Asset Pricing under Asymmetric Information
Rational Expectations Equilibria

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      - uniform price setting
      - limit order book analysis
    - dynamic sequential trade models with multiple trading rounds
- strategic market order models where the market maker sets prices ex-post
Screening Models à la Glosten

1. Uninformed (risk-neutral) market maker sets whole supply schedule
   - market making sector is *competitive*
   - oligopolistic market making sector
   - market maker is *monopolist*

2. Possibly informed trader submits
   - a single order which is executed at *uniform price*
   - many little orders in order to “walk along the limit order book” (*discriminatory prices*)
Uniform Price Setting - Glosten 1989

- Contrast competitive market maker sector with monopolistic market maker (specialist system NYSE).

- Model setup
  - market maker(s) set price (supply) schedule
  - single trader submits order
    - risk-averse with CARA utility function
    - endowment shock of $u$
    - private signal $S^i = v + \epsilon$
  - two-dimensional screening problem
    Glosten (1989) reduces it to a one-dimensional problem (see later)
Uniform Price Setting - Glosten 1989

- Competitive price schedule: \( P^{CO} = E[v|x] \)
  - Perfect competition
    - \( \Rightarrow \) expected profit for any order size \( x \) is ZERO.
    - Prevents market makers from effectively screening orders
    - \( \Rightarrow \) leads to instability
  - Formally, existence problem for certain parameters
    (Hellwig JET 1994 shows that this is due to unbounded support of type space and the existence problem is different to the one in Rothschild & Stiglitz)

- Monopolistic price schedule:

\[
P^{mo} = \arg \max E[[P^{mo}(x^*(\cdot)) - v]x^*(\cdot)],
\]

where \( x^*(\cdot) \) is the optimal order size.

- Principal-agent problem
- Principal sets menu of contracts \( (x, P^{mo}(x)) \)
- Cross-subsidization: large profit from small trades
  - small (-ve) profit from large trades
- Market with monopolistic setting stays open for larger trade sizes than a market with multiple market makers
Discrim. Pricing (Limit Order Book)
Glosten 1994 - BRM 2000

- “upper tail” conditional expectations for next marginal order $y$
  \[ P^{CO}(y) = E[v| x \geq y] \]
- trader who buy only a tiny marginal quantity have to pay a higher (ask) price $\Rightarrow$ small trade spread
- competitive market makers do not know whether trader only buys first marginal unit or continues to buy further units.
- cross-subsidization from small orders to large orders
- limit order book is immune to “cream skimming” of orders by competing exchanges (no advantage of order splitting).
Discrim. Pricing - Biais, Rochet & Martimort

Oligopolistic Market Makers

- oligopolistic screening game (special cases $I = 1$, $I = \infty$)
- **Stage 1**: risk-neutral market maker(s)
  - set supply schedule $p(x)$ (limit order book)
- **Stage 2**: informed trader buy $x = \sum_i x^i$ shares
  - $x^i$ for market maker $i$
  - transfer to mm $i$: $t^i(x^i) = \int_0^{x^i} p(q) \, dq$, $T(x) = \sum_i t^i(x^i)$
  - trader’s endowment shock $u$
  - trader’s signal $S$, where $v = S + \varepsilon$.
  - $\varepsilon \sim N(0, \sigma^2)$
  - $u$ and $S$ have bounded support.
  - trader’s final wealth $W = v(u + x) - \sum_i t^i(x^i)$
    (conditional on $u$, $S$, wealth $W$ is normally distributed
BRM: One Dimensional Screening

• **Stage 2**: (ctd.) - “Glosten (1989)-trick”
  • with CARA utility function

\[
E [W|u, S] - \frac{\rho}{2} V [W|u, S]
= (x + u) S - T (x) - \frac{\rho}{2} (x + u)^2 \text{Var} [v|S]
\]

\[
= \left( uS - \frac{\rho \sigma^2}{2} u^2 \right) + \left( xS - \rho \sigma^2 xu - \frac{\rho \sigma^2}{2} x^2 - T (x) \right)
\]

- independent of \( x \)
- \( \theta \) \text{ i.e. } \theta := S - \rho \sigma^2 u \)
- \( \text{depends on } x \implies \text{Info-Rent} \)

