Abstract

We present a business cycle model in which Bayesian learning about stock price behavior gives rise to boom and bust cycles in stock prices. Stock price cycles transmit into the real economy by generating inefficient price signals for the desirability of new investment and act as a ‘financial accelerator’ that operates even in the absence of financial frictions. The model successfully replicates the quantitative behavior of U.S. stock prices and business cycles, despite assuming time-separable consumption preferences with low risk aversion, a perfectly flexible labor market, and featuring productivity shocks as the only exogenous driving force. Under rational stock price expectations, the model performs poorly: it generates only small amounts of stock price volatility and fails to match the volatility of investment and hours worked in the absence of investment-specific technology shocks.

JEL Class.No.: E32, E44, G12

1 Introduction

We present a simple economic model that quantitatively replicates the joint behavior of business cycles and stock prices. The model matches standard data moments characterizing business cycle behavior, standard moments capturing stock price behavior, as well as data moments linking stock prices with business cycle variables. The model also gives rise to boom and bust cycles in stock price valuation of the kind observed in the data and features a new form of financial accelerator, which does not rely on the presence of financial frictions.
Quantitatively replicating the behavior of the business cycle and of stock prices within a common modeling framework has proven surprisingly difficult. The main challenge consists of simultaneously replicating the huge volatility differences between the real and the financial side of the economy: while the business cycle is relatively smooth, especially with regard to private consumption, stock prices are very volatile and display cycles that are orders of magnitude larger than the business cycle.

Figure 1 illustrates the presence and the magnitude of U.S. stock price cycles by depicting the S&P500 stock index. Over the thirty-year period covered in the figure, stock prices displayed three large price run-ups and two large price reversals, with the latter amounting to a cumulative price drop of close to 50% each, relative to the previous price peak value. Similar medium-term price run-ups and reversals can be observed in the stock markets of other advanced economies.

Figure 2 presents an alternative approach for quantitatively capturing stock price cycles. It scales stock prices by dividends and depicts the empirical distribution of the quarterly price-dividend (PD) ratio of the S&P500. The figure shows that the empirical PD distribution has a lot of mass around values of 100 to 150, but also displays a long right tail that covers quarterly PD values above 300. The positive skewness and the large support of the empirical PD distribution capture the presence of occasional stock price run-ups and reversals, i.e., periods in which prices grow considerably faster (or slower) than dividends.

The existing literature provides two broad approaches that allow reconciling the large volatility of stock prices with the low volatility of business cycles. The first explanation combines preferences featuring a low elasticity of intertemporal substitution (EIS) with adjustment frictions. A low EIS creates a strong desire for intertemporal consumption smoothing, while adjustment frictions prevent such smoothing from fully taking place. For agents to be willing to accept the observed moderate consumption fluctuations, asset prices then need to adjust strongly in equilibrium. Since labor supply adjustments represent one possible adjustment margin in business cycle models, flexible adjustments in the number of hours worked need to be prevented. EIS-based explanations therefore include some form of labor market frictions (adjustment frictions, inflexible labor supply through preferences, real wage frictions). In fact, labor market frictions become central for explaining stock price behavior. This appears puzzling in light of the fact that labor

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1 Figure 1 depicts the nominal value of the S&P500, but similar conclusions emerge if one deflates the nominal value by the consumer price index.

2 The quarterly PD ratio is defined as the end-of-quarter price over a deseasonalized measure of quarterly dividend payouts, see Appendix A.1 for details.

3 The low elasticity of intertemporal substitution is typically generated via habit preferences (Jermann (1998); Boldrin, Christiano, and Fisher (2001); Uhlig (2007); Jaccard (2014)), but sometimes via recursive preferences featuring a low EIS parameter (Guvenen (2009)).

4 Otherwise the high desire to smooth intertemporal consumption fluctuations leads to a strong adjustment in hours worked and a very smooth consumption profile, which in turn would largely eliminate asset price fluctuations.
market institutions differ considerably across advanced economies, while stock prices are about equally volatile in these economies.\textsuperscript{5}

A second explanation reconciling smooth business cycles with volatile stock prices relies on specifying recursive preferences in conjunction with an additional source of exogenous uncertainty, either long-run growth risk (Croce (2014)) or disaster risk (Gourio (2012)). Such shocks have large pricing implications under recursive preferences, provided the coefficient of risk aversion is larger than the inverse of the EIS. The presence of such shocks then generates a large equity premium in the presence of realistic consumption dynamics, while time variation in the equity premium leads to substantial volatility in stock prices and returns.\textsuperscript{6} While offering a perfectly fine quantitative explanation for business cycle and stock price behavior, the calibration of the recursive preference parameters has recently been criticized on the grounds that they imply very large consumption premia for the early resolution of uncertainty (Epstein, Farhi, and Strzalecki (2014)).

\textsuperscript{5}See table 1 in Campbell (2003) for an international comparison of stock price behavior.

\textsuperscript{6}Labor market frictions or the elasticity of labor supply are usually not discussed in this strand of literature. While they seem less central than for the low-EIS-based explanations, flexible labor supply in a frictionless labor market is still a powerful tool for insuring against consumption fluctuations. The fact that the asset pricing models in this second strand of literature tend to generate too little volatility of hours worked is an indication that the chosen preference specifications make labor supply insufficiently flexible.

\textbf{Figure 1.} Price Cycles in the S&P 500 (Q1:1985-Q4:2014)
Figure 2. Postwar distribution of the quarterly PD ratio of the S&P 500 (kernel density estimate, Q1:1955-Q4:2014)

The present paper proposes an alternative explanation for stock price and business cycle behavior. In contrast to the existing approaches, it relies on a standard time-separable preference specification and features a frictionless labor market, as well as households with infinite Frisch elasticity of labor supply.\footnote{We choose this setup not because we believe labor market rigidities, labor supply rigidities, or time-non-separabilities in preferences to be unimportant, but in order to put maximum distance to the existing explanations. This allows us to isolate the effect of subjective stock price beliefs, as introduced further below.} Despite these features, the model can jointly replicate the quantitative behavior of business cycles and stock prices using (reasonably sized) shocks to total factor productivity as its only source of exogenous random variation. The model also generates a stable risk-free interest rate and gives rise to large and persistent stock price boom-bust patterns of the kind observed in the data. The most notable dimension along which the model falls short of fully matching the data is the equity premium: the model-implied (unlevered) equity premium reaches only about one third of the empirically observed premium.\footnote{Given that we consider a setting with logarithmic consumption preferences, this still represents a respectable achievement.}
Key to the empirical success of the model is a departure from the rational expectations hypothesis (REH). This departure from rational expectations is motivated by data on investor surveys, which clearly show that investors’ expectations about future capital gains deviate systematically from the REH (Adam, Marcet, and Beutel (2017)). In fact, findings in Vissing-Jorgensen (2003), Bacchetta, Mertens, and Van Wincoop (2009), Malmendier and Nagel (2011) and Greenwood and Shleifer (2014) show that investor expectations are pro-cyclical: subjectively expected returns (or capital gains) are higher following high realized stock market returns (or capital gains) and in times of high price-dividend ratios. This expectations pattern, which is inconsistent with the REH and the existing RE asset pricing models, can be parsimoniously captured by modeling investors as subjective Bayesians who filter the long-term trend component of capital gains from observed capital gains.

While the model allows – in line with the survey evidence – for deviations from rational stock price expectations, agents in the model are otherwise standard: (1) they hold rational expectations about all other decision-relevant variables, and (2) they make state-contingent plans to maximize utility given their constraints and the beliefs they entertain about variables beyond their control, i.e., agents are ‘internally rational’ in the sense of Adam and Marcet (2011).

Subjective components in stock price expectations give rise to speculative mispricing of stocks, compared to a setting in which investors hold rational stock price expectations. This mispricing of stocks has real consequences because it affects agents’ optimal choices for investment, consumption and hours worked.

Consider, for instance, a setting in which a sequence of positive fundamental shocks triggers a sequence of positive capital gain surprises. Since investors filter from observed capital gains, these capital gain surprises cause them to become unduly optimistic about the long-run component of capital gains, in line with what the survey data suggests. Optimistic capital gain expectations push up prices even higher, thereby generate additional capital gains. The mutual reinforcement between capital gain expectations and realized capital gains sets in motion a belief-driven stock price boom. Increasing stock prices signal to capital producers that new investment is increasingly profitable and cause them to optimally expand investment. The resulting positive association between asset price increases and investment increases is reminiscent of the investment booms associated with the U.S. tech stock boom in the late 1990s or the U.S. housing boom at the beginning of the new millennium. Belief-driven speculative mispricing of stocks thus gives rise

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9 In subsequent work, Adam, Matveev, and Nagel (2018) show that the failure of the REH in survey data can not be explained away by postulating that survey respondents report risk-adjusted expectations.

10 As should be clear, it is important for this argument that the supply of new capital is not fully elastic, e.g., due to decreasing returns to scale in the production or due to adjustment frictions. Otherwise, the stock price increase would be inconsistent with optimal behavior of capital producers.

11 See Adam and Woodford (2018) for a model in which speculative mispricing in housing markets distorts the supply of new houses.
to a financial accelerator that links financial variables to real variables and that results in a misallocation of resources from the viewpoint of a social planner with fully rational expectations. Yet, unlike in standard models of the financial accelerator, e.g., Bernanke, Gertler, and Gilchrist (1999) or Kiyotaki and Moore (1997), the misallocation due to asset price fluctuations does not rely on the presence of financial frictions.

While the model gives rise to rich business cycle and stock price dynamics, it is sufficiently simple to allow for the analytic derivation of a number of results. In particular, we can show how stock price increases and subjective capital gain optimism can mutually reinforce each other to generate a persistent boom in stock prices and how the resulting expansion of investment and of the capital stock eventually dampen the stock price increase. We can also prove analytically, that there is a globally unique equilibrium with subjective stock price beliefs of the kind we specify and rational expectations about the remaining variables. Finally, the model is sufficiently simple to permit estimation of the model parameters using the fully nonlinear model and the Simulated Method of Moments (SMM). Making use of the fully non-linear model is important for obtaining the correct moment implications in the presence of large stock price fluctuations.

The estimated model shows how the economy can experience a ‘Minsky moment’ in which belief-driven stock price booms are eventually dampened, then reverse and ultimately lead to a stock price crash in which stock prices, investment and hours worked fall persistently below their long-run balanced growth path values. Key to generating a stock price reversal is the fact that the ever increasing capital gain expectations along a stock price boom episode are eventually becoming too optimistic relative to capital gain outcomes. This happens either because the capital stock expands sufficiently rapidly or because the increase in optimism is too weak to generate the high capital gains that investors expect. At this point, outcomes fall short of agents optimistic expectations, which – absent further shocks – sets in motion a stock price reversal.

The estimated model reveals that the empirical improvements associated with a departure from the assumption of rational stock price expectations are large and economically significant, both along the business cycle dimension and – even more importantly – along the stock price dimension. Under fully rational expectations, the model spectacularly fails along the stock price dimension and produces only tiny amounts of stock price volatility. It also fails in fully replicating business cycle moments, as it falls significantly short of matching the observed volatility of investment and of hours worked, unless one augments the model by including investment-specific productivity shocks. In a sense, investment specific shocks substitute for the lack of a large financial accelerator that would be operating in the presence of plausibly sized stock price fluctuations.

Under subjective stock price beliefs, the estimated model performs quantitatively well along both the business cycle and the stock price dimension (except for the equity premium) and does not need to feature investment specific shocks to technology.

\[12\] It still takes about one week to estimate the model, despite extensive parallelization efforts.
Using the estimated subjective belief model, we also determine the welfare effects associated with stock price fluctuations caused by fluctuations in investors’ subjective beliefs. Since we consider a setting with a representative household, the welfare gains associated with eliminating subjective stock price beliefs turn out to be small in absolute terms. They amount to a permanent consumption increase of 0.29%, when measured in terms of ex-post realized utility. The welfare gains are thus in the same order of magnitude as the ones associated with eliminating the business cycle.

The paper also makes a conceptual contribution that should be of interest beyond the present application: it shows how to solve for equilibrium in dynamic decision settings in which agents’ expectations deviate from rational expectations along some dimension (future stock prices), but are consistent with the REH along other dimensions (future wages, future rental rates on capital).

The paper is structured as follows. Section 2 discusses the related literature. Section 3 presents key facts about business cycles, stock prices and their interaction in the United States. Section 4 describes our model with the exception of beliefs, which are discussed in detail in Section 5. All equilibrium conditions of the model are summarized in Section 6. Section 7 derives analytical insights and discusses the dynamics of stock prices, price beliefs and macro aggregates. Section 8 outlines our estimation procedure and assesses the empirical performance of our model under subjective and rational stock price beliefs. Section 9 compares our quantitative results to the findings in Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017). Section 10 discusses the welfare implications of belief-driven stock price cycles and section 11 concludes.

2 Related Literature

We start by discussing the rational expectations literature studying business cycle and stock price behavior. To the extent that these papers produce empirically realistic stock price behavior, i.e., expected stock returns that are counter-cyclical, the implications of these models are inconsistent with the pro-cyclical behavior of survey expectations. Early modeling approaches rely on a combination of habit preferences and adjustment frictions to generate high stock price volatility and plausible business cycle dynamics (Jermann, 1998; Boldrin, Christiano, and Fisher, 2001; Uhlig, 2007). Habit preferences create a low elasticity of intertemporal substitution (EIS) and thereby a strong desire to smooth consumption over time. As a result, even small (business-cycle sized) consumption fluctuations give rise to volatile stock prices. Aversion against intertemporal consumption substitution, however, generates volatility in all assets and thus a counterfactually high volatility for the risk-free interest rate. A notable exception is Uhlig (2007) who considers a setting with external habits. External habits create strong fluctuations in risk.

