The Fiscal Theory of the Price Level with a Bubble*

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Abstract

This paper incorporates a bubble term in the standard FTPL equation to explain why countries with persistently negative primary surpluses can have a positively valued currency and low inflation. It also provides an example with closed-form solutions in which idiosyncratic risk on capital returns depresses the interest rate on government bonds below the economy's growth rate.

JEL: E44, E52, E63

1 Introduction

Different monetary theories emphasize different roles of money and different equilibrium equations to determine the price level. The Fiscal Theory of the Price Level (FTPL) stresses the role of money as a store of value and argues that the real value of all outstanding government debt, i.e., the nominal debt level divided by the price level, is given by the discounted stream of future primary government surpluses. Primary surpluses are the difference between government revenue and expenditures excluding interest payments. Absent government default, an increase in primary deficits leads to an increase in the price level, i.e., inflation, by devaluing outstanding debt.

Critics of the FTPL often point to Japan. Even though Japan has mostly run primary deficits since the 1960s (see Figure 1) and with no primary surpluses in sight, the price level has not risen much. Indeed, inflation levels are depressed even though the government and central bank leave no stone unturned to boost inflation closer to 2%.

In this paper, we revisit the key FTPL equation and argue that including the typically ignored bubble term allows us to reconcile the FTPL with Japan's experience. Indeed, we show that the transversality condition is often insufficient to rule out a bubble on the aggregate economy, refuting the usual justification to simply dismiss the bubble term. While the FTPL literature puts a lot of emphasis on distinguishing between monetary and fiscal dominance, the bubble term cannot be ignored under any policy regime.

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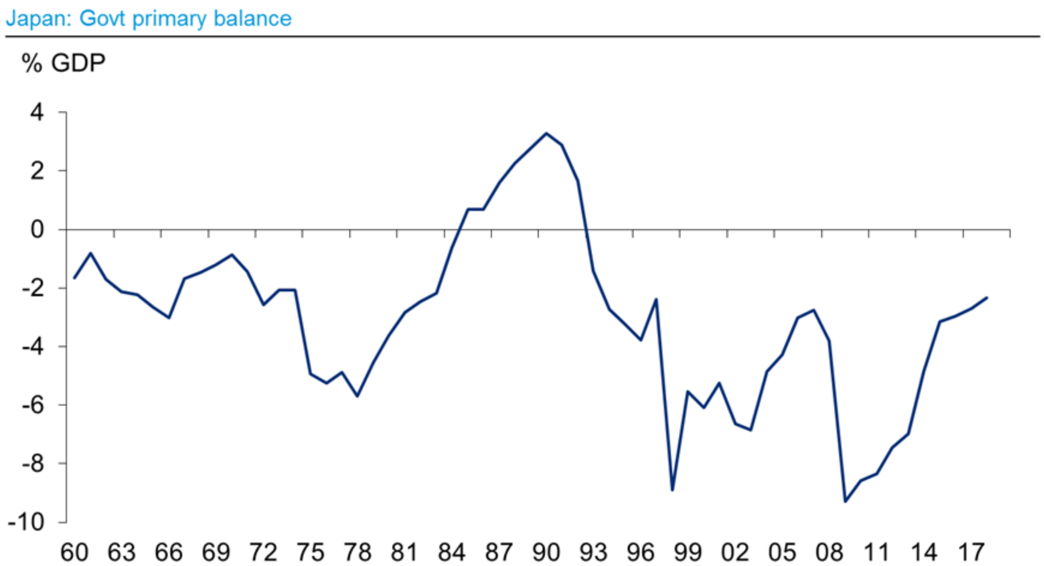

Figure 1: Japanese primary surplus 1960–2017

A bubble term emerges whenever the real rate paid on government debt is persistently below the growth rate of the economy, i.e., whenever $r \leq g$. It is well known that this can be the case in overlapping generations models (Samuelson 1958), models of perpetual youth (Blanchard 1985), and incomplete market models à la Bewley (1980). In this paper we provide another simple example based on Brunnermeier and Sannikov (2016a,b) in which the $r \leq g$ outcome arises naturally and agents can invest in both physical capital and government bonds. Physical capital is subject to uninsurable idiosyncratic return risk. Hence, the expected return on capital exceeds the growth rate g since agents require a risk premium. Government bonds are the safe asset in the economy and allow agents to indirectly share part of their idiosyncratic risk. High idiosyncratic risk makes the government bond more attractive and depresses r below g.

By "printing" bonds at a faster rate, the government imposes an inflation tax that reduces the return on the bonds further. Since government bonds are a bubble, the government in a sense "mines a bubble" to generate seigniorage revenue. The resulting seigniorage revenue can be used to finance government expenditures without ever having to raise extra taxes.

For example, if the primary surpluses are always negative, then their discounted stream is also negative, and only the positive value of the bubble can ensure a positive price level. The size of the bubble, and hence the price level, is determined by wealth effects and goods market clearing. A larger bubble raises agents' wealth and hence their demand for output. To ensure goods market clearing, the bubble has to take on a certain size, which together with the FTPL equation determines the price level.

The price level is uniquely determined if the fiscal authority backs the bubble to rule out equilibria that lead to hyperinflation. Such fiscal backing is only required off-equilibrium.

Literature. Classic references for the FTPL are Leeper (1991), Sims (1994), and Woodford (1995). For more comprehensive treatments see Leeper and Leith (2016) and a recent book draft by Cochrane (2019). All of these references consider bubble-free environments. An exception is Bassetto and Cui (2018) who study the validity of the FTPL in low interest rate environments. Our contribution differs to theirs in two ways. First, they focus exclusively on price determinacy and do not discuss the existence of a bubble and its implication for the government budget. Second, in the context of an example model that does feature the possibility of a bubble, a dynamically inefficient OLG setting, they conclude that the FTPL breaks down while we show in our Section 3.4 how fiscal price level determination can succeed in the presence of a bubble.

There is an extensive literature on rational bubbles. Survey papers include Miao (2014) and Martin and Ventura (2018). More recently, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) provide convincing empirical evidence that U.S. government debt has a bubble component.

2 The FTPL Equation with a Bubble

In this section, we derive the key equation of the fiscal theory of the price level in a generic partial equilibrium setting. We subsequently discuss when there may be a bubble term in this equation that has previously been ignored in the literature and briefly conclude what the general fiscal theory equation tells us about sources of seigniorage. In the following section, we elaborate more on these points in the context of a specific example in general equilibrium.

2.1 Revisiting the Derivation of the FTPL Equation

The derivation of the fiscal theory equation starts with the government flow budget constraint. In discrete time, this constraint is given by

$$\mathcal{B}_t + \mathcal{M}_t + \mathcal{P}_t T_t = (1 + i_{t-1})\,\mathcal{B}_{t-1} + \left(1 + i^m_{t-1}\right)\mathcal{M}_{t-1} + \mathcal{P}_t G_t,$$

where $\mathcal{B}_t$ is the nominal face value of outstanding government bonds, $\mathcal{M}_t$ is the nominal quantity of money in circulation, $\mathcal{P}_t$ is the price level, T_t are (real) taxes, G_t is (real) government spending, and i_t, i^m_t are the nominal interest rates paid on bonds and money, respectively. i^m_t can be smaller than i_t if money provides transaction services. If ξ_t is a real SDF process that prices government bonds, then $1 = \mathbb{E}_t\left[\xi_{t+1}/\xi_t \cdot \mathcal{P}_t/\mathcal{P}_{t+1}\,(1 + i_t)\right]$. Using this property, dividing the government budget constraint by $\mathcal{P}_t$ and rearranging yields