• This reduces it to a one-dimensional screening problem
• function \( v (\theta) = E [v|\theta] \) of (one-dimensional) type \( \theta \)

\[
1 \geq v (\theta) \geq 0
\]
BRM: First Best Benchmark

- ex-ante

optimal trading mechanism

\[
\begin{align*}
\max_{\{\tau(\theta), x(\theta)\}} & \int_{\theta}^{\bar{\theta}} \left( \theta x(\theta) - \frac{\rho \sigma^2}{2} x(\theta)^2 - \tau(\theta) \right) f(\theta) d\theta \\
\text{s.t.} & \int_{\theta}^{\bar{\theta}} (\tau(\theta) - v(\theta) x(\theta)) f(\theta) d\theta = \Pi
\end{align*}
\]

- \(\Pi\) determines how surplus is distributed between P and A

\[
\Rightarrow \max \int_{\theta}^{\bar{\theta}} \left( \theta x(\theta) - \frac{\rho \sigma^2}{2} x(\theta)^2 - v(\theta) x(\theta) - \Pi \right) f(\theta) d\theta
\]
BRM: First Best Benchmark

for a given $\theta$

$$\theta - \rho \sigma^2 x(\theta) - \nu(\theta) = 0$$

$$x^*(\theta) = \frac{\theta - \nu(\theta)}{\rho \sigma^2}$$

$$= E[-u|\theta]$$, since $u = -\frac{\theta - S}{\rho \sigma^2}$

- Assume $x^*(\theta) < 0 < x^*(\theta)$
  $\Rightarrow \exists \theta_0$ s.t. $x^*(\theta_0) = 0$

- almost all $\theta$-types trade
  (see later that $\forall \theta > \theta_0 \Rightarrow$ buy
  $\forall \theta < \theta_0 \Rightarrow$ sell )
BRM: Monopolistic Screening
\( x^*(\theta) \) and \( x_m(\theta) \)

Figure: xxx. xx
BRM: Implementable Allocation under Adverse Selection

- Social planner must elicit information
- Revelation Principle
  Any allocation that can be achieved with non-linear schedules \( T(x) \) can also be achieved with a truthful direct mechanism \( \{\tau(\cdot), x(\cdot)\} \).
- Incentive compatibility

\[
\theta \in \arg \max_{\hat{\theta}} \left( \theta x(\hat{\theta}) - \frac{\rho \sigma^2}{2} x(\hat{\theta})^2 - \tau(\hat{\theta}) \right)
\]

\[
\implies U(\theta) = \max_{\hat{\theta}} \left( \theta x(\hat{\theta}) - \frac{\rho \sigma^2}{2} x(\hat{\theta})^2 - \tau(\hat{\theta}) \right)
\]

\[
\{\tau(\cdot), x(\cdot)\} \text{ transfers and allocation}
\]
BRM: Dual (Mirrlees) Approach

\{U(\cdot), x(\cdot)\} informational rent (see Fudenberg & Tirole Ch. 7)

Lemma 1:

A pair \{U(\cdot), x(\cdot)\} is implementable iff \( U(\cdot) \) is convex on \([\theta, \bar{\theta}]\), and for a.e. \( \theta \),

\[
\frac{dU(\theta, \hat{\theta}(\theta))}{d\theta} \uparrow \frac{\partial U}{\partial \theta} = x(\theta).
\]

envelope theorem
BRM: Monopolistic Screening

m.m.(principal) gets $\int_{\theta}^{\bar{\theta}} \tau(x(\theta)) - v(\theta)x(\theta)$ replacing $\tau$

from information rent $U(\theta) = \theta x(\theta) - \frac{\rho \sigma^2}{2} x^2(\theta) - \tau(x(\theta))$, the m.m.’s objective becomes

$$\max_{\{U(\cdot), x(\cdot)\}} \int_{\theta}^{\bar{\theta}} \left\{ [\theta - v(\theta)]x(\theta) - \frac{\rho \sigma^2}{2} [x(\theta)]^2 - U(\theta) \right\} f(\theta) d\theta$$

subject to

$$\text{IC} \quad \left\{ \begin{array}{l} \dot{U}(\theta) = x(\theta) \quad \forall \theta \ (\text{almost everywhere}) \\ U(\cdot) \text{ is convex on } [\theta, \bar{\theta}] \end{array} \right.$$  

$$\text{ex-post PC} \quad U(\theta) \geq 0 \quad \text{ex-post participation constraints}$$