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13See Adam, Marcet, and Beutel (2017) for details.
aversion\textsuperscript{14} and thereby can give rise to an empirically plausible Sharpe ratio and a stable risk-free interest rate.\textsuperscript{15} Agents in these models have strong desire to intertemporally smooth consumption, but are prevented from doing so, as the model otherwise produces insufficient stock price volatility. This is achieved by introducing labor market frictions in various forms: Jermann (1998) assumes labor supply to be fully inelastic, Boldrin, Christiano, and Fisher (2001) introduce timing frictions that force households to choose labor supply in advance, and Uhlig (2007) introduces real wage rigidity that leads to labor supply rationing following negative productivity shocks.

Subsequently, models with limited stock market participation have been employed to jointly model business cycle dynamics and stock price behavior.\textsuperscript{16} In these settings, a limited set of agents has access to the stock market and in addition insures the consumption of non-participating agents via various other contracts. An early example is Danthine and Donaldson (2002), who consider shareholders and hand-to-mouth workers. Shareholders optimally offer workers a labor contract that insures workers and that gives rise to ‘operating leverage’ for shareholders. The cash flows of shareholders thus become more volatile and procyclical, which gives rise to an equity premium and volatile stock returns, albeit at the cost of creating too much volatility for shareholders’ consumption.\textsuperscript{17} Guvenen (2009) considers a model in which all agents participate in the bond market but only some in the stock market. If stock market participants have a higher EIS than non-participants, then the former optimally insure the latter against income fluctuations via bond market transactions, thereby channeling most labor income risk to stock market participants. As a result, their consumption is strongly procyclical and gives rise to both a high equity premium and high volatility of returns. The model assumes the EIS to be in absolute terms low – even for shareholders – thus generates additional stock price volatility through the same channels as habit models, but generates a more stable risk-free rate. The model performs quantitatively very well along the financial dimension, while on the business cycle dimension consumption tends to be too volatile and investment and hours too smooth.

Tallarini (2000), Gourio (2012), Croce (2014) and Hirshleifer, Li, and Yu (2015) dis-

\textsuperscript{14}See Boldrin, Christiano, and Fisher (1997) for a discussion of the differing risk aversion implications of internal and external habit specifications.

\textsuperscript{15}It remains unclear whether the model separately matches the volatility of stock returns and the size of the equity premium, as only the Sharpe ratio is reported.

\textsuperscript{16}As mentioned in Guvenen (2009), stock market participation increased substantially in the U.S. during the 1990’s. From 1989 to 2002 the number of households who owned stocks increased by 74\%, with half of U.S. households owing stocks by the year 2002.

\textsuperscript{17}Table 6 in Danthine and Donaldson (2002), which displays the specification with the best overall empirical fit, shows that shareholder’s consumption volatility is about 10 times as large as aggregate consumption volatility. Guvenen (2009) reviews the empirical evidence on stockholders’ relative consumption volatility and concludes that stockholders’ consumption is about 1.5-2 times as volatile as non-stockholders’ and thus – in relative terms – even less high when compared to aggregate consumption volatility
cuss the asset pricing predictions of the real business cycle models under Epstein-Zin preferences (Epstein and Zin, 1989), assuming that the coefficient of risk aversion is larger than the inverse of the EIS. Tallarini (2000) shows that increasing risk aversion – while keeping the EIS fixed and equal to one – barely affects business cycle dynamics, but has substantial effects on the price of risk. This allows the model to generate a high Sharpe ratio, although it considerably undershoots the equity premium and the volatility of stock returns. Gourio (2012) considers preferences with a larger EIS and moderate risk aversion and enriches the model by time-varying disaster risk.\footnote{A disaster is a potentially persistent event in which the economy experiences in each disaster period a negative productivity shock and a large capital depreciation shock.} Whereas constant disaster risk has little effect on the model dynamics, time-variation in disaster risk combined with preferences for early resolution of uncertainty generate a high equity premium and high return volatility. At the same time, the business cycle dynamics remain in line with the data, provided the disaster does not realize.\footnote{While framed as a rational expectations model applied to a disaster-free data sample, one may alternatively interpret the model as a subjective belief model in which the exogenous shocks to the disaster probability move agents’ subjective expectations. Under this interpretation, subjective beliefs drive asset price dynamics, not unlike in the present paper.} Croce (2014) considers a production economy with Epstein-Zin preferences and long-run productivity growth risk. The model can match well the equity premium for levered equity returns and produces a low and stable risk-free rate. It also matches business cycle dynamics, but has difficulties in fully replicating the volatility of stock returns. Hirshleifer, Li, and Yu (2015) consider a production economy with preferences for early resolution of uncertainty and assume inelastic labor supply. Agents know the productivity process only imperfectly and over-extrapolate recent productivity observations in their filtering problem. This mechanism endogenously generates long-run variations in perceived technology growth and thus perceived consumption growth, despite there being only short-run technology shocks present in the data-generating process. If the extrapolation bias is small but sufficiently persistent, then the model produces a sizeable equity premium and about 50% of the observed volatility of stock returns.

The present paper is also related to the literature on rational stock market bubbles, as for instance derived in classic work by Froot and Obstfeld (1991). While rational bubbles provide an alternative approach for generating stock market volatility, rational bubbles are inconsistent with empirical evidence along two important dimensions. First, the assumption of rational return expectations is strongly at odds with survey measures of return expectations, as mentioned above. Second, Giglio, Maggiori, and Stroebel (2016) show that there is little evidence supporting the notion that violations of the transversality condition drive asset price fluctuations, unlike suggested by the rational bubble hypothesis.

There is a growing literature introducing subjective belief components into business cycle models. It shows how subjective belief components can improve business cycle
performance, but does not consider stock price implications. Eusepi and Preston (2011) study a setting where agents are learning about the behavior of wages and rental rates, Angeletos, Collard, and Dellas (2018) introduce confidence shocks in the form of autonomous movements in higher-order expectations, and Bhandari, Borovicka, and Ho (2017) consider households with time-varying ambiguity aversion. In our setting, beliefs about business cycle variables (wages, rental rates, etc.) are assumed to be rational, but subjective stock price expectations do affect business cycle outcomes via the investment channel.

The paper is related to prior work of Adam, Marcet, and Nicolini (2016) and Adam, Marcet, and Beutel (2017), who introduce subjective price beliefs into a Lucas type stock price model. The present setup is considerably richer than the endowment settings studied in this earlier work: it features endogenous consumption, labor and investment choices, as well as capital accumulation over time. Moreover, unlike in endowment setups, in the present setting stock price fluctuations have economic consequences, as they affect investment decisions.

In independent and complementary work, Winkler (2018) considers the asset pricing implications of a rich DSGE model featuring subjective stock price beliefs of the kind we also consider. The model contains a range of additional features from which the present paper abstracts: financial frictions, price and wage rigidities, and limited stock market participation. A further distinguishing feature is that the present model assumes all expectations, except for stock price expectations, to be rational. Winkler (2018) considers a setup with ‘conditionally model-consistent expectations’, which allows for additional deviations from the rational expectations hypothesis.

3 Stock Prices and Business Cycles: Key Facts

This section presents key data moments that characterize U.S. business cycles and stock price behavior and that we focus on in our empirical analysis. We consider quarterly U.S. data for the period Q1:1955-Q4:2014. The start date of the sample is determined by the availability of the aggregate hours worked series. Details of the data sources are reported in Appendix A.1.

Table 1 presents a standard set of business cycle moments for output ($Y$), consumption ($C$), investment ($I$) and hours worked ($H$). These quantities have been divided by the working age population so as to take into account demographic changes in the U.S.

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20 In Adam, Marcet, and Nicolini (2016) and Adam, Marcet, and Beutel (2017), stock prices only affect subjectively expected utility, but are irrelevant for ex-post realized utility.

21 As is standard in the business cycle literature, we compute business cycle moments using logged and subsequently HP-filtered data with a smoothing parameter of 1600. All other data moments will rely on unfiltered (level) data. We HP filter model variables when comparing to filtered moments in the data and use unfiltered model moments otherwise.
Table 1  
U.S. business cycle moments (quarterly real values, Q1:1955-Q4:2014)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Data moment</th>
<th>Std. dev. data moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std. dev. of output</td>
<td>$\sigma(Y)$</td>
<td>1.72</td>
</tr>
<tr>
<td>Relative std. dev. of consumption</td>
<td>$\sigma(C)/\sigma(Y)$</td>
<td>0.61</td>
</tr>
<tr>
<td>Relative std. dev. of investment</td>
<td>$\sigma(I)/\sigma(Y)$</td>
<td>2.90</td>
</tr>
<tr>
<td>Relative std. dev. of hours worked</td>
<td>$\sigma(H)/\sigma(Y)$</td>
<td>1.08</td>
</tr>
<tr>
<td>Correlation output and consumption</td>
<td>$\rho(Y,C)$</td>
<td>0.88</td>
</tr>
<tr>
<td>Correlation output and investment</td>
<td>$\rho(Y,I)$</td>
<td>0.86</td>
</tr>
<tr>
<td>Correlation output and hours worked</td>
<td>$\rho(Y,H)$</td>
<td>0.75</td>
</tr>
</tbody>
</table>

The picture that emerges from Table 1 is a familiar one: output fluctuations are relatively small, consumption is considerably less volatile than output, while investment is considerably more volatile; hours worked are roughly as volatile as output. Consumption, investment and hours all correlate strongly with output. A major quantitative challenge will be to simultaneously replicate the relative smoothness of the business cycle with the much larger fluctuations in stock prices to which we turn next.

Table 2 presents a standard set of moments characterizing U.S. stock price behavior. The first three moments summarize the behavior of the PD ratio: the PD ratio is rather large and implies a dividend yield of just 66 basis points per quarter. The PD ratio is also very volatile: the standard deviation of the PD ratio is more than 40% of its mean value and fluctuations in the PD ratio are very persistent, as documented by the high quarterly auto-correlation of the PD ratio. Table 2 also reports the average real stock return, which is high and close to 2% per quarter. Stock returns are also very volatile: the standard deviation of stock returns is about four times its mean value. This contrasts with the behavior of the short-term risk-free interest rate documented in Table 2. The risk-free interest rate is very low and very stable. The standard deviation of the risk-free interest rate in Table 2 is likely even overstated, as ex-post realized inflation rates have been used to transform nominal safe rates into a real rate. Table 2 also reports the

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22 All standard deviations of moments reported in this section are computed by a procedure combining Newey-West estimators with the delta method as in Adam, Marcet, and Nicolini (2016). We refer to Appendix F of that paper for details.

23 The PD ratio is defined as the end-of-quarter stock price divided by dividend payments over the quarter. Following standard practice, dividends are deseasonalized by averaging dividends over the last four quarters.
Table 2

Key moments of stock prices, risk-free rates and dividends (U.S., quarterly real values, Q1:1955-Q4:2014)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Data moment</th>
<th>Std. dev. data moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average PD ratio</td>
<td>$E[P/D]$</td>
<td>152.3</td>
</tr>
<tr>
<td>Std. dev. PD ratio</td>
<td>$\sigma(P/D)$</td>
<td>63.39</td>
</tr>
<tr>
<td>Auto-correlation PD ratio</td>
<td>$\rho(P/D)$</td>
<td>0.98</td>
</tr>
<tr>
<td>Average equity return (%)</td>
<td>$E[r^e]$</td>
<td>1.87</td>
</tr>
<tr>
<td>Std. dev. equity return (%)</td>
<td>$\sigma(r^e)$</td>
<td>7.98</td>
</tr>
<tr>
<td>Average risk-free rate (%)</td>
<td>$E[r^f]$</td>
<td>0.25</td>
</tr>
<tr>
<td>Std. dev. risk-free rate (%)</td>
<td>$\sigma(r^f)$</td>
<td>0.82</td>
</tr>
<tr>
<td>Std. dev. dividend growth (%)</td>
<td>$\sigma(D_{t+1}/D_t)$</td>
<td>1.75</td>
</tr>
</tbody>
</table>

The standard deviation of dividend growth. Dividend growth is relatively smooth, especially when compared to the much larger fluctuation in equity returns. This fact is hard to reconcile with the observed large fluctuations in stock prices (Shiller, 1981).

Table 3 presents data moments that link the PD ratio to business cycle variables. It shows that stock prices are pro-cyclical: (1) the PD ratio correlates positively with hours worked; (2) stock prices also correlate positively with the investment to output ratio, but the correlation is surprisingly weak and also estimated very imprecisely. Table 4 below shows why this is the case: the investment to output ratio correlates positively with the PD ratio over the second half of the sample period (1985-2014), i.e., in the period with large stock price cycles, but negatively in the first half of the sample period (1955-1984).

Table 4 also shows that the overall investment to output ratio correlates much more strongly with the PD ratio if one excludes non-residential investment and investment in non-residential structure. Since our model does not feature real estate investment, we shall use the sample correlation for this reduced investment concept to evaluate the model. The negative correlation in the first half of the sample period, however, appears to be a robust feature of the data.

Table 3 also reports the correlation of the PD ratio with the one-year-ahead expected real stock market return of private U.S. investors. It shows that investors are optimistic about future holding period returns when the PD ratio is high already. As shown in Adam, Marcet, and Beutel (2017), this feature of the data is inconsistent with the notion that investors hold rational return expectations. Since we will consider an asset pricing model with subjective return expectations, we include this data moment into our analysis.

