Asset Pricing under Asymmetric Information
Rational Expectations Equilibrium

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A Classification of Market Microstructure Models

- Simultaneous submission of demand schedules
  - Competitive rational expectation models
  - Strategic share auctions
- Sequential move models
  - Screening models:
    uninformed market maker submits a supply schedule first
    - Static
      - uniform price setting
      - limit order book analysis
    - Dynamic sequential trade models with multiple trading rounds
  - Signalling models:
    informed traders move first, market maker second
Overview

- Competitive REE (Examples)
  - Preliminaries
    - LRT (HARA) utility functions in general
    - CARA Gaussian Setup
      - Certainty equivalence
      - Recall Projection Theorem/Updating
  - REE (Grossman 1976)
  - Noisy REE (Hellwig 1980)
- Allocative versus Informational Efficiency
- Endogenous Information Acquisition
Utility functions and Risk aversion

• Utility functions: $U(W)$
• Risk tolerance, $1/\rho = \text{reciprocal of the Arrow-Pratt measure of absolute risk aversion}$

$$\rho(W) := -\frac{\partial^2 U / \partial W^2}{\partial U / \partial W}$$

• Risk tolerance is linear in $W$ if

$$\frac{1}{\rho} = \alpha + \beta W$$

• Also called hyperbolic absolute risk aversion (HARA) utility functions
## LRT(HARA)-Utility Functions

<table>
<thead>
<tr>
<th>Class</th>
<th>Parameters</th>
<th>$U(W) =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>expo/CARA</td>
<td>$\beta = 0, \alpha = 1/\rho$</td>
<td>$-\exp{-\rho W}$</td>
</tr>
<tr>
<td>gen. power</td>
<td>$\beta \neq 1$</td>
<td>$\frac{1}{\beta - 1}(\alpha + \beta W)^{(\beta - 1)/\beta}$</td>
</tr>
<tr>
<td>a) quadratic</td>
<td>$\beta = -1, \alpha &gt; W$</td>
<td>$-(\alpha - W)^2$</td>
</tr>
<tr>
<td>b) log</td>
<td>$\beta = +1$</td>
<td>$\ln(\alpha + W)$</td>
</tr>
<tr>
<td>c) power/CRRA</td>
<td>$\alpha = 0, \beta \neq 1, -1$</td>
<td>$\frac{1}{\beta - 1}(\beta W)^{(\beta - 1)/\beta}$</td>
</tr>
</tbody>
</table>
Certainty Equivalent in CARA-Gaussian Setup

\[ U(W) = -\exp(-\rho W), \text{ hence } \rho = -\frac{\partial^2 U(W)/\partial (W)^2}{\partial U(W)/\partial W} \]

\[ E[U(W) | \cdot] = \int_{-\infty}^{+\infty} -\exp(-\rho W)f(W|\cdot)dW \]

where: \( f(W|\cdot) = \frac{1}{\sqrt{2\pi\sigma_W^2}} \exp\left[-\frac{(W - \mu_W)^2}{2\sigma_W^2}\right] \)

Substituting it in:

\[ E[U(W) | \cdot] = \frac{1}{\sqrt{2\pi\sigma_W^2}} \int_{-\infty}^{+\infty} -\exp\left(-\frac{\rho z}{2\sigma_W^2}\right)dW \]

where \( z = (W - \mu_W)^2 + 2\rho\sigma_W^2 W \)
Completing square:
\[ z = \left( W - \mu_W + \rho \sigma_W^2 \right)^2 + 2\rho \left( \mu_W - \frac{1}{2} \rho \sigma_W^2 \right) \sigma_W^2 \]

Hence,
\[ E[U(W) \mid \cdot] = -\exp \left[ -\rho \left( \mu_W - \frac{1}{2} \rho \sigma_W^2 \right) \right] \times \]
\[ \times \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \sigma_W^2} \exp \left( -\frac{(W - \mu_W + \rho \sigma_W^2)^2}{2\sigma_W^2} \right) dW \]

\[ = 1 \]

Trade-off is represented by:
\[ V(\mu_W, \sigma_W^2) = \mu - \frac{1}{2} \rho \sigma_W^2 \]
Certainty Equivalent in CARA-Gaussian Setup

More generally, multinomial random variables $\mathbf{w} \sim \mathcal{N}(0, \Sigma)$ with a positive definite (co)variance matrix $\Sigma$. More specifically:

$$E[\exp(\mathbf{w}^T A \mathbf{w} + \mathbf{b}^T \mathbf{w} + d)] =$$

$$= |\mathbf{I} - 2\Sigma A|^{-1/2} \exp \left[ \frac{1}{2} \mathbf{b}^T (\mathbf{I} - 2\Sigma A)^{-1} \Sigma \mathbf{b} + d \right],$$

where:
- $A$ is a symmetric $m \times m$ matrix,
- $\mathbf{b}$ is an $m$-vector, and
- $d$ is a scalar.