(ex-post: since traders decide after knowing $\theta$ whether to participate)
BRM: Monopolistic Screening
Dual Approach

(replace $x(\theta)$ with $\dot{U}(\theta)$)

$$\max_{U(\cdot)} B_m \left( U(\cdot), \dot{U}(\cdot) \right)$$

$$:= \int_{\theta}^{\bar{\theta}} \left( [\theta - \nu(\theta)] \dot{U}(\theta) - \frac{\rho \sigma^2}{2} \dot{U}(\theta)^2 - U(\theta) \right) f(\theta) \, d\theta$$

s.t. $U(\cdot)$ convex
$U(\theta) \geq 0$

Temporarily ignore convexity constraint and check ex-post.
(Sufficient condition: $U(\cdot)$ is convex if

$$\forall \theta > \theta_0 \quad \frac{d}{d\theta} \left( \frac{1-F(\theta)}{f(\theta)} \right) < 0 \quad (18)$$

$$\forall \theta < \theta_0 \quad \frac{d}{d\theta} \left( \frac{F(\theta)}{f(\theta)} \right) > 0 \quad (19)$$
BRM: Monopolistic Screening

\[ \mathcal{L}\left(U, \dot{U}\right) = B_m\left(U, \dot{U}\right) + \int_{\theta}^{\bar{\theta}} U(\theta) \uparrow \sigma \text{ many Lagrange multipliers different from type to type (ex-post constraint)} \]

By complementary slackness condition, support of \( \Lambda \) be constrained in \((U_m)^{-1}(0)\), \((\theta\)-types which get zero info ret) view \( \Lambda(\theta) \) as c.d.f., i.e., \( \exists \) a measure \( \Lambda \)

\[ \Lambda(\theta) = \int_{\theta}^{\bar{\theta}} \frac{d\Lambda(s)}{\int_{\theta}^{\bar{\theta}} d\Lambda(s)} \text{ (slight abuse of notation)} \]
BRM: Monopolistic Screening

Aside: Integrating by parts

$$\int_{\bar{\theta}}^{\theta} U(\theta) \, d[\Lambda(\theta) - F(\theta)] = -\int_{\theta}^{\bar{\theta}} \dot{U}(\theta) (\Lambda(\theta) - F(\theta)) \, d\theta + U(\bar{\theta})$$

Consequently, \( \max \mathcal{L}(U, \dot{U}) = \)

$$= \int_{\theta}^{\bar{\theta}} \left( \left( \theta - v(\theta) + \frac{F(\theta) - \Lambda(\theta)}{f(\theta)} \right) \dot{U}(\theta) - \frac{\rho \sigma^2}{2} \dot{U}(\theta)^2 \right) f(\theta) \, d\theta$$

$$+ U(\bar{\theta}) (\Lambda(\theta) - 1)$$

max only if \( \Lambda(\theta) = 1 \) (since \( U(\bar{\theta}) \) is arbitrary)

pointwise maximization over \( \dot{U}(\theta) \)

$$\forall \theta \in [\theta, \bar{\theta}], \quad x_m(\theta) = \frac{\theta - v(\theta)}{\rho \sigma^2} + \frac{F(\theta) - \Lambda(\theta)}{f(\theta) \rho \sigma^2} \left\{ x^*(\theta) \right\}$$
BRM: Monopolistic Screening

Complementary slackness condition \((d\Lambda = 0 \text{ for some } \theta)\)

\[
\begin{align*}
\forall \theta \in [\underline{\theta}, \theta_m] \quad & \quad \Lambda (\theta) = 0 \\
\forall \theta \in [\theta_m, \bar{\theta}] \quad & \quad \Lambda (\theta) = 1
\end{align*}
\]

\[\implies \text{given (18) & (19), } U (\cdot) \text{ is convex and} \]

**Proposition 2**

\[\exists \theta_a^m > \theta_0 \text{ and } \theta_b^m < \theta \text{ s.t.} \]

(i) for all \(\theta \in [\underline{\theta}, \theta_b^m]\), \(x_m (\theta) = x^* (\theta) + \frac{F(\theta)}{\rho \sigma^2 f(\theta)}\)

