Asset Pricing under Asymmetric Information
Rational Expectations Equilibrium

Markus K. Brunnermeier
Princeton University

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      - uniform price setting
      - limit order book analysis
  - dynamic sequential trade models with multiple trading rounds
- strategic market order models where the market maker sets prices ex-post
Strategic Market Order Models - Overview

- Kyle (1985) model
  - static version
  - dynamic version (in discrete time)
    - Refresher in Dynamic Programming
  - continuous time version (Back 1992)
- Multi-insider Kyle (1985) version
- Other strategic market order models
Kyle 1985 Model

• Model Setup
  • asset return $v \sim \mathcal{N}(\mu_0, \Sigma_0)$
  • Agents (risk neutral)
    • Insider who knows $v$ and submit market order of size $x$
    • Noise trader who submit market orders of exogenous aggregate size $u \sim \mathcal{N}(0, \sigma_u^2)$
    • Market maker sets competitive price after observing net order flow $X = x + u$
  • Timing (order of moves)
    • Stage 1: Insider & liquidity traders submit market orders
    • Stage 2: Market Maker sets the execution price
  • Repeated trading in dynamic version
# Kyle 1985 Model — Static Version

<table>
<thead>
<tr>
<th>Single informed trader</th>
<th>(Competitive) Market Maker</th>
</tr>
</thead>
<tbody>
<tr>
<td>0) <strong>Information</strong></td>
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</tr>
<tr>
<td>( v := \text{asset's payoff} )</td>
<td>( X = x + u \text{ batch net order flow} )</td>
</tr>
<tr>
<td>1) <strong>Conjecture (price-rule)</strong></td>
<td>1) <strong>Conjecture (insider trading rule)</strong></td>
</tr>
<tr>
<td>( p = \mu + \lambda (x + u) )</td>
<td>( x = \alpha + \beta v )</td>
</tr>
<tr>
<td>2) <strong>No Updating</strong></td>
<td>2) <strong>Updating</strong> ( E[v</td>
</tr>
<tr>
<td>3) <strong>Optimal Demand</strong></td>
<td>3) <strong>Price Setting Rule</strong></td>
</tr>
<tr>
<td>( \max_x E[(v - p)</td>
<td>v]x )</td>
</tr>
<tr>
<td>( \max_x E[v - \mu - \lambda x</td>
<td>v]x )</td>
</tr>
<tr>
<td>FOC: ( x = -\frac{\mu}{2\lambda} + \frac{1}{2\lambda} v )</td>
<td>( p = p_0 + \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} {x + u - \alpha + \beta E[v]} )</td>
</tr>
<tr>
<td>SOC: ( \lambda &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>4) <strong>Correct Beliefs</strong></td>
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</tr>
<tr>
<td>( \alpha = -\frac{\mu}{2\lambda}, \beta = \frac{1}{2\lambda} )</td>
<td>( \mu = p_0 \text{ Martingale}, \quad \lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} )</td>
</tr>
</tbody>
</table>
Kyle 1985 Model — Static Version

- solve for unknown coefficients
  - 4 unknown Greeks
  - 4 equations

\[ \lambda = \frac{1}{2} \sqrt{\frac{\Sigma_0}{\sigma_u^2}} \]

- \( \lambda \) (illiquidity) decreases with noise trading, \( \sigma_u^2 \)
- \( \Sigma_0 \) reflect info advantage of insider
Dynamic Programming - A Refresher

• The Problem:
max for several periods \( t = 1, \ldots, T \) (discrete time)

\[
\max_{u_t} E_t \left[ \sum_{s=1}^{T} v_s (\tilde{x}_s, u_s) \right] \quad \forall t \Rightarrow \text{(sequential rationality)}
\]

under the following law of motion

\[
\tilde{x}_{t+1} = f_t (\tilde{x}_t, u_t, \tilde{\varepsilon}_t)
\]

\( \tilde{x}_t \): vector of \underline{state} variables (sufficient state space)
\( u_t \): vector of \underline{control} variables
\( \varepsilon_t \): vector of random \underline{shocks}

• Method
  • Backward Induction
  • Dynamic Programming
Dynamic Programming - A Refresher

- Define **Value Function**

\[
V_t (x_t) := E_t \left[ \sum_{s=t}^{T} v_s (\tilde{x}_s, u^*_s) \right] \quad \uparrow \text{optimal values}
\]

- ⇒ **Bellman Equation**

\[
\max_{u_t} E_t [v_t (x_t, u_t) + V_{t+1} (x_{t+1})]
\]

- Start at final **date** \( T \)

\[
V_{T+1} (\cdot) := 0
\]

⇒ in \( t = T \)

\[
\max_{u_T} E_T [v_T (x_T, u_T)]
\]

\[
\text{FOC} \Rightarrow u^*_T = g_T (x_T)
\]