Note that the left-hand side is only well-defined if $(\mathbf{I} - 2\Sigma A)$ is positive definite.
Demand for a Risky Asset

<table>
<thead>
<tr>
<th>asset</th>
<th>payoff</th>
<th>endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>bond (numeraire)</td>
<td>( R )</td>
<td>( e_0^i )</td>
</tr>
<tr>
<td>stock</td>
<td>( v \sim \mathcal{N}(E[v</td>
<td>\cdot], \text{Var}[v</td>
</tr>
</tbody>
</table>

- Two assets
- \( P x^i + b^i = P z^i + e_0^i \)
- Final wealth:
  \[
  W^i = b^i R + x^i v = (e_0^i + P(z^i - x^i))R + x^i v
  \]

- Mean: \( (e_0^i + P(z^i - x^i))R + xE[v|\cdot] \)
- Variance: \( (x^i)^2 \text{Var}[v|\cdot] \)
Demand for a Risky Asset

\[ V(\mu_W, \sigma^2_W) = \mu_W - \frac{1}{2} \rho^i \sigma^2_W \]

\[ = (e^i_0 + Pz^i)R + x^i(E[v|\cdot] - PR) - \frac{1}{2} \rho^i \text{Var}[v|\cdot](x^i)^2 \]

First order condition: \( E[v|\cdot] - PR - \rho^i \text{Var}[v|\cdot]x^i = 0 \)

\[ x^i(P) = \frac{E[v|\cdot] - PR}{\rho^i \text{Var}[v|\cdot]} \]

Remarks:

- independent of initial endowment (CARA)
- linearly increasing in investor’s expected excess return
- decreasing in investors’ variance of the payoff \( \text{Var}[v|\cdot] \)
- decreasing in investors’ risk aversion \( \rho^i \)
- for \( \rho^i \to 0 \) investors are risk-neutral, and \( x^i \to +\infty \) or \( -\infty \)
Symmetric Info – Benchmark

Model setup:

- \( i \in \{1, ..., I\} \) (types of) traders
- CARA utility function with risk aversion coefficient \( \rho^i \)
  (Let \( \eta^i = \frac{1}{\rho^i} \) be trader \( i \)'s risk tolerance)
- all traders have the same information \( v \sim N(\mu, \sigma_v^2) \)
- aggregate demand: \( \sum_{i=1}^{I} \frac{E[v] - PR}{\rho^i \text{Var}[v]} = \sum_{i=1}^{I} \eta^i \tau_v \{ E[v] - PR \} \)
  Let \( \eta := \frac{1}{I} \sum_{i=1}^{I} \eta^i = \frac{1}{I} \sum_{i=1}^{I} \frac{1}{\rho^i} \) (harmonic mean)
- market clearing: \( \eta \tau_v \{ E[v] - PR \} = X^{\text{supply}} \)

\[
P = \frac{1}{R} \left\{ E[v] - \frac{X^{\text{sup}}}{I \eta \tau_v} \right\}
\]

- Expected excess payoff:

\[
Q := E[v] - PR = \frac{1}{\eta \tau_v} \frac{X^{\text{sup}}}{I}
\]
Symmetric Info – Benchmark

- Trader $i$’s equilibrium demand is:

$$x^i(P) = \frac{\eta^i}{\eta} \frac{X^{\text{sup}}}{l}$$

- Remarks:
  - $\frac{\partial P}{\partial E[v]} = \frac{1}{R} > 0$
  - $\frac{\eta^i}{\eta}$ risk sharing of aggregate endowment:

$$\frac{x^i}{x^i'} = \frac{\eta^i}{\eta'}$$

- no endowment effects
REE – Grossman (1976)

Model setup:

- $i \in \{1, \ldots, I\}$ traders
- CARA utility function with risk aversion coefficient $\rho = \rho^i$ (Let $\eta^i = \frac{1}{\rho^i}$ be trader $i$’s risk tolerance)
- information is dispersed among traders: trader $i$’s signal is $S^i = v + \epsilon^i_S$, where $\epsilon^i_S \sim i.i.d. \mathcal{N}(0, \sigma^2_{\epsilon})$
REE – Grossman (1976)

Step 1: Conjecture price function

\[ P = \alpha_0 + \alpha S \bar{S}, \text{ where } \bar{S} = \frac{1}{I} \sum_{i} S^i \text{ (sufficient statistics)} \]

Step 2: Derive posterior distribution

\[
E[v|S^i, P] = E[v|\bar{S}] = \lambda E[v] + (1 - \lambda) \bar{S} \\
= \lambda E[v] + (1 - \lambda) \frac{P - \alpha_0}{\alpha S}
\]

\[
Var[v|S^i, P] = Var[v|\bar{S}] = \lambda Var[v],
\]

where \( \lambda := \frac{Var[\epsilon]}{lVar[v] + Var[\epsilon]} \)

Step 3: Derive individual demand

\[
x^i*(P) = \frac{E[v|S^i, P] - P(1 + r)}{\rho^i Var[v|S^i, P]}
\]
**REE – Grossman (1976)**