(ii) for all \(\theta \in [\theta_b^m, \theta_a^m]\), \(x_m (\theta) = 0 \) (no info rent)

(iii) for all \(\theta \in (\theta_a^m, \bar{\theta}]\), \(x_m (\theta) = x^* (\theta) - \frac{1-F(\theta)}{\rho \sigma^2 f(\theta)}\)
BRM: Monopolistic Screening

$x^*(\theta)$ and $x_m(\theta)$

Figure: xxx. xx
BRM: Monopolistic Screening
Price Schedule

for $\theta > \theta_a^m$ we know

(1) $\theta \geq \theta_a^m \ U (\theta) = 0 + \int_{\theta_a^m}^{\theta} \dot{U}' (s) \, ds = \int_{\theta_a^m}^{\theta} x (s) \, ds$

(2) $U (\theta) = \theta x_m (\theta) - \frac{\rho \sigma^2 x_m(\theta)^2}{2} - T (x (\theta))$

(1)=(2) $T (x (\theta)) = \theta x_m (\theta) - \frac{\rho \sigma^2 x_m(\theta)^2}{2} - \int_{\theta_a^m}^{\theta} x (s) \, ds$

Differentiate w.r.t. $\theta$

\[
\frac{\partial T}{\partial x} \frac{\partial x}{\partial \theta} = x_m (\theta) + \theta \frac{\partial x}{\partial \theta} - \rho \sigma^2 x_m (\theta) \frac{\partial x_m}{\partial \theta} - x_m (\theta)
\]

\[
\frac{\partial T}{\partial x} = \theta - \rho \sigma^2 x_m (\theta)
\]

We have

\[
x_m (\theta) = x^* (\theta) + \frac{F (\theta)}{\rho \sigma^2 f (\theta)}
\]
BRM: Monopolistic Screening
Price Schedule

\[ \frac{\partial T}{\partial x} = \theta - \theta + v(\theta) - \frac{F(\theta)}{f(\theta)} \]

\[ t_m(x) = \frac{\partial T}{\partial x} = v(\theta) - \frac{F(\theta)}{f(\theta)} \]

for \( \theta < \theta^m \) similar steps

\[ \frac{\partial T}{\partial x} = v(\theta) + \frac{1 - F(\theta)}{f(\theta)} \]

Note that

\[ t_m(x = 0^+) = \theta^m_a > \theta^m_b = t_m(x = 0^-) \]

“small trade spread”
BRM: Oligopolistic Screening
Limit Order Book vs. Uniform Pricing
Röell (1998)

• Model setup
  • order size of trader is *exogenous*
  • is double exponentially distributed $f(x) = \frac{1}{2}ae^{-a|x|}$
  • conditional expectations
    • $E[\cdot|x \geq y] \Rightarrow$ linear schedule in limit order book
    • $E[v|x] = v_0 + \gamma x \textit{ assumed} \Rightarrow$ linear uniform price schedule
  • $p^u(x) = v_0 + \frac{l-1}{l-2}\gamma x$ versus $p^d(x) = v_0 + \frac{l}{l-1}\frac{\gamma}{a} + \gamma x$
Limit Order Book vs. Uniform Pricing
Röell (1998)

Figure: Limit Order Book.
Limit Order Book vs. Uniform Pricing
Röell (1998)

Figure: Limit Order Book and Uniform Pricing.
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Sequential Trade Models à la Glosten & Milgrom (1985)

- order size is restricted to $x \in \{-1, +1\}$

Figure: Bid-Ask Spread.
Sequential Trade Models à la Glosten & Milgrom (1985)

- Monopolistic Market Maker - Copeland & Galai (1983)
  - bid-ask spread is partially due to monopoly power
    partially due to adverse selection
  - difficult to handle in multi-period setting
- Competitive Market Makers - Glosten & Milgrom (1985)
  - bid-ask spread is only due to adverse selection
  - multi-period setting
Glosten & Milgrom (1985)

- Model Setup
  - value of the stock $v$ and $\bar{v}$
  - with probability $\alpha$ an informed trader shows up
  - with probability $(1 - \alpha)$ an uninformed trader shows up
  - all traders are chosen from a pool of a continuum of traders, i.e., the probability that they will trade a second time is zero (rule out strategic considerations as in Kyle’s)
  - informed traders know true $\tilde{v}$: buys if $v > a$ sells if $v < b$.
  - uninformed traders buy with probability $\mu$ and sell with probability $1 - \mu$.
  - Note: Traders can only buy or sell 1 unit (No-Trade is also not allowed!)
Glosten & Milgrom (1985)