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}(1 + i_{t-1}) = T_t - G_t + \overbrace{\left(i_{t-1} - i^m_{t-1}\right)}^{\Delta i_{t-1} :=}\frac{\mathcal{M}_{t-1}}{\mathcal{P}_t} + \mathbb{E}_t\left[\frac{\xi_{t+1}}{\xi_t}(1 + i_t)\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_{t+1}}\right].$$

Iterating this forward until period T implies

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}(1 + i_{t-1}) = \mathbb{E}_t\left[\sum_{s=t}^{T}\frac{\xi_s}{\xi_t}(T_s - G_s)\right] + \mathbb{E}_t\left[\sum_{s=t}^{T}\frac{\xi_s}{\xi_t}\Delta i_{s-1}\frac{\mathcal{M}_{s-1}}{\mathcal{P}_s}\right] + \mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right].$$

Up to this point, we have merely rearranged and iterated the government budget constraint and assumed that there is some SDF process ξ_t that prices government bonds in equilibrium. To derive the fiscal theory equation, the literature now typically proceeds by invoking a private-sector transversality condition to eliminate a terminal value of government debt when passing to the limit $T \to \infty$. In this paper, we focus on environments where the transversality condition does not eliminate the terminal value in the limit. When taking the limit $T \to \infty$, we therefore arrive at the more general equation[1]

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}(1 + i_{t-1}) = \underbrace{\mathbb{E}_t\left[\sum_{s=t}^{\infty}\frac{\xi_s}{\xi_t}(T_s - G_s)\right]}_{\text{PV of primary surpluses}} + \underbrace{\mathbb{E}_t\left[\sum_{s=t}^{\infty}\frac{\xi_s}{\xi_t}\Delta i_{s-1}\frac{\mathcal{M}_{s-1}}{\mathcal{P}_s}\right]}_{\text{PV of future transaction services}} + \underbrace{\lim_{T\to\infty}\mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right]}_{\text{bubble}}.$$

From now on, we switch to continuous time in order to make our formal arguments more elegant. The continuous-time version of the last equation is given by[2]

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \mathbb{E}_t\left[\int_t^{\infty}\frac{\xi_s}{\xi_t}(T_s - G_s)\,ds\right] + \mathbb{E}_t\left[\int_t^{\infty}\frac{\xi_s}{\xi_t}\Delta i_s\frac{\mathcal{M}_s}{\mathcal{P}_s}ds\right] + \lim_{T\to\infty}\mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right]. \qquad (1)$$

This equation for the real value of government debt holds in any monetary model. While most conventional monetary models treat this equation as an intertemporal government budget constraint that holds on- and off-equilibrium, in the FTPL it is an equilibrium condition that determines the price level.

2.2 When Can a Bubble Exist?

Equation (1) differs from the standard fiscal theory only by the presence of an additional bubble term. When can this bubble term be nonzero? Well-known examples are bubbles in OLG (Samuelson 1958) and perpetual youth (Blanchard 1985) models. In Section 3 we present another example with incomplete idiosyncratic risk sharing. Here, we make some generic points that apply to all example models. For tractability, let us focus on environments with a stationary debt-to-GDP ratio and no risk. In this case, the real value of government debt $\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}$ is $\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}e^{g(T-t)}$, where g is the growth rate of the economy, and $\xi_T/\xi_t = e^{-r^f(T-t)}$ with r^f denoting the real risk-free rate. By substituting these expressions into equation (1), we see that the bubble term does not vanish in the limit if $r^f \leq g$. More generally, the correct risk-adjusted discount rate compensating for the real risk inherent in $\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}$ must be used in the comparison instead of the risk-free rate to determine whether a bubble is possible.

For any agent with recursive isoelastic utility (which includes CRRA utility) that is marginal in the market for government debt, the risk-free rate is (for environments with non-stochastic investment opportunities)

$$r^f = \rho + \psi^{-1}\mu^c - \frac{\gamma\left(1 + \psi^{-1}\right)}{2}\|\sigma^c\|^2, \qquad (2)$$

where $\rho > 0$ is the agent's time preference rate, γ is the relative risk aversion coefficient, ψ is the EIS, μ^c is the growth rate of agent-specific consumption, and σ^c is a vector or relative risk exposures of agent-specific consumption to Brownian risk factors.[3] $\|\cdot\|$ denotes the standard Euclidian norm. This

[1] Mathematically, the sum of the three limits in the decomposition below may not be well-defined, even if the limit of the sum is. In this case, the right-hand side should be interpreted as the limit of the sum. For instance, this can happen if the bubble term is ∞, but the present value of surpluses is $-\infty$. While this may seem a pathological case, it can make sense economically because the bubble and surpluses are not separately tradeable, but necessarily bundled together to one asset: government debt. As long as the value of this asset is well-defined and finite, infinite subcomponents do not imply arbitrage opportunities or infinite utility.

[2] A formal derivation can be found in Online Appendix A.1.

[3] Here, we assume that all risk takes the form of Brownian risk. The intuition derived from the argument is unaltered if more general sources of consumption risk are permitted.

equation is linked to the growth rate of the economy through individual consumption growth μ^c – e.g., in a representative agent economy with a balanced growth path $\mu^c = g$.

Equation (2) suggests three reasons for the growth rate to exceed the risk-free rate. First, a higher growth rate g may not always imply higher individual consumption growth μ^c. This is the case, for example, in an OLG model with population growth. Second, if agents do not have strong desires to smooth consumption (large ψ), then the risk-free rate is very insensitive to changes in consumption growth and an increase in growth does not translate into an increase in the risk-free rate. Third, large individual risk exposure (large $\|\sigma^c\|^2$) or risk aversion (large γ) may depress the risk-free rate through the last term in equation (2) and offset any positive effects of growth g on r^f through $\psi^{-1}\mu^c$. In the example we provide below, the latter channel generates a bubble in government debt. However, the main insights we generate from this example are insensitive to this choice and would equally apply to other environments in which a bubble term in equation (1) is possible.

The possibility of $r^f \leq g$ is not merely a theoretical curiosity. Historically, real interest rates on government bonds of advanced economies have mostly been below the growth rate. Even Abel, Mankiw, Summers, and Zeckhauser (1989), who are often cited as providing evidence against the existence of rational bubbles, report that the safe interest rate r^f is smaller than g. With the more recent decline in r^f, as stressed by Blanchard (2019), the evidence for $r^f < g$ has become more clear-cut. See also Geerolf (2013).

2.3 Three Forms of Seigniorage

Equation (1) suggests three forms of seigniorage, which here we define simply as government spending that is not backed by offsetting future taxes. The first takes the form of a dilution of private claims to future primary surpluses through surprise devaluations of existing government debt or money.[4] Under rational expectations, this cannot be a regular source of revenue for governments. For the U.S., Hilscher, Raviv, and Reis (2014) assess the possibility of future surprise devaluations based on option-implied (risk-neutral) probabilities and conclude that this form of seigniorage is perceived to be a negligible source of revenue. The likelihood of a devaluation exceeding 5% of GDP is less than 1%.

A second form of seigniorage comes from exploiting the liquidity benefits (convenience yield) of "narrow" money ($\mathcal{M}$ in equation (1)). This form of seigniorage can only be extracted from the portion of government debt that takes the form of "narrow" money and provides liquidity services. It depends on the interest rate differential $\Delta i = i - i^m$ between illiquid and liquid government debt. It is small if either that differential is small or if the stock of "narrow" money is only a small part of total government debt.[5] This form of seigniorage is not an important funding source for advanced economies. For example, in the U.S., Reis (2019) reports a flow revenue of approximately 0.36% of GDP and estimates a present value of $\approx$ 20% and, at most, 30% of GDP. Moreover, in the future the Δi term is likely to decline, because central banks pay interest on reserves and as money becomes more digitalized, its velocity rises.