**Step 4: Impose market clearing**

\[
\sum_{i} x^{i*}(P) = X^{\text{supply}}
\]

\[
P = \frac{\lambda}{1 + r} \left( E[v] - \rho^i \text{Var}[v] \frac{1}{l} X^{\text{supply}} \right) + \frac{1 - \lambda}{1 + r} \bar{S},
\]

where \( \bar{S} = \frac{P - \alpha_0}{\alpha_S} \)

**Step 5: Impose rationality**

(determine undetermined coefficients \( \alpha_0, \alpha_S \))

N.B.: Price fully reveals sufficient statistic!
## Informational (Market) Efficiency

### Empirical Literature:

<table>
<thead>
<tr>
<th>Form</th>
<th>Price reflects</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>strong</strong></td>
<td>all private and public information</td>
</tr>
<tr>
<td><strong>semi strong</strong></td>
<td>all public information</td>
</tr>
<tr>
<td><strong>weak</strong></td>
<td>only (past) price information</td>
</tr>
</tbody>
</table>

### Theoretical Literature:

<table>
<thead>
<tr>
<th>Form</th>
<th>Price aggregates/reveals</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>fully revealing</strong></td>
<td>all private signals</td>
</tr>
<tr>
<td><strong>informational efficient</strong></td>
<td>sufficient statistic of signals</td>
</tr>
<tr>
<td><strong>partially revealing</strong></td>
<td>a noisy signal of pooled private info</td>
</tr>
<tr>
<td><strong>privately revealing</strong></td>
<td>with one signal reveals suff. stat.</td>
</tr>
</tbody>
</table>
Informational (Market) Efficiency

- $\bar{S}$ sufficient statistic for all individual info sets $\{S^1, ..., S^l\}$
- Illustration: if one can view price function as
  \[ P(\cdot) : \{S^1, ..., S^l\} \xrightarrow{g(\cdot)} \bar{S} \xrightarrow{f(\cdot)} P \]
- if $f(\bar{S})$ is invertible, then price is informationally efficient
- if $f(\cdot)$ and $g(\cdot)$ are invertible, then price is fully revealing
Remarks & Paradox

• Grossman (1976) solved it via “full communication equilibria” (Radner 1979’s terminology)

• ‘unique’ info efficient equilibrium (DeMarzo & Skiadas 1998)

• As $I \to \infty$ (risk-bearing capacity), $P \to \frac{1}{R} E[v]$

• **Grossman Paradox:**
  Individual demand does not depend on individual signal $S_i$’s. How can all information be reflected in the price?

• **Grossman-Stiglitz Paradox:**
  Nobody has an incentive to collect information?

• individual demand is independent of wealth (CARA)

• in equilibrium individual demand is independent of price

• equilibrium is not implementable
Noisy REE – Hellwig 1980

Model setup:

- \( i \in \{1, \ldots, I\} \) traders
- CARA utility function with risk aversion coefficient \( \rho = \rho^i \)
  (Let \( \eta^i = \frac{1}{\rho^i} \) be trader \( i \)'s risk tolerance)
- Information is dispersed among traders:
  trader \( i \)'s signal is \( S^i = v + \epsilon_S^i \), where \( \epsilon_S^i \sim \text{ind } \mathcal{N}(0, (\sigma^i_{\epsilon})^2) \)
- noisy asset supply \( X^{\text{Supply}} = u \)
- Let \( \Delta S^i = S^i - E[S^i] \), \( \Delta u = u - E[u] \) etc.
Noisy REE – Hellwig (1980)

Step 1: Conjecture price function

\[ P = \alpha_0 + \sum_i^{l} \alpha_i^S \Delta S^i + \alpha_u \Delta u \]

Step 2: Derive posterior distribution (let’s do it only half way through)

\[ E[v | S^i, P] = E[v] + \beta_S^i(\alpha) \Delta S^i + \beta_P(\alpha) \Delta P \]

\[ \text{Var}[v | S^i, P] = \frac{1}{\tau_{i}[v | S^i, P]} \text{ (independent of signal realization)} \]

Step 3: Derive individual demand

\[ x^{i^*}(P) = \eta^i \tau_{i}[v | S^i, P] \{ E[v | S^i, P] - P(1 + r) \} \]
Noisy REE – Hellwig (1980)

**Step 4: Impose market clearing**

Total demand = total supply (let $r = 0$)

$$\sum_{i}^{l} \eta^{i} \tau_{v|S^{i},P}^{i} (\alpha) \{ E[v] + \beta^{i}_{S} (\alpha) \Delta S^{i} - \alpha_{0} \beta^{i}_{P} (\alpha) + [\beta^{i}_{P} (\alpha) - 1] P \} = u$$

... 