⇒ \( V_T (x_T) = E_T [v_T (x_T, u^*_T)] \)
Dynamic Programming - A Refresher

- at date $T - 1$

$$
\max_{u_{T-1}, u_T} \mathbb{E}_{T-1} \left[ \sum_{s=T-1}^{T} v_s \left( \tilde{x}_s, u_s \right) \right]
$$
given $V_T (x_T)$

$$\Leftrightarrow \max_{u_{T-1}} \mathbb{E}_{T-1} \left[ v_{T-1} (x_{T-1}, u_{T-1}) + V_T (x_T) \right]$$
given law of motion

$$\Leftrightarrow \max_{u_{T-1}} \mathbb{E}_{T-1} \left[ v_{T-1} (x_{T-1}, u_{T-1}) + V_T (f_{T-1} (x_{T-1}, u_{T-1}, \tilde{\epsilon}_{T-1})) \right]$$

$$\Rightarrow u^*_{T-1} = g_{T-1} (x_{T-1})$$

$$\Rightarrow V_{T-1} = \mathbb{E}_{T-1} \left[ v_{T-1} (x_{T-1}, u^*_{T-1}) + V_T (f_{T-1} (x_{T-1}, u^*_{T-1}, \tilde{\epsilon}_{T-1})) \right]$$

- and so on for date $T - 2$ etc. (and if they didn’t die in the uncertainties they are still solving ...)

- This process is quite time consuming.
Dynamic Programming - A Refresher

Alternative way:

- **Step 1:** “Guess” the general form of the value function

\[
V_{t+1}(x_{t+1}) = H_{t+1}(x_{t+1})
\]

e.g. \( H_{t+1}(x_{t+1}) = \alpha_{t+1}x_{t+1}^2 \)

- **Step 2:** Derive optimal level of current control

\[
\max_{u_t} E_t \left[ v_t(x_t, u_t) + H_{t+1}(\tilde{x}_{t+1}) \right]
\]

\[
\max_{u_t} E_t \left[ v_t(x_t, u_t) + H_{t+1}(f_t(x_t, u_t, \varepsilon_t)) \right]
\]

\[
\Rightarrow u^*_t = \cdots
\]

- **Step 3:** Derive value function and check whether it coincides with general value function

\[
V_t(x_t) = E_t \left[ v_t(x_t, u^*_t) + H_{t+1}(f_t(x_t, u^*_t, \tilde{\varepsilon}_t)) \right]
\]

\[
? H_t(x_t) = \alpha_t x_t^2
\]
Kyle (1985) — Dynamic Version

Insider

- **Step 1:** Conjectured price setting strategy (pricing rule)

\[ p_n = p_{n-1} + \lambda_n \Delta X_n \]

\[ = p_{n-1} + \lambda_n (\Delta x_n + \Delta u_n) \quad \left( \frac{1}{\lambda_t} \approx \text{Liquidity} \right) \]

- **Step 2:** ‘Guess’ Value function for insider’s profit pricing rule is linear \( \rightarrow \) guess quadratic value function)

\[ E[\pi_{n+1} | \tilde{p}_1, \ldots, p_n, v] = \alpha_n (v - p_n)^2 + \delta_n \]

Information set up to \( n \)

(expected profit from time \( n + 1 \) onwards)

\[ \pi_n = E_n [\pi_{n+1} + (v - p_n) \Delta x_n^i] \]
Kyle (1985) — Dynamic Version

Insider ctd.

• **Step 3:** Write Bellman Equation

\[
\max_{\Delta x^i_n} E \left[ (v - p_n) \Delta x^i_n + \alpha_n (v - p_n)^2 + \delta_n | p_1, \ldots, p_{n-1}, v \right]
\]

• **Step 4:** Given insider’s beliefs \( p_n = p_{n-1} + \lambda_n \Delta X_n \)

\[
\max_{\Delta x^i_n} E \left[ \left( v - p_{n-1} - \lambda_n \Delta x^i_n - \lambda_n \Delta u_n \right) \Delta x^i_n + \alpha_n \left( v - p_{n-1} - \lambda_n \Delta x^i_n - \lambda_n u_n \right)^2 + \delta_n \right] | I_n
\]

Take expectations

\[
\max_{\Delta x^i_n} E \left[ \left( v - p_{n-1} - \lambda_n \Delta x^i_n \right) \Delta x^i_n + \alpha_n \left( v - p_{n-1} - \lambda_n \Delta x^i_n \right)^2 + \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n \right] \]

\( u \Rightarrow p_n \) noisy
Kyle (1985) — Dynamic Version

Insider ctd.

