Asset Pricing under Asymmetric Information
Modeling Information & Solution Concepts

Markus K. Brunnermeier
Princeton University

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References

Books:

Articles:
many others - see syllabus
Two Interpretations of Asymmetric Information

- different information
- different interpretation of the same information (different background information)
Modeling information I

- State space $\Omega$
  - state $\omega \in \Omega = \text{full description of reality}$
    - fundamentals
    - signals
  - state space is common knowledge and fully agreed among agents
Modeling information II

- Partition
  - \((\omega_1, \omega_2, \omega_3), (\omega_4, \omega_5), (\omega_6, \omega_7, \omega_8)\)
  - \(P_1^i, P_2^i, P_3^i\) (partition cells)
  - later more about ‘knowledge operators’ etc.

- Field (Sigma-Algebra) \(\mathcal{F}^i\)

- Probability measure/distribution \(P\)
Modeling information III

- Prior distribution
  - Common prior assumption (CPA) (Harsanyi doctrine)
    - any difference in beliefs is due to differences in info
    - has strong implications
  - Rational Expectations
    - prior\(^i = \) objective distribution \( \forall i \)
    - implies CPA
  - Non-common priors
    - Problem: almost everything goes
    - Way out: Optimal Expectations
      (structure model of endogenous priors)

- Updating/Signal Extraction
Modeling information III

• Updating (general)
  • Bayes’ Rule

  \[ P^i (E_n|D) = \frac{P^i (D|E_n) P^i (E_n)}{P^i (D)} , \]

  • if events \( E_1, E_2, ..., E_N \) are a partition

  \[ P^i (E_n|D) = \frac{P^i (D|E_n) P^i (E_n)}{\sum_{n=1}^{N} P^i (D|E_n) P^i (E_n)} , \]
Updating - Signal Extraction - general case

- **Updating - Signal Extraction**
  - \( \omega = \{ \nu, S \} \)
  - desired property: signal realization \( S^H \) is always more favorable than \( S^L \)
  - formally: \( G(\nu|S^H) \) FOSD \( G(\nu|S^L) \)
  - Milgrom (1981) shows that this is equivalent to \( f_S(S|\nu) \) satisfies monotone likelihood ratio property (MLRP)
  - \( f_S(S|\nu)/f_S(S|\bar{\nu}) \) is increasing (decreasing) in \( S \) if \( \nu > (<)\bar{\nu} \)
    \[
    \frac{f_S(S|\nu)}{f_S(S'|\nu')} > \frac{f_S(S'|\nu)}{f_S(S'|\nu')} \quad \forall \nu' > \nu \text{ and } S' > S.
    \]
  - another property: hazard rate \( \frac{f_S(S|\nu)}{1-F(S|\nu)} \) is declining in \( \nu \)
Updating - Signal Extraction - Normal distributions

- updating normal variable $X$ after receiving signal $S = s$

$$E[X|S = s] = E[X] + \frac{\text{Cov}[X,S]}{\text{Var}[S]} (s - E[S])$$

$$\text{Var}[X|S = s] = \text{Var}[X] - \frac{\text{Cov}[X,S]^2}{\text{Var}[S]}$$

- $n$ multidimensional random variable $(\vec{X}, \vec{S}) \sim \mathcal{N} (\mu, \Sigma)$

$$\mu = \begin{bmatrix} \mu_X \\ \mu_S \end{bmatrix}_{n \times 1} ; \Sigma = \begin{bmatrix} \Sigma_{X,X} & \Sigma_{X,S} \\ \Sigma_{S,X} & \Sigma_{S,S} \end{bmatrix}_{n \times n}$$

- Projection Theorem $(X|S = s)$

$$\sim \mathcal{N} \left( \mu_X + \Sigma_{X,S} \Sigma_{S,S}^{-1} (s - \mu_S), \Sigma_{X,X} - \Sigma_{X,S} \Sigma_{S,S}^{-1} \Sigma_{S,X} \right)$$
Special Signal Structures