$$P \left( S^{1}, ..., S^{l}, u \right) =$$

$$\sum_{i} \left( \eta^{i} \tau_{v|S^{i},P}^{i} (\alpha) \right) \left[ E[v] - \alpha_{0} \beta^{i}_{P} (\alpha) + \beta^{i}_{S} (\alpha) \Delta S^{i} \right] - E[u] - \Delta u$$

$$\sum_{i} \left( 1 - \beta^{i}_{P} (\alpha) \right) \eta^{i} \tau_{v|S^{i},P}^{i} (\alpha)$$
Step 5: Impose rationality

\[
\alpha_0 = \frac{\sum_i \left( \eta^i \tau^i_{[v|S^i,P]}(\alpha) \right) \left[ E[v] - \alpha_0 \beta^i_P(\alpha) \right] - E[u]}{\sum_i \left( 1 - \beta^i_P(\alpha) \right) \eta^i \tau^i_{[v|S^i,P]}(\alpha)}
\]

\[
\alpha^i_S = \frac{\eta^i \tau^i_{[v|S^i,P]}(\alpha)}{\sum_i \left( 1 - \beta^i_P(\alpha) \right) \eta^i \tau^i_{[v|S^i,P]}(\alpha)} \beta^i_S(\alpha)
\]

\[
\alpha^i_u = \frac{-1}{\sum_i \left( 1 - \beta^i_P(\alpha) \right) \eta^i \tau^i_{[v|S^i,P]}(\alpha)}
\]

Solve for root \( \alpha^* \) of the problem \( \alpha = G(\alpha) \)
Noisy REE – Hellwig 1980

Simplify model setup:

- All traders have identical risk aversion coefficient $\rho = 1/\eta$
- Error of all traders’ signals $\epsilon_S$ are i.i.d.

**Step 1: Conjecture price function** simplifies to:

$$\Delta P = \alpha_S \sum_i \frac{1}{i} \Delta S^i + \alpha_u \Delta u$$

**Step 2: Derive posterior distribution**:

$$E[v|S^i, P] = E[v] + \beta_S(\alpha) \Delta S^i + \beta_P(\alpha) \Delta P$$

$$Var[v|S^i, P] = \frac{1}{\tau} \quad \text{(independent of signal realization)}$$

where $\beta$’s are projection coefficients.
Noisy REE – Hellwig (1980)

Previous fixed point system simplifies to:

\[
\alpha_S = \frac{1}{\sum_i (1 - \beta_P(\alpha))} \beta_S(\alpha)
\]

\[
\alpha_u = \frac{-1}{\eta_\tau(\alpha) \sum_i (1 - \beta_P(\alpha))}
\]

To determine \(\beta_S\) and \(\beta_P\), invert Co-variance matrix

\[
\Sigma \left( S^i, P \right) = \begin{pmatrix}
\sigma_v^2 + \sigma_\varepsilon^2 & \alpha_S \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) \\
\alpha_S \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) & \alpha_S^2 \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) + \alpha_u^2 \sigma_u^2
\end{pmatrix}
\]

\[
\Sigma^{-1} \left( S^i, P \right) = \frac{1}{D} \begin{pmatrix}
\alpha_S^2 \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) + \alpha_u^2 \sigma_u^2 & -\alpha_S \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) \\
-\alpha_S \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) & \sigma_v^2 + \sigma_\varepsilon^2
\end{pmatrix}
\]

\[
D = \alpha_S^2 \frac{l - 1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \left( \sigma_v^2 + \sigma_\varepsilon^2 \right)
\]
Noisy REE – Hellwig (1980)

Since \( \text{Cov}[v, P] = \alpha_S \sigma_v^2 \) and \( \text{Cov}[v, S^i] = \sigma_v^2 \) leads us to:

\[
\beta_P = \frac{1}{D} \frac{\alpha_S}{\alpha} \frac{l - 1}{l} \sigma_v^2 \sigma_\varepsilon^2 \\
\beta_S = \frac{1}{D} \frac{\alpha_u^2}{\alpha} \sigma_u^2 \sigma_v^2
\]

For conditional variance (precision) from projection theorem:

\[
\text{Var}[v|S^i, P] = \frac{1}{D} \left[ D \sigma_v^2 - \left( \alpha_u^2 \sigma_u^2 + \alpha_S^2 \frac{l - 1}{l} \sigma_\varepsilon^2 \right) \sigma_v^2 \right] \\
= \frac{1}{D} \left[ \alpha_S^2 \frac{l - 1}{l^2} \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \right] \left( \sigma_\varepsilon^2 \right) \sigma_v^2
\]

Hence:
Noisy REE – Hellwig (1980)

\[
\alpha_S = \frac{\alpha_u^2 \sigma_v^2 \sigma_u^2}{(D - \alpha_S \frac{l-1}{l} \sigma_\varepsilon^2 \sigma_v^2)l}
\]

\[
\alpha_u = -\rho \frac{(\alpha_u^2 \sigma_u^2 + \alpha_S^2 \frac{l-1}{l^2} \sigma_\varepsilon^2) \sigma_\varepsilon^2 \sigma_v^2}{(D - \alpha_S \frac{l-1}{l} \sigma_\varepsilon^2 \sigma_v^2)l}
\]