Besides these standard forms of seigniorage, equation (1) suggests a third form of seigniorage that has remained unexplored in prior work and is the focus of this paper. The government can "mine" the bubble by using its outstanding government debt to run an ever-expanding Ponzi scheme: letting the stock of

[4] Without long-term debt as in equation (1) such dilution must work through a sudden surprise inflation (an unexpected upward jump in $\mathcal{P}_t$). In a more realistic setting with long-term debt, news of higher inflation going forward would have similar effects and work through bond prices instead of the general price level.

[5] In reality, one has to distinguish between reserves, whose quantity is nonnegligible, but which pay interest and have therefore a small Δi_t, and cash, which has a much larger Δi_t, but whose quantity is almost negligible relative to the overall stock of government debt.

government debt grow generates a steady revenue flow that does not have to be paid for by future taxes as long as a bubble term is present in equation (1). Unlike a surprise dilution through inflation, dilution of the bubble value is even feasible if it is fully expected by the private sector. This form of seigniorage is arguably larger than the officially measured seigniorage from growing narrow money $\mathcal{M}$ because all revenue from growing $\mathcal{B}+\mathcal{M}$ is relevant for bubble mining.

3 A Simple Example with a Bubble: "I Theory without I"

There are several model structures in which rational bubbles can exist and thus the bubble term in equation (1) does not necessarily disappear. We illustrate this in a simple example with incomplete idiosyncratic risk sharing, where a bubble and a productive asset (capital) can coexist, yet the model is sufficiently simple to have a closed-form solution. The model is a streamlined version of the "I theory of money" (Brunnermeier and Sannikov, 2016a) without banks and has been previously analyzed in Brunnermeier and Sannikov (2016b) and Di Tella (2019). Here, we add fiscal policy and reinterpret money in their model as bonds. For simplicity, we abstract from the presence of additional "narrow" money that yields transaction benefits.[6] After introducing the model, we discuss under which conditions the private-sector transversality condition is insufficient to rule out the bubble, how the standard fiscal theory argument for price level determination has to be adjusted to still guarantee a uniquely determined price level, and how the bubble can be "mined" for fiscal spending.

3.1 A Model with Idiosyncratic Return Risk

Environment. There is a continuum of households indexed by $i \in [0,1]$. All households have identical logarithmic preferences

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t^i dt\right]$$

with discount rate ρ.[7]

Each agent operates one firm that produces an output flow $ak_t^i dt$, where k_t^i is the capital input chosen by the firm. Absent market transactions of capital, capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \tilde{\sigma}d\tilde{Z}_t^i,$$

where $\iota_t^i k_t^i dt$ are physical investment expenditures of firm i (in output goods), Φ is a concave function that captures adjustment costs in capital accumulation, δ is the depreciation rate, and $\tilde{Z}^i$ is an agent-specific Brownian motion that is i.i.d. across agents i. $\tilde{Z}^i$ introduces firm-specific idiosyncratic risk. To obtain simple closed-form expressions, we choose the functional form $\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.

The key friction in the model is that agents are not able to share idiosyncratic risk. While they are allowed to trade physical capital and risk-free assets, they cannot write financial contracts contingent on

[6] Other than adding an additional source of seigniorage, including transaction benefits into the analysis does not substantially alter our conclusions. For the FTPL with transaction benefits but no bubble, see, e.g., Sims (2019).

[7] We assume log utility for maximum simplicity. At the expense of heavier notation and more involved algebra, the model can be solved in closed form with isoelastic recursive utility with EIS kept at 1 and arbitrary coefficient of relative risk aversion γ. The expressions for the model solution p, q, ι, ϑ remain identical under these generalized preferences, except that $\tilde{\sigma}$ has to be replaced everywhere with $\sqrt{\gamma}\tilde{\sigma}$.

individual $\tilde{Z}^i$ histories. As a consequence, all agents have to bear the idiosyncratic risk inherent in their physical capital holdings.

Besides households, there is a government that funds government spending, imposes taxes on firms, and issues nominal government bonds. The government has an exogenous need for real spending $\mathfrak{g}K_t dt$, where K_t is the aggregate capital stock and $\mathfrak{g}$ is a model parameter. The government imposes a proportional output tax (subsidy, if negative) τ_t on firms. Outstanding nominal government debt has a face value of $\mathcal{B}_t$ and pays nominal interest i_t. $\mathcal{B}_t$ follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}}\mathcal{B}_t dt$, where the growth rate $\mu_t^{\mathcal{B}}$ is a policy choice of the government. In short, the government chooses the policy instruments τ_t, i_t, $\mu_t^{\mathcal{B}}$ contingent on histories of prices taking $\mathfrak{g}$ as given and subject to the nominal budget constraint[8]

$$i_t\mathcal{B}_t + \mathcal{P}_t\mathfrak{g}K_t = \mu_t^{\mathcal{B}}\mathcal{B}_t + \mathcal{P}_t\tau_t a K_t, \tag{3}$$

where $\mathcal{P}_t$ denotes the price level as in Section 2.

The model is closed by the aggregate resource constraint

$$C_t + \mathfrak{g}K_t + \iota_t K_t = aK_t, \tag{4}$$

where $C_t := \int c_t^i di$ is aggregate consumption and $\iota_t = \int \iota_t^i k_t^i / K_t di$ is the average investment rate.

Price Processes and Returns. Let q_t^K be the market price of a single unit of physical capital. Then, $q_t^K K_t$ is private capital wealth. Let further $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$ be the ratio of the real value of government debt to total capital in the economy.[9] Then, the real value of the total stock of government bonds is $q_t^B K_t$ and the real value of a single government bond is $\frac{q_t^B K_t}{\mathcal{B}_t}$. It is convenient to define the share of total wealth in the economy that is due to bond wealth,

$$\vartheta_t := \frac{q_t^B K_t}{\left(q_t^B + q_t^K\right) K_t}.$$

We postulate that q_t^B and q_t^K have a generic deterministic evolution

$$dq_t^B = \mu_t^{q,B} q_t^B dt, \qquad dq_t^K = \mu_t^{q,K} q_t^K dt.$$

Whenever $q_t^B, q_t^K \neq 0$, the unknown drifts $\mu_t^{q,B}, \mu_t^{q,K}$ are uniquely determined by the local behavior of q_t^B and q_t^K, respectively. In the following, we also use the notation $\mu_t^{\vartheta} := \frac{\dot{\vartheta}_t}{\vartheta_t} = (1-\vartheta_t)\left(\mu_t^{q,B} - \mu_t^{q,K}\right)$.

Households can trade two assets in positive net supply (if $q_t^B \neq 0$), bonds and capital. Assume that in equilibrium $\iota_t = \iota_t^i$ for all i (to be verified below) such that aggregate capital grows deterministically at rate $\Phi(\iota_t) - \delta$. Then, the return on bonds is

$$dr_t^{\mathcal{B}} = i_t dt + \frac{d\left(q_t^B K_t/\mathcal{B}_t\right)}{q_t^B K_t/\mathcal{B}_t} = \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} - \overbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}^{=:\breve{\mu}_t^{\mathcal{B}}} dt = \left(\Phi(\iota_t) - \delta + \mu_t^{q,B} - \breve{\mu}_t^{\mathcal{B}}\right) dt. \tag{5}$$

[8]At this point, we do not impose additional restrictions on government policy. In particular, policy can be characterized by either monetary dominance or fiscal dominance and the choice of policy regime is irrelevant for most of our discussion. We do make more restrictive assumptions on policy in Section 3.4 where we explain how one can adjust the fiscal theory arguments for price level determination based on fiscal dominance if government debt has a bubble component.