The correlation of the hours worked series with the PD ratio is positive in the first and second half of the sample period.
Table 3
Comovement of stock prices with real variables and survey return expectations (U.S., quarterly real values, Q1:1955-Q4:2014)

<table>
<thead>
<tr>
<th>Correlations</th>
<th>Symbol</th>
<th>Data moment</th>
<th>Std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hours &amp; PD ratio</td>
<td>$\rho(H, P/D)$</td>
<td>0.51</td>
<td>0.17</td>
</tr>
<tr>
<td>Investment-output &amp; PD ratio</td>
<td>$\rho(I/Y, P/D)$</td>
<td>0.19</td>
<td>0.31</td>
</tr>
<tr>
<td>Survey expect. &amp; PD ratio</td>
<td>$\rho(E[r], P/D)$</td>
<td>0.79</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Table 4
Stock prices and investment: alternative measures and sample periods (U.S., quarterly real values)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed investment</td>
<td>0.19</td>
<td>-0.64</td>
<td>0.40</td>
</tr>
<tr>
<td>Fixed investment, less residential inv. and nonresidential structures</td>
<td>0.58</td>
<td>-0.66</td>
<td>0.77</td>
</tr>
</tbody>
</table>

4 Asset Pricing in a Production Economy

We build our analysis on a stripped-down version of the representative agent model of Boldrin, Christiano, and Fisher (2001). This model features a consumption goods producing sector and an investment goods producing sector. Both sectors produce output using a neoclassical production function with capital and labor as input factors. Output from the investment goods sector can be invested to increase the capital stock.

We deviate from Boldrin, Christiano, and Fisher (2001) by using a standard time-separable specification for consumption preferences instead of postulating consumption habits. In addition, we remove all labor market frictions on the firm side by making hours worked perfectly flexible.

Given a linear specification for the disutility of labor, the labor market is then perfectly flexible and competitive. While frictions are arguably present in U.S. labor markets, we prefer the fully flexible specification to illustrate that our asset pricing implications do not depend on assuming labor market frictions. This feature distinguishes the present analysis from much of the earlier work.

We furthermore simplify our setup relative to the one studied in Boldrin, Christiano, and Fisher (2001) by specifying an exogenous capital accumulation process in the investment goods sector, in line with a balanced growth path solution. This helps with analytical tractability of the model, but also insures that the supply of new capital goods is sufficiently inelastic, so that the model has a chance of replicating the large persistent
swings in stock prices that can be observed in the data.\textsuperscript{25}

\section*{4.1 Production Technology}

There are two sectors, one producing a perishable consumption good (consumption sector), the other producing an investment good that can be used to increase the capital stock in the consumption sector (investment sector). The representative firm in each sector hires labor and rents capital, so as to produce its respective output good according to standard Cobb-Douglas production functions,

\[ Y_{c,t} = K_{c,t}^{\alpha_c} (Z_t H_{c,t})^{1-\alpha_c}, \quad Y_{i,t} = K_{i,t}^{\alpha_i} (Z_t H_{i,t})^{1-\alpha_i}, \quad (1) \]

where $K_{c,t}, K_{i,t}$ denote capital inputs and $H_{c,t}, H_{i,t}$ labor inputs in the consumption and the investment sector, respectively, and $\alpha_c \in (0, 1)$ and $\alpha_i \in (0, 1)$ the respective capital shares in production. $Z_t$ is an exogenous labor-augmenting level of productivity and the only source of exogenous variation in the model. Productivity follows

\[ Z_t = \gamma Z_{t-1} \varepsilon_t, \quad \log \varepsilon_t \sim i.i.d. \mathcal{N}\left(-\frac{\sigma^2}{2}, \sigma^2\right), \quad (2) \]

with $\gamma \geq 1$ denoting the mean growth rate of technology and $\sigma > 0$ the standard deviation of log technology growth.

Labor is perfectly flexible across sectors, but capital is sector-specific. The output of investment goods firms increases next period’s capital in the consumption goods sector, so that

\[ K_{c,t+1} = (1 - \delta_c) K_{c,t} + Y_{i,t}, \quad (3) \]

where $\delta_c \in (0, 1)$ denotes the depreciation rate. Capital in the investment goods sector evolves according to

\[ K_{i,t+1} = (1 - \delta_i) K_{i,t} + X_t \quad (4) \]

where $X_t$ is an exogenous endowment of new capital in the investment goods sector and $\delta_i \in (0, 1)$ the capital depreciation rate. We set $X_t$ such that $K_{i,t+1} \propto Z_t$, which insures that the model remains consistent with balanced growth.\textsuperscript{26} The assumed capital stock dynamics in the investment goods sector insures that capital good production that deviates from the balanced growth path is subject to decreasing returns to scale. This allows for persistent price fluctuations in the price of consumption capital around the balanced growth path.

\textsuperscript{25}In Boldrin, Christiano, and Fisher (2001), capital prices fail to display large and persistent fluctuations.

\textsuperscript{26}The model allocation is then identical to a setting in which investment is produced with labor only, i.e., $Y_{i,t} \propto Z_t \varepsilon_t^{-\alpha_i} (H_{i,t})^{1-\alpha_i}$. We prefer a specification that includes capital, capital depreciation and an exogenous investment input, as this allows us to define capital values in both sectors in a symmetric fashion.
An alternative interpretation for the decreasing returns to scale in the production of new investment goods is that decreasing returns reflect capital adjustment costs. In Appendix A.7 we provide a model specification with capital adjustment costs and a linear technology in the production of not yet installed capital goods. This alternative specification is isomorphic to the setup presented here.

4.2 Households

The representative household each period chooses consumption $C_t \geq 0$, hours worked $H_t \geq 0$, the end-of-period capital stocks $K_{c,t+1} \geq 0$ and $K_{i,t+1} \geq 0$ to maximize

$$E_P^0 \left[ \sum_{t=0}^{\infty} \beta^t (\log C_t - H_t) \right],$$

where the operator $E_P^0$ denotes the agent’s expectations in some probability space $(\Omega, S, \mathcal{P})$. Here, $\Omega$ is the space of realizations, $S$ the corresponding $\sigma$-algebra, and $\mathcal{P}$ a subjective probability measure over $(\Omega, S)$. As usual, the probability measure $\mathcal{P}$ is a model primitive and given to agents. The special case with rational expectations is nested in this specification, as explained below.

Household choices are subject to the flow budget constraint

$$C_t + K_{c,t+1}Q_{c,t} + K_{i,t+1}Q_{i,t} = W_t H_t + X_t (Q_{i,t} ((1 - \delta_i)R_{i,t} + R_{i,t})) + K_{i,t} ((1 - \delta_i)Q_{i,t} + R_{i,t}),$$

for all $t \geq 0$, where $Q_{c,t}$ and $Q_{i,t}$ denote the prices of consumption-sector and investment-sector capital, respectively, and $R_{c,t}$ and $R_{i,t}$ the rental rates earned by renting out capital to firms in the consumption and investment sector, respectively; $W_t$ denotes the wage rate and $X_t$ the endowment of new investment-sector capital.

To allow for subjective price beliefs, we shall consider an extended probability space relative to the case with rational expectations. In its most general form, households’ probability space is spanned by all external processes, i.e. by all variables that are beyond their control. These are given by the process $\{Z_t, X_t, W_t, R_{c,t}, R_{i,t}, Q_{c,t}, Q_{i,t}\}_{t=0}^{\infty}$, so that the space of realizations is

$$\Omega := \Omega_Z \times \Omega_X \times \Omega_W \times \Omega_{R,c} \times \Omega_{R,i} \times \Omega_{Q,c} \times \Omega_{Q,i},$$

where $\Omega_X = \prod_{t=0}^{\infty} \mathbb{R}$ with $X \in \{Z, X, W, R_c, R_i, Q_c, Q_i\}$. Letting $S$ denote the sigma-algebra of all Borel subsets of $\Omega$, beliefs will be specified by a well-defined probability measure $\mathcal{P}$ over $(\Omega, S)$. Letting $\Omega^t$ denote the set of all partial histories up to period $t$, households’ decision functions can then be written as

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \Omega^t \rightarrow \mathbb{R}^4.$$
We assume that households choose these functions, so as to maximize (5) subject to the constraints (6).

In the special case with rational expectations, \((X, W, R_c, R_i, Q_c, Q_i)\) are typically redundant elements of the probability space \(\Omega\), because households are assumed to know that these variables can at time \(t \geq 0\) be expressed as known deterministic equilibrium functions of the history of fundamentals \(Z_t\) only. Without loss of generality, one can then exclude these elements from the probability space and write:

\[
(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \Omega_Z^t \rightarrow \mathbb{R}^4,
\]

where \(\Omega_Z^t = \prod_{s=0}^t \mathbb{R}\) is the space of all realizations of \(Z_t = (Z_0, Z_1, ..., Z_t)\). This routinely performed simplification implies that households perfectly know how the markets determine the excluded variables as a function of the history of shocks. By introducing subjective beliefs, we will step away from this assumption.

To insure that the household’s maximization problem remains well-defined in the presence of the kind of subjective price beliefs introduced below, we impose additional capital holding constraints of the form \(K_{c,t+1} \leq \overline{K}_{c,t+1}\) and \(K_{i,t+1} \leq \overline{K}_{i,t+1}\), for all \(t \geq 0\), where the bounds \((\overline{K}_{c,t+1}, \overline{K}_{i,t+1})\) are assumed to increase in line with the balanced growth path and are assumed sufficiently large, such that they never bind in equilibrium. The bounds also need to be sufficiently tight such that the transversality condition holds.

Their precise choice, however, does not affect equilibrium outcomes.

### 4.3 Competitive Equilibrium

The competitive equilibrium of an economy in which households hold subjective beliefs is defined as follows:

**Definition 1.** For given initial conditions \((K_{c,-1}, K_{i,-1})\), a competitive equilibrium with subjective household beliefs \(P\) consists of allocations \(\{C_t, H_t, H_{c,t}, H_{i,t}, K_{c,t+1}, K_{i,t+1}\}_{t=0}^\infty\) and prices \(\{Q_{c,t}, Q_{i,t}, R_c, R_i, W_t\}_{t=0}^\infty\), all of which are measurable functions of the process \(\{Z_t\}_{t=0}^\infty\), such that for all partial histories \(Z'_t = (Z_0, Z_1, ..., Z_t)\) and all \(t \geq 0\), prices and allocations are consistent with

1. profit maximizing choices by firms,
2. the subjective utility maximizing choices for households decision functions (7), and

\(^{27}\)This assumes that there are no sunspot fluctuations in the rational expectations equilibrium.

\(^{28}\)Sunspot fluctuations require either keeping some of the endogenous variables or including the sunspot variables into the probability space.

\(^{29}\)These capital holding constraints are required for subjective price beliefs to be consistent with internal rationality in a setting with an effectively exogenous stochastic discount factor, see Adam and Marcet (2011) for details.

\(^{30}\)Appendix A.4 presents an example for bounds that satisfy both requirements simultaneously.
3. market clearing for consumption goods \( (C_t = Y_{c,t}) \), hours worked \( (H_t = H_{c,t} + H_{i,t}) \), and the two capital goods (equations (3) and (4)).

The equilibrium requirements are weaker than what is required in a competitive rational expectations equilibrium, because household beliefs are not restricted to be rational. For the special case where \( \mathcal{P} \) incorporates rational expectations, the previous definition defines a standard competitive rational expectations equilibrium.

4.4 Connecting Model Variables to Data Moments

In order to compare our model to the data, we need to define the real variables (investment, output) and stock market variables (stock prices, dividends) in our production economy. For the real variables, this is relatively straightforward. We follow Boldrin, Christiano, and Fisher (2001) and define investment as being proportional to the quantity of capital produced and use the steady state capital price \( Q^{ss}_c \) as a base price, so that fluctuations in the price of capital do not contribute to fluctuations in real investment. Investment is thus given by

\[
I_t = Q^{ss}_c Y_{i,t}
\]

and output correspondingly by

\[
Y_t = C_t + I_t.
\]

To define stock prices and dividends, we consider a setup where (investment- and consumption-sector) capital can be securitized via shares and where shares and capital can be jointly created or jointly destroyed at no cost. The absence of arbitrage opportunities then implies that the price of shares is determined by the price of the capital it securitizes. We consider a representative consumption-sector share and a representative investment-sector share. The only free parameter in this extended setup is then the dividend policy of stocks, which is indeterminate (Miller and Modigliani, 1961). To obtain a parsimonious setting, we assume a time-invariant profit payout share \( p \in (0, 1) \): a share \( p \) of rental income/profits per share is paid out as dividends each period and in both sectors, with the remaining share \( 1 - p \) being reinvested in the capital stock that the share securitizes.

We now describe the setting for the consumption sector in greater detail. The setup for the investment sector is identical up to an exchange of subscripts. Let \( k_{c,t} \) denote the units of (beginning-of-period \( t \)) capital held per unit of shares issued in the consumption sector. The capital is used for production and earns a rental income/profit of \( k_{c,t} R_{c,t} \). Given the payout ratio \( p \in (0, 1) \), dividends per share are given by

\[
D_{c,t} = pk_{c,t} R_{c,t}.
\]

Retained profits are reinvested to purchase \((1 - p) k_{c,t} R_{c,t} / Q_{c,t}\) units of new capital per share.\(^{31}\) The end-of-period capital per share then consists of the depreciated beginning-of-

\(^{31}\)In case the aggregate capital supply differs from capital demand implied by the existing number of shares, new shares are created or existing shares repurchased to equilibrate capital demand and supply.
period capital stock and purchases of new capital from retained profits. The end-of-period share price $P_{c,t}$ is thus equal to$^{32}$

$$
P_{c,t} = (1 - \delta)k_{c,t}Q_{c,t} + (1 - p)k_{c,t}R_{c,t},
$$

and the end-of-period PD ratio is given by

$$
\frac{P_{c,t}}{D_{c,t}} = \frac{1 - \delta_c}{p} \frac{Q_{c,t}}{R_{c,t}} + \frac{1 - p}{p}.
$$

Since the last term is small for reasonable payout ratios $p$, the end-of-period PD ratio is approximately proportional to the capital price over rental price ratio ($Q_{c,t}/R_{c,t}$). Moreover, as is easily verified, the equity return per unit of stock $R_{e,c,t} = (P_{c,t} + D_{c,t})/P_{c,t-1}$ is equal to the return per unit of capital $R^k_{c,t} = ((1 - \delta_c)Q_{c,t} + R_{c,t})/Q_{c,t-1}$.