Figure: Tree.
Glosten & Milgrom (1985)  
Calculating Bid-Ask Spread

- Buy order

\[
\begin{align*}
    P(\overline{v}) &= \theta \\
    P(\text{buy}|\overline{v}) &= \alpha + (1 - \alpha) \mu \\
    P(\text{buy}|\overline{v}) &= (1 - \alpha) \mu \\
\end{align*}
\]

Bayes’ Rule

\[
\begin{align*}
    P(\overline{v}|\text{buy}) &= \frac{(\alpha + (1 - \alpha) \mu) \theta}{(\alpha + (1 - \alpha) \mu) \theta + (1 - \alpha) \mu (1 - \theta)} \\
    P(\overline{v}|\text{buy}) &= 1 - P(\overline{v}|\text{buy})
\end{align*}
\]
Glosten & Milgrom (1985) Calculating Bid-Ask Spread

- Sell order $P(\overline{v}|\text{sell}) = \frac{(1-\alpha)(1-\mu)\theta}{(1-\alpha)(1-\mu)\theta + [\alpha + (1-\alpha)(1-\mu)](1-\theta)}$

$P(\overline{v}|\text{buy}) > P(\overline{v}) > P(\overline{v}|\text{sell})$

$P(v|\text{buy}) < P(v) < P(v|\text{sell})$

- Market Maker makes zero expected profit (potential Bertrand competition)

\[ b = \text{bid} = E[v|\text{sell}] = \overline{v}P(\overline{v}|\text{sell}) + vP(v|\text{sell}) \]

\[ a = \text{ask} = E[v|\text{buy}] = \overline{v}P(\overline{v}|\text{buy}) + vP(v|\text{buy}) \]
Remarks to Glosten & Milgrom (1985)

1. Quotes are regret free
2. $\bar{v} < b < a < \bar{v}$
3. $(a - b) \rightarrow$ gain from liquidity traders $= \text{loss to insider}$
4. Bid-ask spread $(a - b)$ increases with $\alpha$
5. Over time price converge to true value
6. Prices follow a martingale: $E_t[p_{t+1} | I_t] = p_t$
   (Changes in prices are uncorrelated)
7. Simple setting price at $t$ depends only on $\#$ buy orders $-$ $\#$ sell orders
   (Sequence of trades does not matter)
8. Mid point of bid ask spread $\frac{a + b}{2}$ is not current market maker’s expectation.
• Easley and O’Hara (1987)
  • ‘small and large’ order size
    • noise traders submit randomly a small or a large sized order
    • informed traders always prefer large order size (if bid and ask is the same for both order sizes)
      ⇒ m.m. will set larger spread for large orders
  • Separating equilibrium
    • Informed traders’ order size is 2
    • Uninformed traders’ order size is 1 and 2 (exogenously given)
      ⇒ Spread for small orders = 0
  • Pooling equilibrium
    • Informed traders’ order size is 1 and 2
    • Uninformed traders’ order size is 1 and 2 (exogenously given)
      ⇒ Larger spread for larger orders
Extensions

• “event uncertainty” (also in Easley & O’Hara (1992))
  • with prob $\gamma$ info is like in Glosten & Milgrom
  • with prob $(1 - \gamma)$ no news event occurs
    (nobody receives a signal)

• No-Trade $\rightarrow$ signals that nothing has occurred!
  $\Rightarrow$ quotes will pull towards $\frac{1}{2}$
  updating
    1. whether event has occurred AND
    2. about true value of the stock

• transaction price is still a Martingale
  but no longer Markov!
Herding - Avery & Zemsky (1998)

- Relates Glosten-Milgrom model to herding models (BHW 1992)
- Price adjustment eliminates herding and informational cascades if market maker learns at the same speed as other informed traders.
- Herding can still arise in a more general setting with event uncertainty and a more complicated information structure which guarantees that the market maker learns at a slower speed compared to other traders.
1987-Crash
Jacklin, Kleiden & Pfleiderer (1992)

Figure: Underestimating portfolio insurance traders $\theta$. 