[9]It is more convenient to work with this normalized version of the inverse price level $1/\mathcal{P}_t$, because the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, whereas q_t^B does not.

The return on agent i's capital is

$$dr_t^{K,i}\left(\iota_t^i\right) = \frac{(1-\tau_t)\,a - \iota_t^i}{q_t^K} + \frac{d(q_t^K k_t^i)}{q_t^K k_t^i} = \left(\frac{(1-\tau_t)\,a - \iota_t^i}{q_t^K} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right)dt + \tilde{\sigma}d\tilde{Z}_t^i.$$

Using the government budget constraint (3) to substitute out $\tau_t a$ yields

$$dr_t^{K,i}\left(\iota_t^i\right) = \left(\frac{a - \mathfrak{g} - \iota_t^i}{q_t^K} + \frac{q_t^B}{q_t^K}\breve{\mu}_t^{\mathcal{B}} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right)dt + \tilde{\sigma}d\tilde{Z}_t^i.$$

Household Problem and Equilibrium. Denote by n_t^i the net worth of household i and let θ_t^i be the fraction of net worth invested into bonds. Then net worth evolves according to

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + dr_t^{\mathcal{B}} + \left(1-\theta_t^i\right)\left(dr_t^{K,i}\left(\iota_t^i\right) - dr_t^{\mathcal{B}}\right). \tag{6}$$

The household chooses consumption c_t^i, real investment ι_t^i, and the portfolio share θ_t^i to maximize utility V_0^i subject to (6). The HJB equation for this problem is (using the returns expressions from the previous paragraph)

$$\rho V_t\left(n^i\right) - \partial_t V_t\left(n^i\right) = \max_{c^i,\theta^i,\iota^i}\left\{\log c^i + V_t'\left(n^i\right)\left[-c^i + n^i\left(\frac{dr_t^{\mathcal{B}}}{dt} + \left(1-\theta^i\right)\overbrace{\left(\frac{a-\mathfrak{g}-\iota^i}{q_t^K} + \Phi\left(\iota^i\right) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right)}^{=\frac{\mathbb{E}_t\left[dr_t^{K,i}\left(\iota_t^i\right)\right]}{dt} - \frac{dr_t^{\mathcal{B}}}{dt}}\right)\right]\right.$$
$$\left. + \frac{1}{2}V_t''\left(n^i\right)\left(n^i\right)^2\left(1-\theta^i\right)^2\tilde{\sigma}^2\right\}.$$

This is a standard consumption-portfolio-choice problem, so we conjecture a functional form $V_t\left(n^i\right) = \alpha_t + \frac{1}{\rho}\log n_t^i$ for the value function,[10] where α_t depends on (aggregate) investment opportunities, but not on individual net worth n^i. Substituting this into the HJB and taking first-order conditions yields[11]

$$q_t^K = \frac{1}{\Phi'\left(\iota_t^i\right)}, \qquad \text{Tobin's } q$$
$$c_t^i = \rho n_t^i, \qquad \text{permanent income consumption}$$
$$\frac{a-\mathfrak{g}-\iota_t}{q_t^K} - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t} = \left(1-\theta_t^i\right)\tilde{\sigma}^2. \qquad \text{Merton portfolio}$$

Using the functional form $\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota)$ and goods market clearing (4), the first two equations aggregated across agents imply

$$\iota_t = \frac{(1-\vartheta_t)(a-\mathfrak{g}) - \rho}{1-\vartheta_t+\phi\rho},$$
$$q_t^B = \vartheta_t\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho},$$
$$q_t^K = (1-\vartheta_t)\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho},$$

[10] We relegate the technical but standard verification argument to Online Appendix A.2.

[11] In particular, the first condition verifies $\iota_t^i = \iota_t$, and the last condition already uses this fact to eliminate $\Phi\left(\iota^i\right) - \Phi(\iota_t)$.

which determines the equilibrium uniquely up to the nominal wealth share ϑ_t. Bond market clearing and the fact that all households choose the same θ_t^i imply $\theta_t^i = \vartheta_t$ and substituting this and goods market clearing into the first-order condition for θ^i gives the additional condition (after solving for μ^ϑ)

$$\mu_t^\vartheta = \rho + \breve{\mu}_t^{\mathcal{B}} - (1 - \vartheta_t)^2 \tilde{\sigma}^2.$$

This is a backward equation for ϑ_t that has been derived under the assumption that bonds have a positive value ($\vartheta_t > 0$). In particular, in these cases multiplying the equation by ϑ_t represents an equivalence transformation. Furthermore, if $\vartheta_t = 0$, then by no arbitrage agents must expect also $\dot{\vartheta}_t = 0$; otherwise, they could earn an infinite risk-free return from investing into bonds. Consequently, the ODE

$$\dot{\vartheta}_t = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - (1 - \vartheta_t)^2 \tilde{\sigma}^2\right) \vartheta_t \tag{7}$$

must hold along any equilibrium path, regardless of whether bonds have positive value or not.

Steady-State Equilibria. We now focus on government policies that hold $\breve{\mu}^{\mathcal{B}}$ and τ constant over time and consider steady-state equilibria with constant q^B and q^K – and thus constant ϑ. All such equilibria must solve equation (4) with $\dot{\vartheta} = 0$. The right-hand side is a third-order polynomial, so there are three solutions to this equation. $\vartheta = 0$ is always a valid equilibrium. In this equilibrium, government bonds have no value, $q^B = 0$, the price level is infinite, $\mathcal{P} = \infty$, and the government does not raise primary surpluses, $\tau a = \mathfrak{g}$. Substituting $\vartheta = 0$ into the equations for ι, q^B, and q^K yields the nonmonetary steady state

$$\iota^0 = \frac{a - \mathfrak{g} - \rho}{1 + \phi\rho}, \qquad q^{B,0} = 0, \qquad q^{K,0} = \frac{1 + \phi\,(a - \mathfrak{g})}{1 + \phi\rho}.$$

The other stationary solutions to (7) are $1 - \vartheta = \pm\frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$. The "$-$" solution has always the property $\vartheta > 1$ and would therefore imply either a negative capital price (if $q^B > 0$) or a negative value of government bonds. Both cases violate free disposal and therefore this solution cannot be a valid equilibrium. The remaining solution $\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$ corresponds to a valid equilibrium, if $\tilde{\sigma} \geq \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}$. In this case, there is a second monetary steady state

$$\iota = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(a - \mathfrak{g}) - \rho\tilde{\sigma}}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \qquad q^B = \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \qquad q^K = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}.$$

In the following we focus exclusively on this monetary steady state. In particular, from now on we make the assumption $\tilde{\sigma} \geq \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}$. In Section 3.4 we show that an off-equilibrium modification to the fiscal policy rule can select this steady state as the unique equilibrium.

3.2 Transversality Condition and Existence of a Bubble

In our model, government debt can have value even in the absence of primary surpluses ($\breve{\mu}^{\mathcal{B}} \geq 0$) because it has a bubble component. It provides a store of value that is free of idiosyncratic risk, which allows agents to self-insure against their risk exposure. In this subsection we discuss why the private-sector transversality condition may not rule out the existence of a bubble.