Given sectoral stock prices and PD ratios, we can define the aggregate PD ratio using a value-weighted portfolio of the sectoral investments. Let

$$
w_{c,t-1} = \frac{Q_{c,t-1}K_{c,t}}{Q_{c,t-1}K_{c,t} + Q_{i,t-1}K_{i,t}}
$$

denote the end-of-period $t-1$ value share of the consumption sector. The value share of the investment sector is then $1 - w_{c,t-1}$. A portfolio with total value $P_{t-1}$ at the end of period $t-1$ and value shares $w_{c,t-1}$ and $1 - w_{c,t-1}$ in the consumption and investment-sector, respectively, must contain $w_{c,t-1}P_{t-1}/P_{c,t-1}$ consumption shares and $(1 - w_{c,t-1})P_{t-1}/P_{i,t-1}$ investment shares. The end-of-period $t$ value of this portfolio is then given by

$$
P_t = w_{c,t-1}\frac{P_{t-1}}{P_{c,t-1}}P_{c,t} + \frac{(1 - w_{c,t-1})P_{t-1}}{P_{i,t-1}}P_{i,t},
$$

(8)

and period $t$ dividend payments for this portfolio are

$$
D_t = w_{c,t-1}\frac{P_{t-1}}{P_{c,t-1}}D_{c,t} + \frac{(1 - w_{c,t-1})P_{t-1}}{P_{i,t-1}}D_{i,t},
$$

(9)

Using the previous two equations, the aggregate PD ratio can be expressed as a weighted mean of the sectoral PD ratios where the weights are given by the share of portfolio dividends coming from each sector:

$$
\frac{P_t}{D_t} = \frac{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}}}{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}} + (1 - w_{c,t-1})\frac{D_{c,t}}{P_{c,t-1}}} \frac{P_{c,t}}{D_{c,t}} + \frac{(1 - w_{c,t-1})\frac{D_{i,t}}{P_{i,t-1}}}{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}} + (1 - w_{c,t-1})\frac{D_{c,t}}{P_{c,t-1}}} \frac{P_{i,t}}{D_{i,t}}.
$$

Note that the PD ratio is independent of the initial portfolio value $P_{t-1}$. Aggregate dividend growth can similarly be expressed using equations (9) and (8) as a weighted average of the sectoral dividend growth rates. This completes our definition of model variables.$^{32}$We compute end-of-period share prices, because this is the way prices have been computed in the data.
5 Price Beliefs, Probability Space and State Space

In specifying household beliefs $\mathcal{P}$, we shall consider two alternative belief specifications: (1) a standard setting in which all expectations are rational and (2) a setting in which households hold subjective beliefs about future capital prices $(Q_{c,t+j}, Q_{i,t+j})$, in line with the belief setup considered in Adam, Marcet, and Beutel (2017), but rational expectations about all remaining variables $(Z_{t+j}, X_{t+j}, W_{t+j}, R_{c,t+j}, R_{i,t+j})$.

We consider the setting with fully rational expectations as a point of reference. It is well known that under rational expectations the asset pricing implications of the model are strongly at odds with the data. Considering this setup, however, allows highlighting the empirical improvements achieved by introducing subjective price beliefs.

Two considerations motivate us to keep rational expectations about variables other than prices: first, we do not want to deviate from the rational expectations assumption by more than in Adam, Marcet, and Beutel (2017), so as to illustrate that the same deviation that can be used to explain stock price behavior in an endowment setting can explain stock price behavior and business cycle dynamics; second, investor expectations about future stock prices can be observed relatively easily from investor survey data, which allows disciplining the subjective belief choice. Observing beliefs about future values of $(X_{t+j}, W_{t+j}, R_{c,t+j}, R_{i,t+j})$ is a much harder task, which prompts us to keep the standard assumption of rational expectations.

Under rational expectations, decision functions in period $t$ depend only on the history of fundamental shocks $Z_t$, as discussed in Section 4.2. Given the Markov-structure for shocks $Z_t$, there is furthermore a recursive time-invariant form of the decision functions, where decisions depend only on current shock $Z_t$ and the beginning-of-period capital stocks $(Z_t, K_{c,t}, K_{i,t})$. Using this fact, we can standardly solve for the nonlinear rational expectations equilibrium using global approximation methods.\textsuperscript{33}

We consider subjective capital price expectations of the form previously introduced in Adam, Marcet, and Beutel (2017). Specifically, we assume that agents perceive the end-of-period capital prices $Q_{s,t}$ ($s = c, i$) to evolve as

$$\log Q_{s,t} = \eta_Q \log Q_{s,t-1} + (1 - \eta_Q) \log \bar{Q}_s + \log \beta_{s,t} + \log \nu_{s,t},$$

(10)

where $\log \bar{Q}_s$ is the perceived long-run mean of $\log Q_{s,t}$, $\nu_{s,t}$ denotes a transitory shock to price growth and $\beta_{s,t}$ a persistent component, which is given by

$$\log \beta_{s,t} = \eta_\beta \log \beta_{s,t-1} + \log \nu_{s,t}.$$ 

(11)

$\eta_Q, \eta_\beta \in [0, 1]$ are given parameters that capture the degree of persistence in beliefs about the level and growth rate of prices. To keep the belief parameterization as parsimonious as possible, we consider in this work the limiting case without belief mean reversion.

\textsuperscript{33}This requires a standard transformation of variables, so as to render them stationary.
In this case, the precise choice of the perceived long-run mean $\bar{Q}_s$ becomes irrelevant.\(^{35}\)

The innovations ($\varepsilon_{s,t}, \nu_{s,t}$) are independent of each other, with $\log \varepsilon_{s,t} \sim \text{i}i\text{N}(-\sigma^2_\varepsilon/2, \sigma^2_\varepsilon)$ and $\log \nu_{s,t} \sim \text{i}i\text{N}(-\sigma^2_\nu/2, \sigma^2_\nu)$, and also independent of all other variables.\(^{36}\)

Each period $t$, agents only observe capital prices $\log Q_s$ up to period $t$ and estimate the persistent component driving capital gains. Let $\log \beta_{s,t-1} \sim \mathcal{N}(\log m_{s,t-1}, \sigma^2)$ denote the period $t-1$ prior belief about the persistent component, where $\sigma$ denotes the steady state Kalman filter uncertainty.\(^{37}\)

In period $t$, agents observe the new capital price $Q_{s,t}$ and update their beliefs using Bayes’ law. Their period $t$ beliefs are then given by $\log \beta_{s,t} \sim \mathcal{N}(\log m_{s,t}, \sigma^2)$ with

$$
\log m_{s,t} = \log m_{s,t-1} - \frac{\sigma^2_\varepsilon}{2} + g \left( \log Q_{s,t} - \log Q_{s,t-1} + \frac{\sigma^2_\varepsilon}{2} - \log m_{s,t-1} \right),
$$

where the Kalman gain is given by

$$
g = \frac{\sigma^2_\varepsilon}{\sigma^2_\varepsilon + \sigma^2_\nu}.
$$

Agents’ beliefs can thus be parsimoniously summarized by the variables $(m_{c,t}, m_{i,t})$, which capture agents’ degree of optimism about future capital gains. Equation (12) shows how agents’ capital gain expectations are driven by observed past capital gains, whenever $g > 0$. Adam, Marcet, and Beutel (2017) show that for values of $g$ around 2% this delivers an empirically plausible specification describing the dynamics of investors’ subjective capital gain beliefs over time.

To avoid simultaneous determination of price beliefs and prices, we follow Adam, Marcet, and Beutel (2017) and use a slightly modified information setup in which agents receive at time $t$ information about the transitory component in $t-1$, $\log \varepsilon_{s,t-1}$. The modification causes the updating equation (12) to contain only lagged price growth and no variance correction terms. Capital gain beliefs then evolve according to

$$
\log m_{s,t} = \log m_{s,t-1} + g (\log Q_{s,t-1} - \log Q_{s,t-2} - \log m_{s,t-1}) + g \log \varepsilon_{s,t}^{1},
$$

\(34\) Technically choosing $\eta_Q = \eta_\beta = 1$ would lead to a violation of households’ transversality condition. We therefore have to consider the limit. In Appendix A.4 we show that arbitrary small degrees of belief mean reversion are sufficient for households’ transversality condition to hold.

\(35\) The following presentation formally puts $\eta_Q = \eta_\beta = 1$ for ease of exposition. Discussion of the full belief specification is relegated to the appendix.

\(36\) The random variables $\varepsilon_{s,t+1}$, $\nu_{s,t+1}$ are not defined on the probability space $\Omega$, but on a larger auxiliary space $\bar{\Omega}$, as they contain random variation that is independent of variation in both prices and other observables in equilibrium. Decision function can still be specified to be functions of $\Omega$ only.

\(37\) We have $2\sigma^2 \equiv -\sigma^2_\nu + \sqrt{(\sigma^2_\nu)^2 + 4\sigma^2_\nu\sigma^2_\varepsilon}$. 

20
where $\log \epsilon_{s,t} \sim \mathcal{N}(\frac{-\sigma^2}{2}, \sigma^2)$ is a time $t$ innovation to agents’ information set (unpredictable using information available to agents up to period $t-1$), which captures information that agents receive in period $t$ about the transitory price growth component $\log \epsilon_{s,t-1}$.\(^{38}\) Agents’ expectations are then given by

$$E_t^\mathcal{P} \left[ \frac{Q_{s,t+1}}{Q_{s,t}} \right] = m_{s,t}.$$  \hspace{1cm} (15)

Given the specific subjective price beliefs and the assumption of rational expectations about $(Z_{t+j}, X_{t+j}, W_{t+j}, R_{c,t+j}, R_{i,t+j})$, we can work with the probability space

$$\tilde{\Omega} \equiv \Omega_Z \times \Omega_{Q,c} \times \Omega_{Q,i}$$

and consider decision functions of the form

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \tilde{\Omega}^t \rightarrow R^4.$$  \hspace{1cm} (16)

Moreover, the subjective belief setup allows us to summarize the history of capital prices in the two sectors using the two state variables $(m_{c,t}, m_{it}) \in \mathbb{R}^2$, so that household decision functions are time-invariant function of an extended set of state variables $(Z_t, K_{c,t}, K_{i,t}, m_{c,t}, m_{it})$. The state $S_t$ describing the aggregate economy at time $t$ can be summarized by the vector $S_t = (Z_t, K_{c,t}, K_{i,t}, m_{c,t}, m_{it}, Q_{c,t-1}, Q_{i,t-1})$, where the latter two variables are required to be able to describe the equilibrium dynamics of beliefs over time, as implied by equation (14). The economy’s equilibrium dynamics can then be described by a nonlinear state transition function $G(\cdot)$ mapping current states and future technology into future states

$$S_{t+1} = G(S_t, Z_{t+1}),$$

and by an outcome function $F(\cdot)$ mapping states into economic outcomes for the remaining variables

$$(W_{t}, R_{c,t}, R_{i,t}, Y_{t}, C_{t}, I_{t}, H_{t}) = F(S_t).$$

We endow households with beliefs about future values $(W_{t+j}, R_{c,t+j}, R_{i,t+j})$ consistent with these equilibrium mappings.\(^{39}\) Since the mappings depend themselves on the solution

$$\log m_{s,t} = \log m_{s,t-1} + g (\log Q_{s,t-1} - \log Q_{s,t-2} - \log m_{s,t-1})$$

and the reduced outcome space $\tilde{\Omega}$ below, which does not include observable innovations $\epsilon_{c,t}^1$ and $\epsilon_{i,t}^1$.\(^{39}\) That is, households beliefs $\mathcal{P}$ over the full set of exogenous variables $\Omega$ are jointly described by the equilibrium mappings $F$ and $G$ and a measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$ that is consistent with the exogenous law of motion for $Z$ and the perceived laws of motion for $Q_i$ and $Q_c$ implied by the learning specification above. Appendix A.3.1 provides details on how $\mathcal{P}$ can be recovered from $\tilde{\mathcal{P}}$ and the mappings $F$ and $G$. Appendix A.3.2 proves that there is a unique measure $\tilde{\mathcal{P}}$ consistent with the subjective belief description given in this section.

\(^{38}\)However, we follow the baseline specification of Adam, Marcet, and Beutel (2017) and assume that along the equilibrium path no actual information is observed, instead set $\log \epsilon_{s,t}^1$ to 0 each period. It is easy to see in our specific model, that – given the solution to the filtering problem – beliefs about future $m$ values do not matter for policy functions. For this reason we work with a reduced updating equation.