- $N$-Signals of form: $S_n = X + \varepsilon_n$
  
  (Let $X$ be a scalar and $\tau_y = \frac{1}{\text{Var}[y]}$),

\[
E[X|s_1, \ldots, s_N] = \mu_X + \frac{1}{\tau_X + \sum_{n=1}^{N} \tau_{\varepsilon_n}} \sum_{n=1}^{N} \tau_{\varepsilon_n} (s_n - \mu_X)
\]

\[
\text{Var}[X|s_1, \ldots, s_N] = \frac{1}{\tau_X + \sum_{n=1}^{N} \tau_{\varepsilon_n}} = \frac{1}{\tau_X|s_1, \ldots, s_N}
\]

- If, in addition, all $\varepsilon_n$ i.i.d. then

\[
E[X|s_1, \ldots, s_N] = \mu_X + \frac{1}{\tau_X + N\tau_{\varepsilon_n}} N\tau_{\varepsilon_n} \left( \sum_{n=1}^{N} \frac{1}{N} s_n - \mu_X \right),
\]

\[
\text{Var}[X|s_1, \ldots, s_N] = \frac{1}{\tau_X + N\tau_{\varepsilon_n}} \left( \sum_{n=1}^{N} \frac{1}{N} s_n - \mu_X \right)
\]

where $\bar{s} := \sum_{n=1}^{N} \left( \frac{1}{N} \right) s_n$ is a sufficient statistic
Special Signal Structures

• $\mathcal{N}$-Signals of form: $X = S + \varepsilon$

\[
E[X|S = s] = s \\
Var[X|S = s] = Var[\varepsilon]
\]

• Binary Signal: Updating with binary state space/signal
  • $q = \text{precision} = \text{prob}(X = H|S = S^H)$

• “Truncating signals”: $v \in \overline{S}, \underline{S}$
  • $v$ is Laplace (double exponentially) distributed or uniform
  • posterior is a truncated exponential or uniform
  (see e.g. Abreu & Brunnermeier 2002, 2003)
Solution/Equilibrium Concepts

- **Rational Expectations Equilibrium**
  - Competitive environment
  - agents take prices as given (price takers)
  - Rational Expectations (RE) $\Rightarrow$ CPA
  - *General Equilibrium Theory*

- **Bayesian Nash Equilibrium**
  - Strategic environment
  - agents take strategies of others as given
  - CPA (RE) is typically assumed
  - *Game Theory*
  - distinction between normal and extensive form games
    simultaneous move versus sequential move
### The 5 Step Approach

<table>
<thead>
<tr>
<th></th>
<th><strong>REE</strong></th>
<th><strong>BNE</strong> (sim. moves)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1</strong></td>
<td>Specify joint priors &lt;br&gt;Conjecture price mappings &lt;br&gt;$P : {S^1, \ldots, S^l, u} \rightarrow \mathbb{R}_+^J$</td>
<td>Specify joint priors &lt;br&gt;Conjecture strategy profiles</td>
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<tr>
<td><strong>Step 2</strong></td>
<td>Derive posteriors</td>
<td>Derive posteriors</td>
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<td><strong>Step 3</strong></td>
<td>Derive individual demand</td>
<td>Derive best response</td>
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<td><strong>Step 4</strong></td>
<td>Impose market clearing</td>
<td>Impose Rationality</td>
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<tr>
<td><strong>Step 5</strong></td>
<td>Impose Rationality &lt;br&gt;Equate undet. coeff.</td>
<td>No-one deviates</td>
</tr>
</tbody>
</table>
A little more abstract

- **REE**
  Fixed Point of Mapping: $\mathcal{M}_P(P(\cdot)) \mapsto P(\cdot)$

- **BNE** (simultaneous moves)
  Fixed Point of Mapping:
  strategy profiles $\mapsto$ strategy profiles

- What’s different for sequential move games?
  - late movers react to deviation
  - equilibrium might rely on ‘strange’ out of equilibrium moves
  - refinement: subgame perfection

- Extensive form move games with asymmetric information
  - Sequential equilibrium (agents act sequentially rational)
  - Perfect BNE (for certain games)
    - nature makes a move in the beginning (chooses type)
    - action of agents are observable
A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models: (uninformed) market maker submits a supply schedule first
    - static
      - uniform price setting
      - limit order book analysis
  - dynamic sequential trade models with multiple trading rounds
- signalling models: informed traders move first, market maker second