Trick: solve for \( h = -\frac{\alpha_u}{\alpha_S} \) (Recall price signal can be re-written as \( \frac{P - \alpha_0}{\alpha_S} = \sum_i \frac{1}{l} S_i + \frac{\alpha_u}{\alpha_S} u \)) [noise signal ratio]

\[
h = \rho \frac{(h^2 \sigma_u^2 + \frac{l-1}{l^2} \sigma_\varepsilon^2) \sigma_\varepsilon^2 \sigma_v^2}{h^2 \sigma_v^2 \sigma_u^2}
\]

\[
h = \rho \sigma_\varepsilon^2 + \frac{\rho (l - 1) \sigma_\varepsilon^4}{h^2 \frac{l^2 \sigma_u^2}{l^2 \sigma_u^2}} \quad \Rightarrow \quad \text{unique} \quad h > \rho \sigma_\varepsilon^2
\]
Remember that $h$ is increasing in $\rho$. Back to $\alpha_S$:

$$
\alpha_S = \frac{\alpha_u^2 \sigma_v \sigma_u^2}{(D - \alpha_S \frac{l-1}{l} \sigma^2 \epsilon \sigma_v^2)} \left( \frac{\alpha^2_u \sigma_v \sigma_u^2}{D - \alpha_S \frac{l-1}{l} \sigma^2 \epsilon \sigma_v^2} \right)
$$

multiply by denominator:

$$
\alpha_S Dl = \alpha_u^2 \sigma_v \sigma_u^2 + (l - 1) \alpha_S^2 \frac{l-1}{l} \sigma^2 \epsilon \sigma_v^2
$$

$$
\Leftrightarrow \alpha_S = \frac{1}{D} \left[ \frac{1}{l} \alpha_u^2 \sigma_v \sigma_u^2 + \alpha_S^2 \frac{l-1}{l} \sigma^2 \epsilon \sigma_v^2 \right]
$$

Sub in $D = \ldots$

$$
\alpha_S = \frac{\frac{\alpha_u^2}{\alpha_S^2} \sigma_v \sigma_u^2 + (l - 1) \sigma^2 \epsilon \sigma_v^2}{(l - 1) \left( \sigma_v^2 + \frac{1}{l} \sigma^2 \epsilon \right) \sigma_v^2 + l \frac{\alpha_u^2}{\alpha_S^2} \sigma_u^2 \left( \sigma_v^2 + \sigma^2 \epsilon \right)} \Rightarrow \text{unique } \alpha_S
$$

This proves existence and uniqueness of the NREE!
Recall that:

\[
\text{Var} \left[ v \mid S^i, P \right] = \frac{1}{D} \left[ \alpha_S^2 \frac{l - 1}{l^2} \sigma_v^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_v^2 \sigma_v^2, \text{ and} \\
\alpha_S = \frac{1}{D} \left[ \frac{1}{l} \alpha_u^2 \sigma_v^2 \sigma_u^2 + \frac{1}{l} \sigma_v^2 \sigma_u^2 \right],
\]

Hence,

\[
\alpha_S = \text{Var} \left[ v \mid S^i, P \right] \frac{\left[ \frac{1}{l} \alpha_u^2 \sigma_v^2 + \alpha_s^2 \frac{l - 1}{l} \sigma_v^2 \right]}{\left[ \alpha_S^2 \frac{l - 1}{l^2} \sigma_v^2 + \alpha_u^2 \sigma_v^2 \right] \sigma_v^2} \quad \text{(notice } l^2 \text{ square)}
\]

\[
\alpha_S = \text{Var} \left[ v \mid \cdot \right] \frac{1}{\sigma_v^2} \frac{\left[ \frac{l}{l - 1} h^2 \sigma_u^2 + l \sigma_v^2 \right]}{\left[ \sigma_v^2 + \frac{l^2}{l - 1} h^2 \sigma_u^2 \right]}
\]

\[
= \text{Var} \left[ v \mid \cdot \right] \frac{1}{\sigma_v^2} \frac{\left[ \frac{l}{l - 1} h^2 \sigma_u^2 + \frac{1}{l} \sigma_v^2 + (l - \frac{1}{l}) \sigma_v^2 \right]}{\left[ \sigma_v^2 + \frac{l^2}{l - 1} h^2 \sigma_u^2 \right]}
\]
Characterization of NREE

\[
\alpha_S = \text{Var} \left[ v \mid S^i, P \right] \frac{1}{\sigma_v^2} \left[ \frac{1}{l} + \frac{(l - \frac{1}{l}) \sigma_v^2}{\sigma_v^2 + \frac{l^2}{l-1} h^2 \sigma_u^2} \right]
\]

\[
= \text{Var} \left[ v \mid S^i, P \right] \frac{\tau_v}{l} \left[ 1 + (l + 1) \left( (l - 1) \frac{\tau_u}{\tau_u + \frac{l^2}{l-1} h^2 \tau_v} \right) \right] := \theta
\]