For each individual agent, the transversality condition,

$$\lim_{T \to \infty} \mathbb{E}\left[\xi_T^i n_T^i\right] = 0,$$

is necessary for an optimal choice. We can write $n_T^i = n_T^{b,i} + n_T^{k,i}$, where $n_T^{b,i}$ is bond wealth at time T and $n_T^{k,i}$ is capital wealth at time T. Because $n^{b,i}, n^{k,i} \geq 0$, the transversality condition for total wealth also implies individual transversality conditions

$$\lim_{T\to\infty} \mathbb{E}\left[\xi_T^i n_T^{b,i}\right] = 0, \qquad \lim_{T\to\infty} \mathbb{E}\left[\xi_T^i n_T^{k,i}\right] = 0$$

for bond wealth and capital wealth, respectively. The former condition seems to suggest that there is a transversality condition on government bonds that should rule out a bubble. This argument overlooks that individual bond wealth $n_T^{b,i}$ is optimally chosen to be risky, because agents constantly adjust their portfolio in response to idiosyncratic shocks. It is therefore discounted by the individual agent at a discount rate that takes into account the idiosyncratic risk. Specifically, $n_T^{b,i} = \vartheta n_T^i$ and the SDF is $\xi_T^i = e^{-\rho T}\frac{1}{n_T^i}$, so $\mathbb{E}\left[\xi_T^i n_T^{b,i}\right] = e^{-\rho T}\vartheta$ and thus the individual transversality condition on bond wealth is trivially satisfied in the model. Yet, when valuing a marginal additional unit of bonds, the relevant discount rate from the perspective of all agents is the risk-free rate, because government bonds do not have idiosyncratic risk. Formally, the marginal time 0 valuation of any agent i of the entire time T bond wealth in the economy is

$$\mathbb{E}\left[\xi_T^i \int n_T^{b,j} dj\right] = \mathbb{E}\left[\xi_T^i q^B K_T\right] = e^{-r^f T} q^B K_T = e^{\left(g-r^f\right)T} q^B K_0$$

and the latter expression does not converge to zero, if $r^f \leq g$.

Nothing in the model implies that the risk-free rate must necessarily be above the growth rate. Indeed, the growth rate of the economy equals the growth rate of capital, $g = \Phi(\iota) - \delta$, and the risk-free rate equals the return on bonds, by equation (5) (in steady state)

$$r^f = \Phi(\iota) - \delta - \breve{\mu}^{\mathcal{B}} = g - \breve{\mu}^{\mathcal{B}}. \tag{8}$$

Consequently, $r^f \leq g$, if $\breve{\mu}^{\mathcal{B}} \geq 0$. A nonnegative value of $\breve{\mu}^{\mathcal{B}}$ is consistent with the existence condition of a monetary equilibrium, $\tilde{\sigma} \geq \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}$, as long as idiosyncratic risk is sufficiently large.

3.3 Mining the Bubble

In this section, we show how the government can mine a bubble, i.e., finance government expenditures without ever raising taxes for it.

Primary surplus is defined as $T_t - G_t = \tau a K_t - \mathfrak{g} K_t =: sK_t$. Due to our assumptions on fiscal policy, it grows at the same rate as K. From the government budget constraint (3), $s = -\breve{\mu}^{\mathcal{B}} q^B$. Hence, in our model the fiscal theory equation (1) reduces to (this time more precisely with a limit of the sum):[12]

$$q^B K_0 = \lim_{T\to\infty}\left(\underbrace{\int_0^T e^{-(r^f-g)t} sK_0 dt}_{=:PVS_{0,T}} + e^{-(r^f-g)T} q^B K_0\right).$$

Provided $q^B > 0$, equation (8) implies precisely three cases:[13]

[12] Since the model does not include "narrow" money, there is no Δi term.

[13] The apparent dichotomy, a positive bubble value and nonnegative surpluses or positive surpluses and no bubble, is due to the steady-state nature of our analysis. In a more general model, a positive present value of surpluses and a bubble can coexist.

1. $s > 0, \breve{\mu}^{\mathcal{B}} < 0$: then $r^f > g$, $PVS_{0,\infty} > 0$ and a bubble cannot exist. This is the "conventional" situation considered in the literature.

2. $s = \breve{\mu}^{\mathcal{B}} = 0$: then $r^f = g$, $PVS_{0,\infty} = 0$ and there is a finite positive bubble whose value exactly equals $q^B K_0$ and grows over time at the growth rate/risk-free rate.

3. $s < 0, \breve{\mu}^{\mathcal{B}} > 0$: then $r^f < g$ and thus the integral $PVS_{0,T}$ converges to $-\infty$ as $T \to \infty$. Yet, q^B is still positive, which is only possible if there is an offsetting infinite positive bubble. These infinite values do not violate any no-arbitrage condition and are also not otherwise economically problematic, since the bubble cannot be traded separately from the claim on surpluses. Both are necessarily bundled in the form of government bonds. As long as $\frac{\mathcal{B}_t}{\mathcal{P}_t} = q^B K_t$ is determined and finite in equilibrium, the model remains economically and mathematically sensible despite the infinite values in the decomposition of the value of government bonds.[14]

In all three cases, the (possible) presence of a bubble grants the government some extra leeway. Clearly in case 3, the government can run a perpetual deficit, "mine the bubble" and never has to raise taxes to fully fund all government expenditures. In case 2, the existence of the bubble is beneficial, because the value of government debt is positive – allowing agents to share part of their idiosyncratic risk – despite the fact that the present value of primary surpluses is zero. Even in case 1, government debt is more sustainable since an unexpected drop of primary surpluses to zero results in a bubble instead of a total collapse of the value of debt.

3.4 Price Level Determination, Uniqueness, and Off-Equilibrium Policy

The key equation (1) of the fiscal theory of the price level without a bubble term can be solved for the price level as a function of the present value of primary surpluses and the outstanding quantity of nominal government debt. For a given real allocation and initial quantity of debt, this equation alone therefore pins down the price level.[15] In the fiscal theory with a bubble, this is no longer true because the size of the bubble is not determined by the present value identity itself. Instead, goods market clearing determines the price level. A larger real value of bonds, holding taxes constant, means bonds represent more net wealth for the private sector, which increases consumption demand through a wealth effect. The equilibrium price level is the price level at which consumption demand equals consumption supply.[16] The fiscal theory equation itself determines the size of the bubble as the residual value of government debt that is not explained by the present value of primary surpluses.

As a consequence, the presence of a bubble makes price level determination based on fiscal dominance more challenging because it eliminates the simple one-to-one relationship between the present value of primary surpluses and the price level. Even making primary surpluses completely exogenous may not be sufficient to determine the price level uniquely if government policy fails to pin down the value of the

[14]However, surpluses s cannot become arbitrarily small, because q^B is decreasing in $\breve{\mu}^{\mathcal{B}}$ and reaches zero at the finite value $\breve{\mu}^{\mathcal{B}} = \tilde{\sigma}^2 - \rho$; there is a Laffer curve for bubble mining.

[15]This conclusion does not rely on the assumption of fiscal dominance. Assuming fiscal dominance just ensures that surpluses do not react too strongly to the price level to make the "given real allocation" the only possible equilibrium allocation.

