versus more risk-tolerant (i.e., $\gamma_e < \gamma_h$)

- Show everything with a user-friendly web application to solve models, downloadable at https://larspeterhansen.org/mfr-suite/