where \( \theta \) is decreasing in \( \rho \) (\( h \) is increasing)
Characterization of NREE

\[
\text{Var} \left[ v \mid S^i, P \right] = \frac{1}{D} \left[ \alpha_S^2 \frac{l - 1}{l^2} \sigma_e^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_e^2 \sigma_v^2
\]

\[
= \frac{\left[ \alpha_S^2 \frac{l - 1}{l^2} \sigma_e^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_e^2 \sigma_v^2}{\alpha_S^2 \frac{l - 1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma_e^2 \right) \sigma_e^2 + \alpha_u^2 \sigma_u^2 \left( \sigma_v^2 + \sigma_e^2 \right) \left[ \frac{l - 1}{l^2} \sigma_e^2 + h^2 \sigma_u^2 \right] \sigma_e^2 \sigma_v^2}
\]

\[
= \frac{l - 1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma_e^2 \right) \sigma_e^2 + h^2 \sigma_u^2 \left( \sigma_v^2 + \sigma_e^2 \right) = \ldots
\]

\[
\frac{1}{\text{Var} \left[ v \mid S^i, P \right]} = \tau_v + \tau_\varepsilon + \theta \tau_\varepsilon
\]

**Interpretation**

\( \theta = (l - 1) \frac{\tau_u}{(\tau_u + \frac{l^2}{l - 1} h^2 \tau_\varepsilon)} \) measure of info efficiency

\( \sigma_u^2 \to \infty \ (\tau_u \to 0): \ \theta \to 0 \) price is uninformative (Walras. equ.)

\( \sigma_u^2 \to 0 \ (\tau_u \to \infty): \ \theta \to 1 \) price is informationally efficient
Remarks to Hellwig (1980)

- Since $\alpha_u^2 \neq 0$, $\beta_S \neq 0$, i.e. agents condition on their signal
- As risk aversion of trader increases the informativeness of price $\theta$ declines
- Price informativeness increases in precision of signal $\tau_{\varepsilon}$ and declines in the amount of noise trading $\sigma_u^2$
- Negative supply shock leads to a larger price increase compared to a Walrasian equilibrium, since traders wrongly partially attribute it to a good realization of $v$
- Diamond and Verrecchia (1981) is similar except that endowment shocks of traders serve as asymmetric information
Endogenous Info Acquisition

Model setup:

- \( i \in \{1, \ldots, I\} \) traders
- CARA utility function with risk aversion coefficient \( \rho \)
  (Let \( \eta = \frac{1}{\rho} \) be traders’ risk tolerance)
- no information aggregation – two groups of traders
  - Informed traders who have the same signal \( S \):
    \[ S = \nu + \epsilon_S \text{ with } \epsilon_S \sim \mathcal{N}(0, \sigma^2_{\epsilon}) \]
  - Uninformed traders have no signal
- FOCUS on information acquisition
Noisy REE – Grossman-Stiglitz

**Step 1: Conjecture price function**

\[ P = \alpha_0 + \alpha_S \Delta S + \alpha_u \Delta u \]

**Step 2: Derive posterior distribution**

- for informed traders:
  
  \[ E[v|S, P] = E[v|S] = E[v] + \frac{\tau \varepsilon}{\tau_v + \tau \varepsilon} \Delta S \]
  
  \[ \tau[v|S] = \tau_v + \tau \varepsilon \]

- for uninformed traders:
  
  \[ E[v|P] = E[v] + \frac{\alpha_S \sigma_v^2}{\alpha_S^2 (\sigma_v^2 + \sigma \varepsilon^2) + \alpha_u^2 \sigma_u^2} \Delta P \]

  \[ \text{Var}[v|P] = \sigma_v^2 \left( 1 - \frac{\alpha_S^2 \sigma_v^2}{\alpha_S^2 (\sigma_v^2 + \sigma \varepsilon^2) + \alpha_u^2 \sigma_u^2} \right) \]

  or:

  \[ \tau[v|P] = \tau_v + \frac{\tau_u}{\tau_u + h^2 \tau \varepsilon} \tau \varepsilon, \text{ where } h = -\frac{\alpha_u}{\alpha_S} \]

  \[ := \phi \in [0,1] \]
Noisy REE – Grossman-Stiglitz

After some algebra we get:

\[ E[v|P] = E[v] + \frac{1}{\alpha S \tau_v + \phi \tau_\epsilon} \Delta P \]

**Step 3: Derive individual demand**

\[ x^I(P, S) = \eta^I[\tau_v + \tau_\epsilon] \left( E[v] + \frac{\tau_\epsilon}{\tau_v + \tau_\epsilon} \Delta S - P \right) \]

\[ x^U(P) = \eta^U[\tau_v + \phi \tau_\epsilon] \left( E[v] + \frac{1}{\alpha S \tau_v + \phi \tau_\epsilon} \Delta P - P \right) \]