[16]The same mechanism is present in the fiscal theory without a bubble, but it may not be as clearly visible because one can mechanically solve the model by reading off the price level from the fiscal theory equation.

bubble. The same path of surpluses can be consistent with multiple paths for the bubble value and thus with multiple initial price levels.[17]

The steady state equilibria derived above are consistent with a policy that fixes $\breve{\mu}^{\mathcal{B}}$ at a constant level and adjusts taxes τ such that the government budget constraint (3) holds after any price history.[18] While simple, such a policy is clearly inadequate to determine the price level. This is evident from the existence of a nonmonetary equilibrium in which nominal government bonds are worthless ($q^B = 0$). However, government policy can easily be modified off-equilibrium to select the monetary steady state as the unique equilibrium.[19] To see this, recall that along any equilibrium path ODE (7) must hold. The converse is also true, provided ϑ is contained in $[0, 1]$:

Lemma 1. *An absolutely continuous function* $[0, \infty) \to [0, 1], t \mapsto \vartheta_t$ *corresponds to a model equilibrium, if and only if it satisfies equation (7).*

Above, we have already provided the proof that equation (7) is necessary for an equilibrium. The proof of sufficiency requires (1) that equilibrium prices q^B and q^K consistent with market clearing can be expressed as a function of ϑ (see above) and (2) a number of technical arguments verifying that the resulting equilibrium satisfies all optimal choice conditions of households (see Online Appendix A.3).

With constant $\breve{\mu}^{\mathcal{B}}$, there is a continuum of solution paths for ϑ consistent with the requirement in Lemma 1, which can be indexed by the initial value $\vartheta_0 \in [0, \vartheta^*]$, where $\vartheta^* := \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$ denotes the monetary steady-state level of ϑ. For any initial value but the right endpoint, ϑ asymptotically converges to 0.[20] Conversely, if for some reasons agents expected that the equilibrium value of ϑ could never fall below a positive threshold $\underline{\vartheta} > 0$, then all equilibria but the monetary steady state $\vartheta_0 = \vartheta^*$ could be ruled out.

These considerations suggest a simple off-equilibrium modification of the fiscal policy rule to achieve equilibrium uniqueness: fix an arbitrary threshold $0 < \underline{\vartheta} < \vartheta^*$ and, whenever ϑ falls below $\underline{\vartheta}$, switch from a constant debt growth rule (constant $\breve{\mu}^{\mathcal{B}}$) to a positive surplus rule with a constant output tax rate $\tau > \mathfrak{g}/a$ for as long as $\vartheta \leq \underline{\vartheta}$. From the government budget constraint (3) it follows that under this modified fiscal policy ODE (7) becomes

$$\dot{\vartheta}_t = \begin{cases} \left(\rho - (1 - \vartheta_t)^2 \tilde{\sigma}^2 + \breve{\mu}^{\mathcal{B}}\right) \vartheta_t, & \vartheta_t > \underline{\vartheta} \\ \left(\rho - (1 - \vartheta_t)^2 \tilde{\sigma}^2\right) \vartheta_t - (\tau a - \mathfrak{g}) \frac{1 - \vartheta_t + \phi\rho}{1 + \phi \breve{a}}, & \vartheta_t \leq \underline{\vartheta} \end{cases}.$$

It is easy to see that this modified ODE has a strictly negative left-hand side on the interval $[0, \vartheta^*)$ including the left endpoint, and therefore all solutions that start inside this interval turn negative in finite time. By Lemma 1, the only possible equilibrium path for ϑ is therefore the steady-state equilibrium $\vartheta = \vartheta^*$.

The above rule modifies fiscal policy only off-equilibrium, whereas along the equilibrium path of the remaining unique equilibrium the government is free to choose any debt growth rate net of interest

[17] In an augmented version of our model with lump-sum taxes, Ricardian equivalence holds and thus changing the path of lump-sum taxes does not affect the equilibrium allocations otherwise, including the real value of the bubble. To the extent that (exogenous) surpluses are raised using lump-sum taxes, not capital taxation, both the no-bubble steady state and the bubble steady state are valid steady-state equilibria of this augmented model, but they imply different initial price levels.

[18] We have opted not to choose the opposite specification where τ is constant and $\breve{\mu}^{\mathcal{B}}$ adjusts to make the government budget constraint hold because this is only a valid policy specification, if $\tau \geq 0$. For $\tau < 0$, there are histories of prices in which no value of $\breve{\mu}^{\mathcal{B}}$ is consistent with equation (3) (e.g. $\mathcal{P} = \infty$, i.e., the moneyless equilibrium).

[19] Here, we focus on equilibria that are deterministic and feature absolutely continuous price paths. With additional technical arguments, one can also rule out non-time-continuous equilibria and equilibria driven by sunspot noise.

[20] This is implied by the fact that the right-hand side of equation (1) is negative for all $\vartheta_t \in (0, \vartheta^*)$.

payments $\breve{\mu}^{\mathcal{B}}$. This raises questions about the credibility and fiscal capacity to promise off-equilibrium surpluses. These issues are beyond the scope of the present paper.

We close this section by commenting on an apparent disagreement of our result with the analysis of Bassetto and Cui (2018) in the context of a dynamically inefficient OLG model. They study constant tax policies that are not contingent on the price level and conclude that "the FTPL breaks down in [their] OLG economy" (p. 13). This is fully consistent with the discussion in this section which highlights that contingent policy is required to obtain uniqueness of the price level. In order to back the bubble in equilibrium, the government must commit to off-equilibrium taxation that replaces the value of the bubble by a present value of primary surpluses. Constant tax policies are insufficient for this purpose.[21]

4 Conclusion

This paper integrates the typically ignored bubble term in the FTPL equation, which is necessary to explain low inflation in countries with persistently negative primary surpluses. Brunnermeier, Merkel, and Sannikov (2020) argue that this model provides a tractable framework to discuss and evaluate Modern Monetary Theory and enrich the debate on sovereign debt sustainability analysis. The analysis can also be easily extended to include a transaction role of money. In this case "narrow money" has an interest rate advantage $\Delta i = i - i^M$, which constitutes another source of seigniorage besides the "bubble mining" emphasized in this paper.

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[21] In addition, even positive primary surpluses are associated with equilibrium multiplicity in the OLG economy of Bassetto and Cui (2018), whereas in our model the constant tax policy with $\tau > \mathfrak{g}/a$ selects a unique equilibrium. This is due to a difference in assumptions: while we assume here that the government does not lend to the private sector, they allow for such lending, thereby effectively validating additional equilibrium paths that lead to a steady state with a negative value of debt. Because a government can simply refuse to lend to the private sector, this type of multiplicity is easily avoided by modifying government policy.

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A Online Appendix

A.1 Derivation of the Continuous-time Fiscal Theory Equation (Equation (1))

As in the discrete-time case, the derivation of the fiscal theory equation starts with the government flow budget constraint, which here is

$$\left(\mu_t^{\mathcal{B}}\mathcal{B}_t+\mu_t^{\mathcal{M}}\mathcal{M}_t+\mathcal{P}_tT_t\right)dt=\left(i_t\mathcal{B}_t+i_t^m\mathcal{M}_t+\mathcal{P}_tG_t\right)dt,$$

where $\mathcal{B}_t$, $\mathcal{M}_t$, T_t, G_t, i_t and i_t^m have the same meaning as in the main text and $\mu_t^{\mathcal{B}}$, $\mu_t^{\mathcal{M}}$ are the growth rates of nominal bonds and money, respectively.[22] Multiplying the budget constraint by the nominal SDF $\xi_t/\mathcal{P}_t$ and rearranging yields

$$\left(\left(\mu_t^{\mathcal{B}}-i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t+\left(\mu_t^{\mathcal{M}}-i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\right)dt=-\xi_t\left(\left(T_t-G_t\right)+\Delta i_t\frac{\mathcal{M}_t}{\mathcal{P}_t}\right)dt. \tag{9}$$

Next, Ito's product rule implies

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\right)=\left(\mu_t^{\mathcal{B}}-i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_tdt+\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t}+i_tdt\right),$$
$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\right)=\left(\mu_t^{\mathcal{M}}-i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_tdt+\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t}+i_tdt\right).$$