\(^{39}\)That is, households beliefs $\mathcal{P}$ over the full set of exogenous variables $\Omega$ are jointly described by the equilibrium mappings $F$ and $G$ and a measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$ that is consistent with the exogenous law of motion for $Z$ and the perceived laws of motion for $Q_i$ and $Q_c$ implied by the learning specification above. Appendix A.3.1 provides details on how $\mathcal{P}$ can be recovered from $\tilde{\mathcal{P}}$ and the mappings $F$ and $G$. Appendix A.3.2 proves that there is a unique measure $\tilde{\mathcal{P}}$ consistent with the subjective belief description given in this section.
to the household problem and the solution to the household problem on the assumed mappings, one needs to solve for a fixed point. Because the economy features seven state variables, it is generally not feasible to solve for the fixed point of this problem at a speed that would allow estimating the subjective belief model using the simulated methods of moments. Yet, as we explain below, it becomes computationally feasible for our log-linear specification for household preferences.

6 Equilibrium Conditions, Stable Consumption and Financial Accelerator

This section derives the set of equations characterizing the competitive equilibrium. These equations hold independently of the assumed belief structure and show how the model can generate a smooth consumption process, even in the presence of volatile stock prices. They additionally show how subjective price beliefs can contribute to price volatility and give rise to a ‘financial accelerator’ that lets optimism and pessimism in price beliefs affect the dynamics of real variables.

From the household’s first-order conditions we get

\[ C_t = W_t, \quad (17) \]

\[ Q_{c,t} = \beta E_t^P \left[ \frac{W_t}{W_{t+1}} \left( (1 - \delta_c) Q_{c,t+1} + R_{c,t+1} \right) \right], \quad (18) \]

\[ Q_{i,t} = \beta E_t^P \left[ \frac{W_t}{W_{t+1}} \left( (1 - \delta_i) Q_{i,t+1} + R_{i,t+1} \right) \right], \quad (19) \]

for all \( t \geq 0 \). The first equation is due to our log-linear specification of household utility and shows that wages will be smooth, whenever consumption is a smooth process. The last two equations show how optimism/pessimism about future capital prices (high or low values for \( E_t^P Q_{c,t+1} \) or \( E_t^P Q_{i,t+1} \)) affect – ceteris paribus – current capital prices. Variations in subjective price expectations can thus contribute to increasing the variability of capital prices (and thus stock prices), relative to a setting with rational expectations. Optimal decisions by households also imply the transversality condition

\[ \lim_{t \to \infty} \beta^t E_t^P \left[ \frac{1}{W_t} \left( K_{c,t+1} Q_{c,t} + K_{i,t+1} Q_{i,t} \right) \right] = 0. \quad (20) \]

---

40 We only assume that beliefs are such that they are consistent with optimal behavior, i.e. beliefs are such that there exist choices that satisfy the equilibrium/optimality conditions. This is in fact the case for the belief configurations that we shall consider.

41 The household budget constraint will hold automatically, as we will keep all market clearing conditions in the set of equations characterizing equilibrium.

42 See e.g. Kamihigashi (2005) for the formulation of transversality conditions in stochastic problems with inequality constraints.
Appendix A.4 provides conditions under which the transversality condition is satisfied. We verify these conditions for our estimated model parameters.

The first-order conditions of consumption-sector firms deliver

\[
W_t = \frac{(1 - \alpha_c) Y_{c,t}}{H_{c,t}}, \quad (21)
\]

\[
R_{c,t} = \frac{\alpha_c Y_{c,t}}{K_{c,t}}, \quad (22)
\]

and the optimality conditions of investment-sector firms

\[
W_t = (1 - \alpha_i) Q_{c,t} K_{i,t}^{1 - \alpha_i} Z_{1,t}^{1 - \alpha_i} H_{i,t}^{1 - \alpha_i}, \quad (23)
\]

\[
R_{i,t} = \alpha_i Q_{c,t} K_{i,t}^{1 - \frac{1}{\alpha_i}} Z_{1,t}^{1 - \alpha_i} H_{i,t}^{1 - \alpha_i}. \quad (24)
\]

The market clearing conditions are given by

\[
C_t = Y_{c,t}, \quad (25)
\]

\[
H_t = H_{c,t} + H_{i,t}, \quad (26)
\]

\[
K_{c,t+1} = (1 - \delta_c) K_{c,t} + Y_{i,t}, \quad (27)
\]

\[
K_{i,t+1} = (1 - \delta_i) K_{i,t} + X_t. \quad (28)
\]

For the case with subjective beliefs, capital price expectations are given by

\[
E_t^P [Q_{s,t+1}] = m_{s,t} Q_{s,t} \quad \text{for } s = c, i, \quad (29)
\]

where \(m_{s,t}\) evolves according to (14) and expectations about all other variables are rational. For the case with rational expectations, we simply have \(E_t^P [\cdot] = E_t [\cdot]\).

**Stable Consumption.** Equations (17), (21) and (25) show that hours worked in the consumption sector are constant over time:

\[
H_{c,t} = 1 - \alpha_c. \quad (30)
\]

This is a result of our log-linear utility specification and is important for two reasons.

First, it implies that consumption variations are exclusively driven by productivity changes and by the dynamics of capital accumulation in the consumption sector.\(^{43}\) Since productivity shocks tend to be small and capital accumulation a slowly moving process, it gives the model a chance to replicate the observed smoothness of the aggregate consumption series. Clearly, this would still be approximately the case, if one deviated slightly from the exact log-linear utility specification.

\(^{43}\)This follows from the production function for \(Y_{c,t}\) in equation (1) and the market clearing condition (25), which together with equation (30) imply \(C_t = (1 - \alpha_c) Z_{1,t}^{1 - \delta_c} (K_{c,t})^{\alpha_c}\).
Second, the fact that $H_{c,t}$ is constant, allows expressing the future wage $W_{t+1}$ showing up in the capital pricing equations (18) and (19) as a function of current variables and future exogenous variables. In particular, combining equations (1), (21) and (30) for period $t + 1$, we obtain

$$W_{t+1} = ((1 - \alpha_c) Z_{t+1})^{1 - \alpha_c} K_{c,t+1}^{\alpha_c}.$$  

(31)

The log-linear specification thus allows for a simple and fast computation of wage expectations $E_t W_{t+1}$ as a function of the time $t$ capital choice $K_{c,t+1}$, without having to explicitly solve the nonlinear household optimization problem. This feature is absolutely crucial for being able to efficiently solve for the equilibrium dynamics of the nonlinear model under subjective price expectations and allows us to estimate the nonlinear model using the Simulated Method of Moments.\textsuperscript{44} For the estimation, it is thus key to use the exact log-linear specification.\textsuperscript{45}

**Financial Accelerator.** From equation (23) follows that hours worked in the investment sector are given by

$$H_{i,t} = K_{i,t} Z_t^{\frac{1 - \alpha_i}{\alpha_i}} \left(1 - \alpha_i \right) \left(\frac{Q_{c,t}}{W_t}\right)^{\frac{1}{\alpha_i}},$$

(32)

which shows that high prices for consumption sector capital ($Q_{c,t}$) induce - ceteris paribus - high labor demand by firms in the investment sector and thus drive up investment in the consumption sector. Since investment sector firms produce new consumption sector capital, a high value for $Q_{c,t}$ signals to them that it pays to expand production, even in the presence of decreasing returns to scale.\textsuperscript{46} Since $Q_{c,t}$ is potentially affected by subjective capital price expectations, as discussed before, belief-driven fluctuations in the price of consumption sector capital give rise to fluctuations in hours worked and in investment. We thus have a financial accelerator that translate belief-driven price fluctuations into variations of real variables. Importantly, the accelerator is such that it gives rise to positive comovement between capital prices, hours worked and investment, akin to the empirical evidence presented in section 3.

\textsuperscript{44}It still takes about one week to estimate the model parameters, despite parallelization efforts.

\textsuperscript{45}When deviating from this specification - even if only slightly - one has to first solve the dynamic household optimization problem. As explained in the previous section, this is a nonlinear problem in which the two capital stocks and the two beliefs variables are endogenous state variables and in which the postulated beliefs about future wages and rental rates for capital must be consistent with the resulting equilibrium outcome. The latter requires solving for a fixed point in this nonlinear optimization problem. This is computationally so expensive that it prevents carrying out a formal estimation approach.

\textsuperscript{46}There are decreasing returns in the short-run because capital in the investment sector is predetermined within the period. In the long-run investment sector capital grows in line with the balanced growth path.
7 Equilibrium Dynamics under Subjective Beliefs

This section provides analytic insights into the equilibrium dynamics of the model under subjective beliefs. It shows how the model can give rise to persistent swings in capital prices (and thus stock prices) and how these swings translate into investment and capital accumulation dynamics. It also illustrates the quantitative importance of the price and investment dynamics using the estimated model from the next section.

We start by studying the price dynamics of consumption-sector capital. Given the assumption of rational dividend and wage expectations and given the assumed subjective price beliefs (29), we obtain from equation (18)

\[ Q_{c,t} = \beta E_t \left[ \frac{W_t}{W_{t+1}} \right] (1 - \delta_c) m_{c,t} Q_{c,t} + \beta E_t \left[ \frac{W_t}{W_{t+1}} R_{c,t+1} \right]. \]  

(33)

Using the equilibrium relationships (17), (22) and (25) allows expressing the discounted expected rental rate (the last term in equation (33)) using period \( t \) variables:

\[ E_t \left[ \frac{W_t}{W_{t+1}} R_{c,t+1} \right] = \alpha_c W_t K_{c,t+1} \]  

(34)

Substituting the previous expression into equation (33), solving for \( Q_{c,t} \) and using equation (31) to substitute \( W_t \) and \( W_{t+1} \), delivers

\[ Q_{c,t} = \frac{\beta \alpha_c (1 - \alpha_c)^{1-\alpha_c}}{1 - \beta (1 - \delta_c) e} \left( \frac{K_{c,t}}{K_{c,t+1}} \right)^\alpha_c m_{c,t} \left( \frac{Z_t}{K_{c,t+1}} \right)^{1-\alpha_c} \left( \frac{K_{c,t}}{K_{c,t+1}} \right)^{\alpha_c}, \]  

(35)

where \( e = E_t [(\gamma \varepsilon_{t+1})^{\alpha_c-1}] > 0 \). Equation (35) reveals that time variation in capital price \( Q_{c,t} \) is driven by time-variation in three key variables: the belief variable \( m_{c,t} \), the (gross) growth rate of the capital stock \( K_{c,t+1}/K_{c,t} \) and the capital to technology ratio \( K_{c,t+1}/Z_t \).

To understand the price dynamics implied by equation (35), it is useful to consider the deterministic balanced growth path under rational expectations as a reference point. This reference point is achievable under the subjective belief setting when \( m_{c,t} = m_{i,t} = 1 \) for all \( t \); moreover, in this reference point the variables \( K_{c,t+1}/K_{c,t} \) and \( K_{c,t+1}/Z_t \) assume constant values, so that the capital price \( Q_{c,t} \) is also constant. These features allow for a clear understanding of the pricing contributions associated with variations in subjective beliefs. One difficulty with assessing these contributions is, however, the fact that variations in subjective beliefs \( m_{c,t} \) not only have a direct effect on capital prices, as suggested by equation (35), but also an indirect effect through the financial accelerator effects discussed in the previous section, which imply that the terms \( K_{c,t+1}/K_{c,t} \) and \( K_{c,t+1}/Z_t \) are also affected by subjective beliefs.

To analyze the joint dynamics of capital prices, price beliefs and the consumption sector capital \( (Q_{c,t}, m_{c,t}, K_{c,t+1}) \), it is useful to observe that the dynamics of capital accumulation tend to be a slow moving process. This is so because the production of new
capital is subject to decreasing returns to scale and because the rate of depreciation is relatively low on a quarterly basis. The gross growth rate of the capital stock \( \frac{K_{c,t+1}}{K_{c,t}} \) will thus deviate only little from its balanced growth path value \( \gamma \). While we will discuss the pricing effects associated with time variation in \( \frac{K_{c,t+1}}{K_{c,t}} \) further below, we proceed for now by using the approximation \( \frac{K_{c,t+1}}{K_{c,t}} \approx \gamma \). The pricing equation (35) the simplifies to

\[
Q_{c,t} = \frac{\beta \alpha_c (1 - \alpha_c)^{1-\alpha_c} \gamma^{-\alpha_c}}{1 - B \cdot m_{c,t}} \left( \frac{Z_t}{K_{c,t+1}} \right)^{1-\alpha_c},
\]

where

\[
B \equiv \frac{\beta (1 - \delta_c)}{\gamma} E_t \left[ (\varepsilon_{t+1})^{\alpha_c-1} \right] < 1.
\]

Since \( B \) is relatively close one and \( m_{c,t} \) fluctuates around a value of one (the deterministic balanced growth path value), equation (36) shows that even small variations in beliefs \( m_{c,t} \) generate large pricing implications, while correspondingly small variations in \( \frac{K_{c,t+1}}{Z_t} \) generate only small prices effects. We can thus proceed by keeping \( \frac{K_{c,t+1}}{Z_t} \) fixed at its (constant) balanced growth path value.\(^{47}\) We shall consider the additional effects arising from variations in \( \frac{K_{c,t+1}}{Z_t} \) further below.

With a constant value for \( \frac{K_{c,t+1}}{Z_t} \), equation (36) simplifies further to

\[
Q_{c,t} = \frac{A}{1 - B \cdot m_{c,t}},
\]

where \( A \) is a positive constant. In this simplified setting, capital price variations arise exclusively from belief variations. In fact, combining the previous equation with the belief updating equation (14) delivers a second order difference equation that uniquely determines the dynamics of beliefs and thereby the dynamics of capital prices according to equation (37).