**Step 4: Impose market clearing**

\[ \lambda^I \eta^I[\tau_v + \tau_\epsilon] \left( E[v] + \frac{\tau_\epsilon}{\tau_v + \tau_\epsilon} \Delta S - P \right) + \]

\[ := \nu^I \]

\[ (1 - \lambda^I) \eta^U[\tau_v + \phi \tau_\epsilon] \left( E[v] + \frac{1}{\alpha S \tau_v + \phi \tau_\epsilon} \Delta P - P \right) = u \]

\[ := \nu^U \]
Noisy REE – Grossman-Stiglitz

\[
P(S, u) = \left(\nu^I + \nu^U\right) E[v] + \nu^I \frac{\tau_v}{\tau_v + \tau_{\epsilon}} \Delta S - \frac{1}{\alpha_S \tau_v + \phi \tau_{\epsilon}} \alpha_0 \nu^U - E[u] - \Delta u
\]

\[
\nu^U \left(1 - \frac{1}{\alpha_S \tau_v + \phi \tau_{\epsilon}}\right) + \nu^I
\]

Hence,

\[
h = -\frac{\alpha_U}{\alpha_S} = \left[\nu^I \frac{\tau_{\epsilon}}{\tau_v + \tau_{\epsilon}}\right]^{-1} = \frac{1}{\lambda^I \eta^I \tau_{\epsilon}}
\]

\[
\phi = \frac{\tau_u \tau_{\epsilon}}{\tau_u \tau_{\epsilon} + \frac{1}{(\lambda^I \eta^I)^2}}
\]

Remarks:
- As \(\text{Var}[u] \downarrow 0\), \(\phi \uparrow 1\)
- If signal is more precise (\(\tau_{\epsilon}\) is increasing) then \(\phi\) increases (since informed traders are more aggressive)
- Increases in \(\lambda^I\) and \(\eta^I\) also increase \(\phi\)
Noisy REE – Grossman-Stiglitz

Step 5: Impose rationality

Solve for coefficients:

\[ \alpha_0 = E[v] - \frac{1}{\nu^I + \nu^U} E[u] \]
\[ \alpha_S = \frac{1}{\nu^U \left(1 - \frac{1}{\alpha_S \frac{\Phi \tau_\varepsilon}{\tau_V + \Phi \tau_\varepsilon}}\right) + \nu^I \tau_V + \tau_\varepsilon} \frac{\tau_\varepsilon}{\nu^I} = \frac{\lambda' \eta^I + \lambda^U \eta^U \Phi}{\nu^I + \nu^U} \tau_\varepsilon \]
\[ \alpha_u = -\frac{1}{\nu^I + \nu^U} \left(1 + \frac{\lambda^U \eta^U}{\lambda' \eta^I} \Phi\right) \]

Finally let’s calculate:

\[ \frac{\tau[v | S]}{\tau[v | P]} = \frac{\tau_V + \tau_\varepsilon}{\tau_V + \phi \tau_\varepsilon} = 1 + \frac{(1 - \phi) \tau_\varepsilon}{\tau_V + \phi \tau_\varepsilon} \]
Information Acquisition Stage – Grossman-Stiglitz (1980)

- **Aim**: endogenize $\lambda^I$
- **Recall**:
  \[ x^i = \eta^i \tau_{[Q|S]} E[Q|S], \text{ where } Q = v - R P \text{ is excess payoff} \]
- **Final wealth**:
  \[ W^i = \eta^i Q \tau_{[Q|S]} E[Q|S] + (Pu^i + e^i_0)R \]
  (CARA $\Rightarrow$ we can ignore second term)
  Note $W^i$ is product of two normally distributed variables.
  Use formula of Slide 7 or follow following steps:
- **Conditional on $S$**, wealth is normally distributed:
  \[ E[W|S] = \eta \tau_{[Q|S]} E[Q|S]^2 \]
  \[ Var[W|S] = \eta^2 \tau_{[Q|S]} E[Q|S]^2 \]
Expected utility conditional on $S$:

\[
E[U(W)|S] = - \exp \left\{ - \frac{1}{\eta} [\eta \tau_{Q|S}] E[Q|S]^2 - \frac{1}{2} \eta \tau_{Q|S} E[Q|S]^2 \right\}
\]

\[
= - \exp \left\{ - \frac{1}{2} \tau_{Q|S} E[Q|S]^2 \right\}
\]

Integrate over possible $S$ to get the ex-ante utility:

w.l.o.g. we can assume that $S = Q + \epsilon$

Normal density $f(S) = \sqrt{\frac{\tau_S}{2\pi}} \exp\{-\frac{1}{2} \tau_S [\Delta S]^2\}$

\[
E[U(W)] =
\]

\[
- \int_S \sqrt{\frac{\tau_S}{2\pi}} \exp \left\{ - \frac{1}{2} \left[ \tau_{Q|S} E[Q|S]^2 + \tau_S (\Delta S)^2 \right] \right\} dS
\]
Information Acquisition Stage – Grossman-Stiglitz (1980)