Solving these last two equations for $\left(\mu_t^{\mathcal{B}}-i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_tdt$ and $\left(\mu_t^{\mathcal{M}}-i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_tdt$, respectively, and substituting the results into equation (9) yields (after rearranging)

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\left(\mathcal{B}_t+\mathcal{M}_t\right)\right)=-\xi_t\left(\mathcal{P}_t\left(T_t-G_t\right)+\Delta i_t\mathcal{M}_t\right)dt+\xi_t\frac{\mathcal{B}_t+\mathcal{M}_t}{\mathcal{P}_t}\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t}+i_tdt\right),$$

or in integral form

$$\xi_T\frac{\mathcal{B}_T+\mathcal{M}_T}{\mathcal{P}_T}-\xi_t\frac{\mathcal{B}_t+\mathcal{M}_t}{\mathcal{P}_t}=-\int_t^T\xi_s\left(T_s-G_s\right)ds-\int_t^T\xi_s\Delta i_s\frac{\mathcal{M}_s}{\mathcal{P}_s}ds+\int_t^T\xi_s\frac{\mathcal{B}_s+\mathcal{M}_s}{\mathcal{P}_s}\left(\frac{d\left(\xi_s/\mathcal{P}_s\right)}{\xi_s/\mathcal{P}_s}+i_sdt\right).$$

Up to this point, we have merely rearranged and integrated the government budget constraint. To derive the fiscal theory equation, the literature proceeds by using two equilibrium conditions. First, if the nominal SDF $\xi/\mathcal{P}$ prices the government bonds, then its expected rate of change must be the negative of the nominal interest rate. Then, the last stochastic integral on the right must be a martingale and disappears when taking conditional time-t expectations $E_t[\cdot]$. Second, a private-sector transversality condition is invoked to eliminate a terminal value of government debt when passing to the limit $T\to\infty$. We perform the first operation, but do not want to restrict attention to environments where transversality can rule out a nonzero discounted terminal value. When taking the limit $T\to\infty$, we therefore arrive at the more general equation (1).

[22]Here we abstract from long-term bonds and the possibility of taxes, spending, and adjustments in $\mathcal{B}$ and $\mathcal{M}$ that are not absolutely continuous over time (e.g., lumpy adjustments in response to a Poisson shock). Such elements could be easily added, but require more complicated notation without generating additional insights for our purposes.

A.2 Verification Argument for the Value Function

While one could adjust standard verification arguments for HJB equations, we present here a more direct verification proof for the functional form $V_t\left(n^i\right) = \alpha_t + \frac{1}{\rho}\log n^i$ that does not make reference to the HJB equation. In what follows, we drop all i superscripts as we always consider a single household's decision problem. Taking return processes $dr_t^{\mathcal{B}}, dr_t^K\left(\iota_t\right)$ as given, the household chooses c, θ, ι to maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t dt\right]$$

subject to the net worth evolution (6). For any $n > 0$, denote by $V_0\left(n\right) := \sup_{\{c_t,\theta_t,\iota_t\}}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t dt\right]$ the optimal objective at $t = 0$[23], if $n_0 = n$. To make the dependence of n_t on n_0 and the chosen policy $\{c_t, \theta_t, \iota_t\}$ explicit, we write furthermore $n_t^{c,\theta,\iota}(n_0)$ instead of just n_t.

We first prove the following auxiliary result: if $n^{(1)}, n^{(2)} > 0$ and $\{c_t^{(1)}, \theta_t^{(1)}, \iota_t^{(1)}\}$ is an admissible policy for initial wealth $n_0 = n^{(1)}$, then $c_t^{(2)} := \frac{n^{(2)}}{n^{(1)}}c_t^{(1)}$, $\theta_t^{(2)} := \theta_t^{(1)}$, $\iota_t^{(2)} := \iota_t^{(1)}$ is an admissible policy for initial wealth $n_0 = n^{(2)}$ and $n_t^{c^{(2)},\theta^{(2)},\iota^{(2)}}\left(n^{(2)}\right) = \frac{n^{(2)}}{n^{(1)}}n_t^{c^{(1)},\theta^{(1)},\iota^{(1)}}\left(n^{(1)}\right)$. Indeed, if $\{c_t^{(1)}, \theta_t^{(1)}, \iota_t^{(1)}\}$ is admissible for $n_0 = n^{(1)}$, then $n_t^{(1)} := n_t^{c^{(1)},\theta^{(1)},\iota^{(1)}}\left(n^{(1)}\right)$ must be strictly positive and satisfy equation (6), i.e.,

$$dn_t^{(1)} = -c_t^{(1)}dt + n_t^{(1)}\left(dr_t^{\mathcal{B}} + \left(1-\theta_t^{(1)}\right)\left(dr_t^K\left(\iota_t^{(1)}\right) - dr_t^{\mathcal{B}}\right)\right).$$

Then $n_t^{(2)} := \frac{n^{(2)}}{n^{(1)}}n_t^{(1)}$ is also strictly positive and satisfies the equation

$$dn_t^{(2)} = \frac{n^{(2)}}{n^{(1)}}dn_t^{(1)} = -\underbrace{\frac{n^{(2)}}{n^{(1)}}c_t^{(1)}}_{=c_t^{(2)}}dt + \underbrace{\frac{n^{(2)}}{n^{(1)}}n_t^{(1)}}_{=n_t^{(2)}}\left(dr_t^{\mathcal{B}} + \left(1-\theta_t^{(1)}\right)\left(dr_t^K\left(\iota_t^{(1)}\right) - dr_t^{\mathcal{B}}\right)\right),$$

which is exactly equation (6) for the policy choice $\{c_t^{(2)}, \theta_t^{(2)}, \iota_t^{(2)}\}$. Because in addition $n_0^{(2)} = \frac{n^{(2)}}{n^{(1)}}n_0^{(1)} = n^{(2)}$, $\{n_t^{(2)}\}$ is the solution to (6) with policy $\{c_t^{(2)}, \theta_t^{(2)}, \iota_t^{(2)}\}$ and initial condition $n_0 = n^{(2)}$, that is $n_t^{c^{(2)},\theta^{(2)},\iota^{(2)}}\left(n^{(2)}\right) = \frac{n^{(2)}}{n^{(1)}}n_t^{(1)}$, which proves the auxiliary result.

Next, let $\varepsilon > 0$ be arbitrary and again $n^{(1)}, n^{(2)} > 0$. By the definition of $V_0\left(n^{(1)}\right)$, we can choose admissible policies $\{c_t, \theta_t, \iota_t\}$ for $n_0 = n^{(1)}$, such that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t dt\right] + \varepsilon \geq V_0\left(n^{(1)}\right). \tag{10}$$

[23] The verification argument for any initial time $t > 0$ is identical. For notational convenience, we therefore restrict attention to $t = 0$.