Figure 3 illustrates these dynamics using a 2-dimensional phase diagram for the variables \( (m_{c,t}, m_{c,t-1}) \).\(^{48}\) The two solid black lines indicate the points in the space along which the dynamics imply \( m_{c,t} = m_{c,t-1} \) and \( m_{c,t+1} = m_{c,t} \), respectively. The blue arrows in the figure indicate the directions in which the belief variables adjust from one period to the next, when being away from those two lines. They show that beliefs have a tendency to move counterclockwise around the rational expectations balance growth path point \( (m_{c,t}, m_{c,t-1}) = (1, 1) \), which is marked by a black dot in the figure.

Since the Kalman gain \( g \) in the belief updating equation (14) is small, in line with what the survey data suggests, beliefs will tend to move – for the most part – slowly around the balance growth path outcome.\(^{49}\) This is indicated by the red dotted line. The

\(^{47}\)This amounts to effectively shutting down any financial accelerator effects arising from subjective belief variations.

\(^{48}\)See appendix A.5 for details on how the phase diagram can be constructed.

\(^{49}\)Absent further shocks, beliefs would over time locally converge to the point \( (m_{c,t}, m_{c,t-1}) = (1, 1) \).
The red line shows how beliefs evolve when starting from an initial belief below the balanced growth value ($m_{c,t} < 1$) and a situation in which beliefs have been rising ($m_{c,t} > m_{c,t-1}$). It shows that beliefs keep rising further over time and well beyond the point ($m_{c,t} = 1$). This is so because the mere fact that beliefs increase generates additional capital gains, see equation (37), which cause upward revisions in beliefs beyond the balanced growth path outcome. If the increase in beliefs becomes too weak, i.e., if the belief pair $(m_{c,t}, m_{c,t-1})$ approaches the 45 degree line from below, then $m_{c,t}$ starts reversing direction. At this point, the economy experiences a ‘Minsky moment’: beliefs are high but stop increasing further, so that realized capital gain outcomes disappoint.\textsuperscript{50} Beliefs will then jump across the 45 degree line and enter a period of persistent reversal, as indicated in the figure.

We can now consider the additional effects associated with variations in $K_{c,t+1}/Z_t$ in equation (36). Combining equation (32), which captures the financial accelerator effect, \textsuperscript{\textsuperscript{50}With $m_{c,t} \leq m_{c,t-1}$, it follows from equation (37) that $Q_{c,t}/Q_{c,t-1} \leq 1$, so that outcomes disappoint whenever $m_{c,t} > 1$.}
with equation (31) for period \( t \) to substitute wages, we obtain

\[
H_{it} = C_t \cdot \left( \frac{Q_{c,t}}{(K_{c,t}/Z_t)^{\alpha_c}} \right)^{\frac{1}{\alpha_i}}
\]

(38)

where \( C_t \equiv \frac{K_{i,t}}{Z_t} \left( \frac{(1-\alpha_i)}{(1-\alpha_c)} \right)^{\frac{1}{\alpha_i}} \) is constant in the absence of productivity shocks. Since \( K_{i,t} \) and \( Z_t \) are exogenous processes, it follows from the production function for \( Y_{i,t} \) that variations in \( H_{it} \) capture the amount of new investment relative to the deterministic balanced growth path (along which \( H_{it} \) is constant). Equation (32) thus reveals that investment is positively associated with \( Q_{c,t} \) but negatively associated with \( K_{c,t}/Z_t \). During a purely belief-driven run-up in capital prices, as illustrated in figure 3, it follows from equation (38) that investment expands, provided capital prices \((Q_{c,t})\) increase faster than the capital stock \((K_{c,t}/Z_t)\). This will be the case in our estimated model because the capital stock will adjust only slowly over time. A persistent run-up in capital prices is then associated with a persistent run-up in investment, causing prices and investment to comove. It also causes the ratio \( K_{c,t+1}/Z_t \) to slowly increase over time relative to its deterministic balanced growth path value. Equation (36) show that this increase dampens the stock price run-up, but as discussed before, the effect is - at least initially - quantitatively small. The dampening occurs because the build-up in consumption sector capital reduces the marginal productivity of capital and thereby lowers the rental rates of capital. The capital build-up will also lead to an increase in the ratio \( K_{c,t+1}/K_{c,t} \) relative to its balanced growth path value. It follows from equation (35) that this excerts an additional dampening effect on the capital price increase.

It turns out that the dampening forces associated with rising capital stocks are not always strong enough to terminate capital price booms.\(^{52}\) To insure that capital prices remain finite we therefore impose a projection facility that eventually bounds upward belief revisions.\(^{53}\) The projection facility can be interpreted as an approximate implementation of a Bayesian updating scheme where agents have a truncated prior about the support of \( \beta_{s,t} \). The details of the projection facility are spelled out in Appendix A.2.

\(^{51}\)The latter emerges because an increase in consumption sector capital \((K_{c,t}/Z_t)\) above its balanced growth path value increases private consumption, which decreases marginal utility and thereby drives up the wage \( W_t \), see equation (17). This makes investment more costly and reduces labor demand in the investment sector.

\(^{52}\)The upward belief dynamics depicted in figure 3 may not always be sufficiently close to the 45 degree line, so that they enter the area of the space where \( m_{c,t} \) starts reverting its direction.

\(^{53}\)The issue of bounding beliefs so as to ensure that expected utility remains finite is present in many applications of both Bayesian and adaptive learning to asset prices. The literature has typically dealt with this issue by using a projection facility, assuming that agents simply ignore observations that would imply updating beliefs beyond the required bound. See Timmermann (1993, 1996), Marcet and Sargent (1989), or Evans and Honkapohja (2001). This approach has two problems. First, it does not arise from Bayesian updating. Second, it introduces a discontinuity in the simulated moments and creates
Figure 4. Impulse response to a +10 bps shock to expected capital gains ($m_{c,t}$) in consumption-sector capital. (Variables are normalized relative to the deterministic balanced growth path value emerging in the absence of the shock.)

Figure 4 illustrates the quantitative model dynamics by displaying the impulse responses to an exogenous one-time increase in the quarterly expected growth rate of the price of consumption sector capital ($m_{c,t}$) by 10 basis points above its deterministic balanced growth path value in period zero. The figure uses the estimated parameters from Section 8 and normalizes all variables relative to the deterministic balanced growth path values that would emerge in the absence of the shock. Figure 4 shows that the period zero increase in optimism and the associated small increase in the capital price is followed by eight quarters of further increases in optimism and capital prices. The increased capital price leads to an increase in hours worked and in the stock of consumption-sector capital, as discussed before. These reactions eventually dampen the price increase and capital prices start to revert direction. Prices significantly undershoot their balanced growth path value before slowly recovering. As capital prices become depressed, investment falls and consumption-sector capital persistently undershoots its balanced growth path value.

difficulties for our MSM estimation. To avoid this, we follow the differentiable approach to bounding beliefs used in Adam, Marcet, and Nicolini (2016).

54 We report the impulse response to a technology shock in Section 8.
Figure 5. Impulse response to a +10bps shock to expected capital gains \((m_{c,t})\) in consumption-sector capital. (Variables are normalized relative to the deterministic balanced growth path value emerging in the absence of the shock.)

Figure 5 illustrates how the dynamics in the consumption sector spill over into the investment sector. To understand the channels through which this spillover happens, note that we obtain from equation (19) and the assumed subjective price beliefs (29) the following relationship:

\[
Q_{i,t} = \frac{\beta E_t \left[ \frac{W_t}{W_{t+1}} R_{i,t+1} \right]}{1 - \beta (1 - \delta_i) E_t \left[ \frac{W_t}{W_{t+1}} \right] m_{i,t}}. \tag{39}
\]

Since wages increase during the price boom in consumption sector capital, see Figure 4, the drop of \(E [W_t/W_{t+1}]\) exerts downward pressure on the price of investment-sector capital in both the numerator and the denominator of equation (39). Yet, equation (24) shows that expected rental rates \(R_{i,t+1}\) increase strongly following the increase in the price of consumption-sector capital (and also as a result of the increase in hours worked in the investment sector). Due to the strong response of \(R_{i,t+1}\), the overall price effect at time zero is positive and the price boom in the consumption-sector capital spills over into an increase in the price of investment-sector capital. The initial price increase then
propagates via the belief updating mechanism (14) and the positive dependence of capital prices on $m_{i,t}$ (equation (39)) into a persistent boom for the price of investment-sector capital. This shows that capital prices in the two sectors have a tendency to comove over time.

![Figure 6. Impulse response to a +10bps shock to expected capital gains in consumption-sector capital ($m_{c,t}$)](image)

Figures 4 and 5 show that capital prices in the investment sector move considerably more than in the consumption sector. Figure 6 shows that this difference is far less pronounced for a standard stock market valuation measure such as the PD ratio. Since dividends in the investment sector increase strongly during the price boom (equation (24)) the difference in the response of the PD ratio across sectors is much more muted. Overall, the model is consistent with empirical observation that stock prices in the investment sector are more volatile than stock prices in the consumption sector.

While the PD ratios approximately double in response to the considered optimism shock, the response of real quantities is one or two orders of magnitude smaller. Consumption, for instance, increases in line with the increase in the stock of consumption sector capital and moves by less than ±1% over the boom-bust cycle.\(^{55}\) Hours worked

\(^{55}\)The elasticity of consumption with respect to changes in the capital stock is 0.36 in the estimated model, which is equal to the capital share in consumption sector production.
are more variable and increase by around 13% while investment increases by up to 25%. This relatively muted response of real variables in the presence of a very large stock price cycle allows the model to reconcile the relative smoothness of the business cycle with the observed high volatility of stock prices.

8 Quantitative Performance: Subjective versus Rational Price Beliefs

This section estimates the model using the simulated methods of moments (SMM). In the setting with subjective price beliefs, the model has nine parameters that need to be estimated, \((\beta, \alpha_c, \alpha_i, \delta_c, \delta_i, \gamma, \sigma, p, g)\). There is one parameter less under fully rational expectations, as the Kalman gain \(g\) then drops out.

In our baseline approach, we estimate the model using 13 target moments and a diagonal weighting matrix with the data-implied variance of the estimated data moment as diagonal entries. For the subjective belief model, these target moments include the seven business cycle moments listed in Table 1 and all asset pricing moments listed in Table 2, except for the mean and standard deviation of the risk-free interest rate. We exclude the risk-free rate moments from the set of targeted moments because the model has a hard time fully matching the equity premium. We shall nevertheless report the risk-free rate moments implied by the estimated model. For the rational expectations model, we furthermore exclude the equity return moments \((E[r^e], \sigma(r^e))\) and the autocorrelation of the PD ratio \((\rho(P/D))\), as including these variables significantly deteriorates the model fit along the business cycle dimension without improving it substantially for the excluded moments.\(^{56, 57}\)

With fully rational expectations, equilibrium prices and equilibrium quantities are unique.\(^{58}\) For the case with subjective price beliefs, this is far from obvious, but the following result shows that we have indeed uniqueness of equilibrium allocations and prices in our setting with subjective price beliefs:

**Proposition 1.** There are unique functions \(G\) and \(F\) and a unique measure \(\tilde{P}\) on \(\tilde{\Omega}\), such that

\(^{56}\)Excluding these moments does not conceal the true potential of the rational expectations model to match the equity return moments. Even if we give the model the best chance to match \(E[r^e]\) and \(\sigma(r^e)\) by making them the only estimation targets, the resulting model moments are just 0.74% and 0.50%, respectively, as opposed to the values 0.77% and 0.16% reported in Table 6. Relative to their empirical counterparts (1.87% and 7.98%), these numbers are small and similar in magnitude.

\(^{57}\)We keep the level and standard deviation of the PD ratio as estimation targets for the RE model, because otherwise the payout ratio \(p\) would be unidentified. Excluding them has almost no impact on the remaining estimated parameters.

\(^{58}\)As is well-known, the market allocation is equivalent to the solution of a social planning problem, with the latter featuring a concave objective function and a convex set of constraints.
1. $\tilde{P}$ describes the joint beliefs of households about technology and capital prices as defined in Section 5.

2. $G$ is a state transition function, $S_{t+1} = G(S_t, Z_{t+1})$, and $F$ an outcome function, $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t) = F(S_t)$, such that $S_t$, $S_{t+1}$ and $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t)$ are consistent with:

   (a) All equilibrium conditions (see Section 6)
   (b) Households’ belief updating equations (equation (14) for $s \in \{c, i\}$).

The proof of Proposition 1 is given in Appendix A.3.3. There, we also provide an explicit characterization of the mappings $G$ and $F$ based on the equations collected in Section 6. Essentially, this characterization determines those functions up to a static equation system (stated in Lemma 4 in the appendix). In our numerical solution procedure, we solve this equation system and construct the mapping $G$ mirroring the construction in the proof of Proposition 1. Having recovered $G$ and $F$, this essentially determines equilibrium.

When estimating the learning model we impose the additional restriction that the impulse responses of capital prices to fundamental shocks display an exponential decay rate of 1.16% per quarter. We do so to avoid that the estimation selects parameter values that would imply deterministic equilibrium cycles. Clearly, imposing this additional restriction can only deteriorate the fit of the targeted moments. The model moments that we present thus represent a lower bound on the fit that could be achieved in the absence of this restriction.

Table 5 reports the estimated parameter values and Table 6 target moments. We first discuss business cycle moments, which are reported in the upper half of Table 6. Along this dimension, our estimated subjective belief model matches the targets very well. Instead, the rational expectations model displays somewhat too much consumption volatility and considerably too little investment and hours volatility. In addition, the degree of comovement of the macro aggregates is higher in the rational expectations model than both in the data and in the subjective belief model.