• **Step 5:** maximize

  FOC: \[(v - p_{n-1}) - 2\lambda_n \Delta x^i_n - 2\alpha_n \lambda_n (v - p_{n-1}) + 2\alpha_n \lambda_n^2 \Delta x^i_n = 0\]

  \[
  \Delta x^i_n = \frac{1 - 2\alpha_n \lambda_n}{2\lambda_n (1 - \alpha_n \lambda_n)} (v - p_{n-1})
  \]

  \[:= \beta_n \Delta t_n\]

  SOC: \[\lambda_n (1 - \alpha_n \lambda_n) > 0\]

• **Step 6:** Check whether ‘guessed’ value fcn is correct

  \[E [\pi | I_{n-1}] = \max_{\Delta x^i_n} E \left[ (v - p_n) \Delta x^i_n + \alpha_n (\tilde{v} - \tilde{p}_n)^2 + \delta_n | I_{n-1} \right]\]

  \[= \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1}, \text{ where}\]

  \[\alpha_{n-1} = \frac{1}{4\lambda_n (1 - \alpha_n \lambda_n)}, \delta_{n-1} = \delta_n + \alpha_n (\lambda_n)^2 \sigma_u^2 \Delta t_n\]
Kyle (1985) — Dynamic Version

**Market Maker (Filtering Problem)**

- **Step 1:** MM’s belief about insider’s strategy

  \[ \Delta x^i_n = \beta_n \Delta t_n (v - p_{n-1}) \]
  \[ \Delta X_n = \beta_n \Delta t_n (v - p_{n-1}) + \Delta u_n \]

  \[ \text{Var}[\Delta u_n] = \sigma_u^2 \Delta t_n \]

- **Step 2:** MM’s filtering problem

  By definition:

  \[ p_{n-1} : = E [v | \Delta X_1, \ldots, \Delta X_{n-1}] \]
  \[ \Sigma_{n-1} : = \text{Var} [v | \Delta X_1, \ldots, \Delta X_{n-1}] \]

  \[ E [\Delta X_n | \Delta X_1, \ldots, \Delta X_{n-1}] = \beta_n \Delta t_n E [(v - p_{n-1}) + \Delta u_n | \cdot] \]
  \[ \text{Var} [\Delta X_n | \cdot \cdot] = (\beta_n \Delta t_n)^2 \Sigma_{n-1} + \sigma_u^2 \Delta t_n \]
  \[ \text{Cov} [v, \Delta X_n | \cdot \cdot] = E [v (\beta_n \Delta t_n (v - p_{n-1}) + \Delta u_n | \cdot] \]

  \[ = \beta_n \Delta t_n \Sigma_{n-1} \]
Kyle (1985) — Dynamic Version

Now we have all ingredients for the Projection Theorem

\[ p_n = p_{n-1} + \frac{\beta_n \Delta t_n \sum_{n-1} \Delta X_n}{\left(\beta_n \Delta t_n\right)^2 \sum_{n-1} + \Delta t \sigma_u^2} \]

\[ := \lambda_n \]

\[ \sum_n = V[\nu | \cdots \Delta X_n] = \sum_{n-1} - \frac{\left(\beta_n \Delta t_n\right)^2 \sum_{n-1}^2}{(\beta_n \Delta t_n)^2 \sum_{n-1} + \Delta t \sigma_u^2} \]

\[ = \frac{\sigma_u^2 \sum_{n-1}}{(\beta_n)^2 \Delta t_n \sum_{n-1} + \sigma_u^2} \]

\[ \Rightarrow \]

\[ \lambda_n = \frac{\beta_n \Delta t_n \sum_{n-1}}{(\beta_n \Delta t_n)^2 \sum_{n-1} + \Delta t \sigma_u^2} \]

\[ \sum_n = (1 - \lambda_n \beta_n \Delta t_n) \sum_{n-1} = \frac{\sigma_u^2 \lambda_n}{\beta_n} \]

\[ \Rightarrow \lambda_n = \beta_n \sum_n/\sigma_u^2 \]


Kyle (1985) — Dynamic Version

- **Step 3:** Equate coefficients $\alpha_n, \beta_n, \delta_n, \sum_n$

  \[
  \beta_n \Delta t_n = \frac{1-2\alpha_n \lambda_n}{2\lambda_n(1-\alpha_n \lambda_n)}
  \]

  \[
  \alpha_{n-1} = \frac{1}{4\lambda_n(1-\alpha_n \lambda_n)}
  \]

  \[
  \delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \sigma^2_u \Delta t_n
  \]

  \[
  \sum_n = \sigma^2_u \sum_{n-1}
  \]

  \[
  \lambda_n = \ldots
  \]

  Solve recursive system of equations.

- **Interpretation of Equilibrium**
  - restrain from aggressive trading
    - price impact in current trading round
    - price impact in all future trading rounds
  - ...

...
Generalizations of Kyle (1985)

- Multiple Insiders
  - all have same information
  - all hold different information
  - information is correlated
    ⇒ see Foster & Viswanathan, JF 51, 1437-1478

- Risk averse insiders
  - CARA utility

- etc. etc.