By the previous auxiliary result, $\{\frac{n^{(2)}}{n^{(1)}}c_t, \theta_t, \iota_t\}$ is then an admissible policy for $n_0 = n^{(2)}$ and thus

$$\begin{aligned} V_0\left(n^{(2)}\right) &\geq \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left(\frac{n^{(2)}}{n^{(1)}}c_t\right)dt\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left(\frac{n^{(2)}}{n^{(1)}}\right)dt\right] + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t dt\right] \\ &= \frac{1}{\rho}\log\left(\frac{n^{(2)}}{n^{(1)}}\right) + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t dt\right] \\ &\geq \frac{1}{\rho}\log n^{(2)} - \frac{1}{\rho}\log n^{(1)} + V_0\left(n^{(1)}\right) - \varepsilon, \end{aligned}$$

where the last inequality follows from (10). In total, this argument implies the value function inequality

$$V_0\left(n^{(2)}\right) + \frac{1}{\rho}\log n^{(1)} \geq V_0\left(n^{(1)}\right) + \frac{1}{\rho}\log n^{(2)} - \varepsilon$$

for any arbitrary $n^{(1)}, n^{(2)}, \varepsilon > 0$. Because $\varepsilon > 0$ was arbitrary, the inequality must then also hold for $\varepsilon = 0$. Because $n^{(2)}, n^{(1)}$ were arbitrary, the inequality must still hold, when the roles of $n^{(1)}$ and $n^{(2)}$ are interchanged. Taking both facts together implies the *equality*

$$V_0\left(n^{(2)}\right) + \frac{1}{\rho}\log n^{(1)} = V_0\left(n^{(1)}\right) + \frac{1}{\rho}\log n^{(2)}. \tag{11}$$

To complete the proof, choose $n^{(2)} = n$, $n^{(1)} = 1$ in equation (11), which implies

$$V_0(n) = V_0(1) + \frac{1}{\rho}\log n.$$

Consequently, with the definition $\alpha_0 := V_0(1)$, V_0 has the desired functional form.

A.3 Missing Steps in Proof of Lemma 1

It is to show that any solution $\vartheta : [0, \infty) \to [0, 1]$ to (7) corresponds to a unique equilibrium of the model. For any such function, define ι, q^B, and q^K consistent with the expressions given in the main text, i.e.,

$$\begin{aligned} \iota_t &= \frac{(1-\vartheta_t)(a-\mathfrak{g}) - \rho}{1-\vartheta_t+\phi\rho}, \\ q_t^B &= \vartheta_t \frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho}, \\ q_t^K &= (1-\vartheta_t)\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho}. \end{aligned}$$

Because $\vartheta_t \in [0, 1]$ at all times, $q_t^B, q_t^K \geq 0$, so these expressions are consistent with free disposal of both money and capital. We now verify that ι_t, q_t^B, q_t^K and $\theta_t := \vartheta_t$ satisfy all household choice conditions and the aggregate resource constraint.

One immediately verifies that ι_t and q_t^K satisfy households' optimal investment choice condition, $q_t^K = \frac{1}{\Phi'(\iota_t)} = 1 + \phi\iota_t$. In addition, total wealth of all households is $\left(q_t^B + q_t^K\right)K_t$ and because individual consumption demand $c_t^i = \rho n_t^i$ implies an aggregate consumption demand of $C_t = \rho\left(q_t^B + q_t^K\right)K_t$, we obtain

$$\begin{aligned}
C_t + \mathfrak{g}K_t + \iota_t K_t &= \left(\rho\left(q_t^B + q_t^K\right) + \mathfrak{g} + \iota_t\right)K_t \\
&= \left(\rho\frac{1+\phi\left(a-\mathfrak{g}\right)}{1-\vartheta_t+\phi\rho} + \mathfrak{g} + \frac{\left(1-\vartheta_t\right)\left(a-\mathfrak{g}\right)-\rho}{1-\vartheta_t+\phi\rho}\right)K_t \\
&= \left(\left(\frac{\phi\rho}{1-\vartheta_t+\phi\rho} + \frac{\left(1-\vartheta_t\right)}{1-\vartheta_t+\phi\rho}\right)\left(a-\mathfrak{g}\right) + \mathfrak{g}\right)K_t \\
&= aK_t,
\end{aligned}$$

so this equilibrium candidate satisfies the aggregate resource constraint (4).

It is left to show that at the asset prices q^B and q^K, agents' capital portfolio share $\theta_t = \vartheta_t$ is consistent with their optimal choice condition for θ_t. We consider two cases:

1. If $\vartheta_t > 0$, then equation (7) (that ϑ satisfies by assumption) is equivalent to $\mu_t^\vartheta = \rho + \breve{\mu}_t^{\mathcal{B}} - \left(1-\vartheta_t\right)^2\tilde{\sigma}^2$ and rearranging the latter equation and using $\theta_t = \vartheta_t$ yields

$$1-\theta_t = \frac{1}{\tilde{\sigma}^2}\frac{\rho + \breve{\mu}_t^{\mathcal{B}} - \mu^\vartheta}{1-\vartheta_t}. \tag{12}$$

   Next, by definition of ι_t and q_t^K

$$\begin{aligned}
\frac{a-\mathfrak{g}-\iota_t}{q_t^K} &= \frac{\left(a-\mathfrak{g}\right)\left(1-\vartheta_t+\phi\rho\right) - \left(1-\vartheta_t\right)\left(a-\mathfrak{g}\right)+\rho}{\left(1-\vartheta_t\right)\left(1+\phi\left(a-\mathfrak{g}\right)\right)} \\
&= \frac{\left(1+\phi\left(a-\mathfrak{g}\right)\right)\rho}{\left(1-\vartheta_t\right)\left(1+\phi\left(a-\mathfrak{g}\right)\right)} = \frac{\rho}{1-\vartheta_t},
\end{aligned}$$

   and substituting this into equation (12) yields

$$1-\theta_t = \frac{1}{\tilde{\sigma}^2}\left(\frac{a-\mathfrak{g}-\iota_t}{q_t^K} - \frac{\mu^\vartheta - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right),$$

   which is precisely households' first-order condition with respect to θ_t as stated in the main text.

2. If $\vartheta_t = 0$, then $q_t^B = 0$, hence money has no value and the return on money is not well-defined. Consequently, the household portfolio choice condition as stated in the main text is not directly applicable. Instead, households demand a finite quantity of money (which is consistent with equilibrium and $\theta_t = 0$), if and only if $\dot{q}_t^B \leq 0$, i.e., the value of money is expected to remain nonpositive in the infinitesimal future. Because $p_s \geq 0$ for all s, this condition reduces here to $\dot{q}_t^B = 0 \Leftrightarrow \dot{\vartheta}_t = 0$. We therefore have to show that $\vartheta_t = 0$ implies $\dot{\vartheta}_t = 0$. We do this in two steps.

   First, at $\vartheta_t = 0$ equation (7) is misleading, because it appears that always $\dot{\vartheta}_t = 0$, but this ignores that by the government budget constraint, $\breve{\mu}_t^{\mathcal{B}} = \frac{\mathfrak{g}-\tau_t a}{q_t^B} = \frac{\mathfrak{g}-\tau_t a}{\vartheta_t}\frac{1-\vartheta_t+\phi\rho}{1+\phi(a-\mathfrak{g})}$, which diverges to ∞ as $\vartheta_t \searrow 0$. Nevertheless, the right-hand side of (7) remains well-defined even at the limit point, if we plug in $\breve{\mu}_t^{\mathcal{B}}$,

$$\dot{\vartheta}_t = \left(\rho - \left(1-\vartheta_t\right)^2\tilde{\sigma}^2\right)\vartheta_t + \left(\mathfrak{g}-\tau_t a\right)\frac{1-\vartheta_t+\phi\rho}{1+\phi\left(a-\mathfrak{g}\right)}. \tag{13}$$

We use this representation of ODE (7) in the remaining argument.

Second, substituting $\vartheta_t = 0$ into equation (13) yields

$$\dot{\vartheta}_t = (\mathfrak{g} - \tau_t a) \frac{1 - \vartheta_t + \phi\rho}{1 + \phi (a - \mathfrak{g})} = 0,$$

where the last equality follows from the government budget constraint (3) in the limit $\mathcal{P}_t \to \infty$ (which is equivalent to $q_t^B = \vartheta_t = 0$) and the assumption that government policy is specified to be consistent with the government budget constraint. Consequently, $\vartheta_t = 0$ implies indeed $\dot{\vartheta}_t = 0$.