The poorer performance of the rational expectations model can be fixed by introducing additional investment-specific shocks. While the literature has mostly focused on

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59 With projection adjustments as described in Appendix A.2, where necessary.
60 Moments are computed based on a Monte Carlo simulation of model dynamics using the state transition function $G$. Simulated paths for all parameter combinations start in the balanced growth path and are based on the same random draw of 10,000 productivity shocks. The first 500 observations of the simulation are dropped before computing moments.
61 Details of this restriction are provided in Appendix A.6.
62 Investment-specific technology shocks are found to be important drivers of business cycles in estimated models (compare Justiniano, Primiceri, and Tambalotti, 2010, 2011) and thus are a natural candidate to improve business cycle predictions of the real business cycle model.
### Table 5
Estimated parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Subjective Belief</th>
<th>RE Model w/o inv. shocks</th>
<th>RE Model w inv. shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>0.996</td>
<td>0.997</td>
<td>0.997</td>
</tr>
<tr>
<td>( \alpha_c )</td>
<td>0.36</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td>( \alpha_i )</td>
<td>0.73</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td>( \delta_c )</td>
<td>0.010</td>
<td>0.020</td>
<td>0.010</td>
</tr>
<tr>
<td>( \delta_i )</td>
<td>0.014</td>
<td>0.030</td>
<td>0.030</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1.004</td>
<td>1.005</td>
<td>1.003</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.015</td>
<td>0.018</td>
<td>0.015</td>
</tr>
<tr>
<td>( p )</td>
<td>0.350</td>
<td>0.195</td>
<td>0.337</td>
</tr>
<tr>
<td>( g )</td>
<td>0.019</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( \sigma_i )</td>
<td>–</td>
<td>–</td>
<td>0.015</td>
</tr>
</tbody>
</table>

### Table 6
Model fit for targeted moments (13 / 10 targets, 9/8 model parameters for the subjective belief/RE model)

<table>
<thead>
<tr>
<th>Data (std.dev.)</th>
<th>Subj. Belief Model</th>
<th>RE Model w/o inv. shocks</th>
<th>RE Model w inv. shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Business Cycle Moments</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma(Y) )</td>
<td>1.72 (0.25)</td>
<td>1.83</td>
<td>1.90</td>
</tr>
<tr>
<td>( \sigma(C)/\sigma(Y) )</td>
<td>0.61 (0.03)</td>
<td>0.67</td>
<td>0.75</td>
</tr>
<tr>
<td>( \sigma(I)/\sigma(Y) )</td>
<td>2.90 (0.35)</td>
<td>2.90</td>
<td>1.88</td>
</tr>
<tr>
<td>( \sigma(H)/\sigma(Y) )</td>
<td>1.08 (0.13)</td>
<td>1.06</td>
<td>0.31</td>
</tr>
<tr>
<td>( \rho(Y,C) )</td>
<td>0.88 (0.02)</td>
<td>0.84</td>
<td>0.98</td>
</tr>
<tr>
<td>( \rho(Y,I) )</td>
<td>0.86 (0.03)</td>
<td>0.89</td>
<td>0.97</td>
</tr>
<tr>
<td>( \rho(Y,H) )</td>
<td>0.75 (0.03)</td>
<td>0.70</td>
<td>0.89</td>
</tr>
<tr>
<td><strong>Financial Moments</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E[P/D] )</td>
<td>152.3 (25.3)</td>
<td>150.0</td>
<td>174.6</td>
</tr>
<tr>
<td>( \sigma(P/D) )</td>
<td>63.39 (12.39)</td>
<td>44.96</td>
<td>7.00</td>
</tr>
<tr>
<td>( \rho(P/D) )</td>
<td>0.98 (0.003)</td>
<td>0.97</td>
<td>0.96</td>
</tr>
<tr>
<td>( E[r^e] )</td>
<td>1.87 (0.45)</td>
<td>1.25</td>
<td>0.77</td>
</tr>
<tr>
<td>( \sigma(r^e) )</td>
<td>7.98 (0.35)</td>
<td>7.07</td>
<td>0.16</td>
</tr>
<tr>
<td>( \sigma(D_{t+1}/D_t) )</td>
<td>1.75 (0.38)</td>
<td>2.46</td>
<td>1.19</td>
</tr>
</tbody>
</table>
persistent investment-specific technology processes, we add here just a parsimonious iid component to the investment-sector production function, so as to keep the changes to the model described above minimal and to have only one additional parameter. Specifically, we replace $Y_{i,t}$ from equation (1) by

$$Y_{i,t} = \varepsilon^I_t K_{i,t}^{\alpha_i} (Z_t H_{i,t})^{1-\alpha_i}$$

with $\log \varepsilon^I_t \sim \mathcal{N}\left(-\frac{\sigma^2_i}{2}, \sigma^2_i\right)$. We solve and re-estimate this augmented model under rational expectations using the same procedure and targets as for the model without investment-specific shocks. The resulting parameter estimates and model moments are reported in the third columns of Tables 5 and 6, respectively. With investment shocks, the business cycle predictions of the model are much closer to the data and overall comparable with our subjective belief model. While this approach is in principle capable of generating good business cycle predictions, the required volatility $\sigma_i$ of the investment-specific shock is as high as the volatility $\sigma$ of the neutral shock (see Table 5). Interestingly, this additional source of exogenous randomness is not required in our subjective belief model to generate the same amount of investment volatility (and even more hours volatility), despite $\sigma$ being the same in both specifications.

We next turn to the financial target moments reported in the lower part of Table 6. The model with subjective beliefs is able to match the level and persistence of the PD ratio and the PD ratio in the model also displays substantial variability, although its volatility is somewhat lower than in the data. Equity returns in the model are also very volatile, generating almost 90% of the volatility observed in the data, despite a moderate – though somewhat overstated – dividend growth volatility. The model can match the level of equity returns only partially with the mean equity return in the model being only two thirds of the 1.87% per quarter earned on average on stock market investments in our sample period.

The two versions of the rational expectations model, reported in the last two columns, perform significantly worse along the financial dimension. While the level of the PD ratio (by choice of $p$), its persistence and the volatility of dividend growth are roughly in line with the data, the rational expectations model generates almost no price and return volatility and equity returns are considerably lower than in the subjective belief model, regardless of whether investment shocks are present or not.63

Table 7 reports all moments from Section 3 which where not targeted by the estimation. The first two rows show the mean and volatility of the risk-free rate. The model is consistent with a stable risk-free rate, both under subjective beliefs and rational expectations, but tends to overstate its level.64 Yet, despite the too low mean stock return

---

63The poor ability to match $E[r^e]$ and $\sigma(r^e)$ is not because these are untargeted moments in the rational expectations estimation, compare footnote 56.

64The somewhat lower value for the model with investment shocks is due to the lower growth rate ($\gamma$) in that model.
Table 7
Model fit for untargeted moments

<table>
<thead>
<tr>
<th>Moment</th>
<th>Data</th>
<th>Subjective Belief Model</th>
<th>RE Model w/o inv. shocks</th>
<th>RE Model w inv. shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[r_f]$</td>
<td>0.25 (0.13)</td>
<td>0.78</td>
<td>0.77</td>
<td>0.57</td>
</tr>
<tr>
<td>$\sigma(r_f)$</td>
<td>0.82 (0.12)</td>
<td>0.06</td>
<td>0.09</td>
<td>0.16</td>
</tr>
<tr>
<td>$\rho(H, P/D)$</td>
<td>0.51 (0.17)</td>
<td>0.79</td>
<td>-0.97</td>
<td>-0.95</td>
</tr>
<tr>
<td>$\rho(I/Y, P/D)$</td>
<td>0.58 (0.19)</td>
<td>0.69</td>
<td>-0.97</td>
<td>-0.94</td>
</tr>
<tr>
<td>$\rho(E^{\mu}[r^2], P/D)$</td>
<td>0.79 (0.07)</td>
<td>0.52</td>
<td>-0.99</td>
<td>-0.98</td>
</tr>
</tbody>
</table>

and the too high mean risk-free rate, the subjective belief model generates an equity premium of 0.47%. While only around 30% of the historical equity premium, this result is still remarkable, given the low risk aversion implicit in our preference specification.\(^{65}\) In comparison, there is literally no equity premium in both rational expectations versions of the model.

The subjective belief model also generates positive comovement between the business cycle and stock prices as measured by the correlations of $H$ and $I/Y$ with the PD ratio. In contrast, these correlations are strongly negative under rational expectations. Furthermore, the model matches the positive correlation between expected returns and the PD ratio. While quantitatively this correlation is almost four standard errors lower than in the data, it is still in the range of values that Adam, Marcet, and Beutel (2017) report for other surveys than the UBS Gallup survey, which our data moment is based on, see their Appendix A.2. We interpret this as evidence that our model generates overall realistic comovement of stock prices and expected returns. Again, rational expectations are strongly at odds with the data: because the PD ratio tends to mean-revert and expected returns must on average forecast realized returns, the correlation is strongly negative in both rational expectation specifications.

The central mechanism underlying the dynamics in the subjective belief model has already been discussed in Section 7. To illustrate this mechanism further, Figures 7 and 8 depict impulse response functions of the model with subjective beliefs to a technology shock (solid lines) for macro aggregates and capital prices/expectations, respectively.\(^{66}\) For comparison purposes, the figures also plot impulse responses of the rational expec-

\(^{65}\)Due to linear disutility of labor the effective risk aversion is even lower than what is implied by logarithmic utility in consumption: agents are risk-neutral with respect to all risk that is uncorrelated with wages. Thus only assets that yield low payoffs in states where wages are low carry a positive risk-premium.

\(^{66}\)We start the model in a deterministic balanced growth path and plot deterministic impulse responses to a one-time one-standard-deviation positive innovation to technology, shutting down all randomness in the dynamics thereafter. For trending quantities ($Y, C, I, K$), the plotted dynamics are normalized by dividing by $\gamma^t$ to eliminate the deterministic time trend.
Figure 7. Impulse response functions of macro aggregates to a technology shock for subjective belief model (solid lines) and RE model (dashed lines)
Figure 8. Impulse response functions of prices and beliefs to a technology shock for subjective belief model (solid lines) and RE model (dashed lines)
tations model with an identical parameterization as the subjective belief model (dashed lines). We first discuss the latter as the dynamics under rational expectations are also informative for the response under subjective beliefs and the difference between the two explains well, why our subjective belief model can account better for the joint dynamics of business cycles and stock prices than the rational expectations model.

The dashed lines in Figure 7 show familiar adjustment dynamics present in a real business cycle model with capital adjustment frictions. On impact, the higher technology level leads to an increase in output, consumption and investment and, with one period lag, also in hours worked. The increased technology increases the marginal product of (consumption-sector) capital. This in turn increases capital prices in both sectors above their steady state level (see Figure 8), which sets in motion a capital accumulation process that lasts beyond the impact period (lower panel in Figure 7). As capital increases over time, investment declines and consumption and output increase to their new balanced growth path, on which all three variables are permanently higher (relative to their deterministic trend) than on the pre-shock balanced growth path. As the capital stock expands, marginal products of capital and capital prices return back to steady state. The two lower plots in Figure 8, which show the expected capital price growth, for \( s \in \{c,i\} \), mirror the response of decreasing capital prices along the adjustment path.

Next, we turn to the impulse responses of the subjective belief model. Here, two channels determine the adjustment dynamics after a technology shock. First, the economy reacts on impact almost identical to the rational expectations economy and, again, the shock creates a situation in which the capital stock is below its desired level, so low frequency movements of the economy display an adjustment process very close to the one observed under rational expectations. But second, as capital supply is not fully elastic, the higher capital demand on impact increases capital prices and this has substantially different implications under subjective beliefs than under rational expectations. Under rational expectations, the capital price just indicates the relative scarcity of capital and

\[ E_t\left[\frac{Q_{s,t+1}}{Q_{s,t}}\right] \]

for \( s \in \{c,i\} \), reflects the response of decreasing capital prices along the adjustment path.

\[ \frac{Q_{c,t}}{Q_{c,t}\alpha_c + \alpha_i} \]

for \( \alpha_c < \alpha_i \), with a sufficiently strong increase in \( Q_c \) (recall, that hours in the consumption sector are constant, so all the dynamics in hours are due to adjustments in labor supply of the investment sector).

\[ \frac{Q_{i,t}}{Q_{c,t}\alpha_c + \alpha_i} \]

While capital in the investment sector exogenously appears as a response to the technology shock, \( Q_{i,t} \) still increases due to the increased output price \( Q_{c,t} \) of the investment sector. The response is, however, quantitatively small and thus hardly visible in the figure.

\[ \frac{Q_{c,t}}{Q_{c,t}\alpha_c + \alpha_i} \]

The only difference is that under rational expectations agents expect a subsequent decline in capital prices, which dampens the initial increase. Under subjective beliefs, no such decline is expected, which leads to a larger impact response in \( Q_c \), translating into an additional positive effect on hours worked and investment.

\[ \frac{Q_{c,t}}{Q_{c,t}\alpha_c + \alpha_i} \]
reverts back to steady state as capital is accumulated. Instead, under subjective beliefs, the initial surprise increase in prices $Q_{c,t}, Q_{i,t}$ triggers an upward revision in beliefs $m_{c,t+1}, m_{i,t+1}$ through the belief updating equation (14). The exogenous technology shock thus endogenously triggers a “belief shock” of the kind we have studied in Section 7 to illustrate the equilibrium dynamics. As a result, all impulse responses cyclically fluctuate around the long-run adjustment path under rational expectations. Quantitatively, these cyclical deviations from their long-run paths are small for consumption and capital, somewhat larger for hours and output, even larger for investment and substantially larger for capital prices and price-growth expectations. This explains why the model with subjective beliefs can generate considerably more volatility in investment, hours and, particularly, stock prices, without overstating the overall volatility of consumption and output. The additional belief-driven cyclical fluctuations in the adjustment dynamics after the shock also break the almost perfect comovement of output, consumption, investment and hours under rational expectations, instead bring the correlations $\rho(Y, C), \rho(Y, I)$ and $\rho(Y, H)$ closer to the data.