Term in square bracket is:

\[
\begin{align*}
&\left(\tau_Q + \tau_\epsilon\right) \left( E [Q] + \frac{\tau_\epsilon}{\tau_Q + \tau_\epsilon} \Delta S \right)^2 + \frac{\tau_Q \tau_\epsilon}{\tau_Q + \tau_\epsilon} (\Delta S)^2
\end{align*}
\]

which simplifies to:

\[
\tau_Q E [Q]^2 + \tau_\epsilon (\Delta S + E [Q])^2
\]

Hence:

\[
E \left[ U (W) \right] =
\]

\[
- \exp \left\{ - \frac{\tau_Q E [Q]^2}{2} \right\} \int_s \sqrt{\frac{\tau_S}{2\pi}} e^{-\frac{1}{2} \left[ \tau_\epsilon (\Delta S + E [Q])^2 \right]} \, dS
\]

Define:

\[
y := \sqrt{\tau_\epsilon} (\Delta S + E [Q])
\]
Information Acquisition Stage – Grossman-Stiglitz (1980)

\[
E[U(W)] = -\exp \left\{ -\frac{\tau_Q E[Q]^2}{2} \right\} \sqrt{\frac{\tau_S}{\tau_\varepsilon}} \int_S -\sqrt{\frac{\tau_\varepsilon}{2\pi}} \exp \left\{ -\frac{1}{2} y^2 \right\} dS
\]

= 1

Let: \( k = -\exp \left\{ -\frac{\tau_Q E[Q]^2}{2} \right\} \sqrt{\tau_Q} \)

Note: \( \tau_S = \frac{\tau_Q \tau_\varepsilon}{\tau_Q + \tau_\varepsilon} \)

Hence:

\[
E[U(W)] = \frac{k}{\sqrt{\tau[Q|S]}} = \frac{k}{\sqrt{\tau_Q + \tau_\varepsilon}}
\]
Willingness to Pay for Signal

General Problem (No Price Signal)

• Without price signal $p$ and signal $S$, expected utility:

$$E[U(W)] = \frac{k}{\sqrt{\tau Q}}$$

• If the agent buys a signal at a price of $m_S$ his expected utility is:

$$E[U(W - m_S)] = E[-\exp(-\rho(W - m_S))]$$

$$= E[-\exp(-\rho(W)) \exp(\rho m_S)]$$

$$= \frac{k}{\sqrt{\tau[Q|S]}} \exp(\rho m_S)$$

• Agent is indifferent when:

$$\frac{k}{\sqrt{\tau Q}} = \frac{k}{\sqrt{\tau[Q|S]}} \exp(\rho m_S)$$
Willingness to Pay for Signal

General Problem (No Price Signal)

- Hence willingness to pay is:

\[ m_S = \eta \ln \left( \frac{\sqrt{\tau[Q|S]}}{\tau_Q} \right) \]

Willingness to pay depends on the improvement in precision
Information Acquisition Stage – Grossman-Stiglitz (1980)

- Every agent has to be indifferent between being informed or not. The cost of the signal is:

\[ c = \eta \ln \left( \sqrt{\frac{\tau[v|S]}{\tau[v|P]}} \right) = \eta \ln \left( \sqrt{\frac{\tau v + \tau \epsilon}{\tau v + \phi \tau \epsilon}} \right) \]

(previous slide)

- This determines \( \phi \):

\[
\phi = \frac{\tau u \tau \epsilon}{\tau u \tau \epsilon + \left( \frac{1}{\lambda^I} \eta^I \right)^2}, \text{ which pins down } \lambda^I
\]

- Comparative Statics (using IFT):
  - \( c \uparrow \Rightarrow \phi \downarrow \)
  - \( \eta \uparrow \Rightarrow \phi \uparrow \) (extreme case: risk-neutrality)
  - \( \tau \epsilon \uparrow \Rightarrow \phi \uparrow \)
  - \( \sigma^2_u \uparrow \Rightarrow \phi \rightarrow \) (number of informed traders \( \uparrow \))
  - \( \sigma^2_u \downarrow 0 \Rightarrow \) no investor purchases a signal
Information Acquisition Stage

• Further extensions:
  • Purchase signals with different precisions (Verrecchia 1982)
  • Optimal sale of information
    • Photocopied (newsletter) or individualistic signal (Admati & Pfleiderer)
    • Indirect versus direct (Admati & Pfleiderer)
Endogenizing Noise Trader Demand

- Endowment shocks or outside opportunity shocks that are correlated with asset
- Welfare analysis
  - more private information $\rightarrow$ adverse selection
  - more public information $\rightarrow$ Hirshleifer effect (e.g. genetic testing)
Tips & Tricks

• risk-neutral competitive fringe observing limit order book $L$

$$p = E[v|L(\cdot)]$$

• separates risk-sharing from informational aspects