While these impulse responses display cyclical price variations, the fluctuations in the PD ratio (not shown) implied by a single one-standard-deviation technology shock are quantitatively small. To generate sizeable fluctuations in the PD ratio as in Figure 6, larger amounts of capital gains optimism are required than generated by a single shock. However, a sequence of positive technology shock can occasionally create a sufficient amount of optimism for dynamics to display large stock price cycles. Figure 9, which plots simulated sample paths for output and the PD ratio over 200 quarters, shows two such price cycles in the first half of the sample (lower panel). The first, larger, price cycle is accompanied by an output boom that ends in a deep recession as prices drop sharply, mainly due to a strong decline in investment activity (not shown). The subsequent recovery is followed by a second, smaller cycle in output and prices. In the second half of the sample, prices fluctuate only little and output fluctuations are mainly driven by technology shocks and capital adjustment dynamics. From this discussion, we conclude that our model gives rise to quantitatively realistic stock price cycles, not unlike the ones depicted in Figure 1, and that these cycles are linked to economic activity.

The existence of stock price cycles in our model also implies that the model economy spends a considerably amount of time in states of capital gains optimism, in which PD ratios are substantially above their long-run mean. There is thus a large amount of mass in the right tail of the ergodic P/D distribution, which is illustrated in Figure 10. The figure plots once again the empirical kernel density of the P/D distribution in the

---

70While we have shown there a “large” increase in optimism to illustrate the emergence of a boom-bust cycle, the effect is quantitatively much smaller here. This explains the differences between the Figures 7 and 8 here and the Figures 4 and 5 there.

71Note that the peak level in $m_e$ in Figure 8 is just 5 basis points, whereas we have studied a one-time increase in optimism of 10 basis points in Section 7.

72The figure plots output relative to its stochastic trend.

40
Figure 9. Simulated equilibrium paths of (detrended) output and the price-dividend ratio data as in Figure 2, but now in addition the distribution in the model. As is visible there, the model generates a right tail that is remarkably close to the one observed in the data. However, in the model more mass is concentrated around the mode, while very small PD ratios are rare compared to the data. The reason for this asymmetry is that belief revisions have much larger effects on prices in states of optimism than in states of pessimism as is easily observable in the equations (35) and (39), such that additional negative shocks in a bust do not drive down prices as much as additional positive shocks in a boom drive them further up. The lack of mass in the left tail of the distribution also explains why we cannot match the full scale of the volatility of the PD ratio in Table 5 without overstating the mean.

To further illustrate by how much the shape of the P/D distribution is affected by subjective beliefs, Figure 11 plots the distribution of the PD ratio in the data together with the distributions implied by the two versions of the rational expectations model. In both cases, the distribution is unimodal and symmetric, with a substantial amount of mass closely centered around the mean.
Figure 10. Unconditional density of PD ratio: subjective belief model versus data (not targeted in estimation, kernel estimates)

Figure 11. Unconditional density of PD ratio: RE models vs data (not targeted in estimation, kernel estimates)
9 Comparison with Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017)

In this section we compare the quantitative implications of our model with Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017). We choose to compare our model to Boldrin, Christiano, and Fisher (2001), because it is one of the leading joint explanations of stock prices and business cycles under rational expectations in the literature and the one closest to ours. We compare our model to Adam, Marcet, and Beutel (2017), because they study the same belief specification as we do, but in an endowment economy.

Boldrin, Christiano, and Fisher (2001) calibrate their model to a different data sample than we consider. In order to make the comparison as fair as possible, we use their reported data moments as a benchmark, not the ones reported in Section 3, but re-estimate our model to fit those moments using the procedure outlined in Section 8. As estimation targets we use all moments reported in Table 8 with the exception of the standard deviation of the risk-free rate. Boldrin, Christiano, and Fisher (2001) report annual instead of quarterly return moments despite their model being quarterly, which necessitates another change to make the two models comparable. As return autocorrelations are not fully in line with the data in either model, we transform the annual return moments reported in Boldrin, Christiano, and Fisher (2001) both for the data and the model to quarterly frequency under the assumption of no return autocorrelation at that frequency.

Table 8 shows the estimated parameters in comparison with the parameters used by Boldrin, Christiano, and Fisher (2001). Due to the similar structure of the two models, the parameters in their model have an identical interpretation as in ours. They are also quantitatively not too different. Table 9 reports a standard set of business cycle and financial return moments in the data and in both models. The second column reports

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74 The standard deviation of the risk-free rate is clearly lower in our model than in the data, particularly relative to the sample starting in 1892 used by Boldrin, Christiano, and Fisher (2001). We do not consider this as a serious shortcoming, since the data moment is likely overstated as argued in Section 3. For this reason, we do not attempt to match this number perfectly.
75 In our model, returns at the quarterly frequency are positively autocorrelated. This is a known weakness of subjective price belief models of the kind studied here in the absence of transitory shocks. See Adam, Marcet, and Beutel (2017) Section VIII.A for a discussion and solution of this issue. Instead, in Boldrin, Christiano, and Fisher (2001) quarterly returns are strongly negatively correlated.
76 This assumption is (approximately) correct for the data and it transforms – counterfactually – the good fit of the Boldrin, Christiano, and Fisher (2001) model to quarterly returns, so as to not bias the model comparison towards our model. It implies that means are divided by 4 and standard deviations are divided by \( \sqrt{2} \). Similarly, we divide standard errors of means by 2 and standard errors of standard deviations by \( \sqrt{2} \).
Table 8
Parameters for subjective belief model and model of Boldrin, Christiano, and Fisher (2001)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Subjective Belief Model</th>
<th>BCF</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta)</td>
<td>0.997</td>
<td>0.999999</td>
</tr>
<tr>
<td>(\alpha_c)</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td>(\alpha_i)</td>
<td>0.6</td>
<td>0.36</td>
</tr>
<tr>
<td>(\delta_c)</td>
<td>0.15</td>
<td>0.021</td>
</tr>
<tr>
<td>(\delta_i)</td>
<td>0.01</td>
<td>0.021</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>1.004</td>
<td>1.004</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.015</td>
<td>0.018</td>
</tr>
<tr>
<td>(g)</td>
<td>0.025</td>
<td>–</td>
</tr>
<tr>
<td>(p)</td>
<td>0.4</td>
<td>(0.248)</td>
</tr>
</tbody>
</table>

**Note:** BCF refers to Boldrin, Christiano, and Fisher (2001).

Moments for our model, the third column moments for Boldrin, Christiano, and Fisher (2001). Overall, both models match the set of business cycle moments well. Where there are differences between the two, our model tends to do slightly better. Along the financial dimension, the model by Boldrin, Christiano, and Fisher (2001) is able to match the average levels of the risk-free rate and stock returns perfectly and generates stock return volatility close to the one observed in the data. Our model is less successful with the former two moments, but its prediction lie within a one-standard-error interval around the point estimates in the data. As Boldrin, Christiano, and Fisher (2001), we are able to generate high stock return volatility, but unlike in their paper, this is not achieved by generating counterfactually high volatility in the risk-free rate.

Table 10 reports additional evidence for the two models. The first four moments are statistics reported in Tables 1 and 2 of Boldrin, Christiano, and Fisher (2001) which have not been discussed in Section 3 of the present paper. \(\rho(\Delta Y_t)\), \(\rho(\Delta C_t)\) denote the autocorrelation of log output growth and log consumption growth, respectively, \(\sigma(P_{hp})\) is the standard deviation of logged and HP-filtered quarterly stock prices and \(\rho(Y, P_{hp})\) is the correlation of output and stock prices, both logged and HP-filtered. The two models perform similarly for the first three moments, both matching the persistence of output growth, but underpredicting the persistence of consumption growth and overstating the volatility of HP-filtered stock prices. Our model generates somewhat more comovement of stock prices with output than Boldrin, Christiano, and Fisher (2001), in line with the data.

Table 10 also shows statistics that relate to the behavior of dividends and the PD ratio. These statistics are not reported by Boldrin, Christiano, and Fisher (2001). We therefore report our own estimates discussed in Section 3 in the data column and compute the
### Table 9
Model comparison with Boldrin, Christiano, and Fisher (2001): targeted moments

<table>
<thead>
<tr>
<th>Data (std.dev.)</th>
<th>Subj. Belief</th>
<th>BCF Model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Business Cycle Moments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma(Y)$</td>
<td>1.89 (0.21)</td>
<td>1.83</td>
</tr>
<tr>
<td>$\sigma(C)/\sigma(Y)$</td>
<td>0.40 (0.04)</td>
<td>0.67</td>
</tr>
<tr>
<td>$\sigma(I)/\sigma(Y)$</td>
<td>2.39 (0.06)</td>
<td>2.46</td>
</tr>
<tr>
<td>$\sigma(H)/\sigma(Y)$</td>
<td>0.80 (0.05)</td>
<td>0.80</td>
</tr>
<tr>
<td>$\rho(Y,C)$</td>
<td>0.76 (0.05)</td>
<td>0.91</td>
</tr>
<tr>
<td>$\rho(Y,I)$</td>
<td>0.96 (0.01)</td>
<td>0.93</td>
</tr>
<tr>
<td>$\rho(Y,H)$</td>
<td>0.78 (0.05)</td>
<td>0.74</td>
</tr>
<tr>
<td><strong>Financial Moments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[r^f]$</td>
<td>0.30 (0.41)</td>
<td>0.68</td>
</tr>
<tr>
<td>$E[r^{ce} - r^f]$</td>
<td>1.66 (0.89)</td>
<td>0.79</td>
</tr>
<tr>
<td>$\sigma(r^f)$</td>
<td>2.64 (0.52)</td>
<td>0.07</td>
</tr>
<tr>
<td>$\sigma(r^{ce})$</td>
<td>9.70 (1.10)</td>
<td>9.21</td>
</tr>
</tbody>
</table>

**Notes:** BCF refers to Boldrin, Christiano, and Fisher (2001); data and standard errors are taken from BCF and refer to their sample; moments and standard errors for financial moments are transformed to quarterly frequency by the procedure described in the main text.

### Table 10
Model comparison with Boldrin, Christiano, and Fisher (2001): additional moments

<table>
<thead>
<tr>
<th>Data (std. dev.)</th>
<th>Subj. Belief</th>
<th>BCF Model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Additional Moments reported by BCF</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(\Delta \log Y)$</td>
<td>0.34 (0.07)</td>
<td>0.39</td>
</tr>
<tr>
<td>$\rho(\Delta \log C)$</td>
<td>0.24 (0.09)</td>
<td>-0.02</td>
</tr>
<tr>
<td>$\sigma(P_{hp})$</td>
<td>8.56 (0.85)</td>
<td>19.7</td>
</tr>
<tr>
<td>$\rho(Y,P_{hp})$</td>
<td>0.30 (0.08)</td>
<td>0.32</td>
</tr>
<tr>
<td><strong>Dividend and P/D Moments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma(D_{t+1}/D_t)$</td>
<td>1.75 (0.38)</td>
<td>1.49</td>
</tr>
<tr>
<td>$E[P/D]$</td>
<td>152.3 (25.3)</td>
<td>110.5</td>
</tr>
<tr>
<td>$\sigma(P/D)$</td>
<td>63.39 (12.39)</td>
<td>32.95</td>
</tr>
<tr>
<td>$\rho(P/D)$</td>
<td>0.98 (0.003)</td>
<td>0.91</td>
</tr>
<tr>
<td>$\rho(H,P/D)$</td>
<td>0.51 (0.17)</td>
<td>0.29</td>
</tr>
<tr>
<td>$\rho(I/Y,P/D)$</td>
<td>0.58 (0.19)</td>
<td>0.20</td>
</tr>
</tbody>
</table>

**Notes:** BCF refers to Boldrin, Christiano, and Fisher (2001); $p$ estimated to minimize standard-error-weighted squared distance of mean and std of P/D from data, estimate is $p = 0.24782$; data and standard errors for first four moments are taken from BCF and refer to their sample; data and standard errors for remaining moments are not reported in BCF, we therefore take the ones from Section 3.
respective moments in the Boldrin, Christiano, and Fisher (2001) model ourselves. For the definition of dividends and prices we use the same convention as for our own model. Namely, firms pay out a fixed fraction \( p \) of capital rental income each period as dividends and reinvest the remaining fraction \( 1 - p \) into new capital. Then, dividend growth equals the growth rate of the (sector-weighted) capital rental rates and the PD ratio is an affine linear function of the capital-price-to-rental-rate ratio, as in our model. For this reason, only the moments \( E[P/D] \) and \( \sigma(P/D) \) depend on the value of the parameter \( p \), which is not present in Boldrin, Christiano, and Fisher (2001). We choose this parameter so as to minimize the sum of squared t statistics for two moments. The resulting value is \( p = 0.248 \). In our model, dividends and the PD ratio behave qualitatively as discussed in Section 8, although the overall quantitative fit is somewhat worse, because the PD ratio was not targeted by the estimation. Yet, for any of the reported moments, our model clearly outperforms Boldrin, Christiano, and Fisher (2001). Dividend growth is far too volatile in their model and the PD ratio is neither as volatile nor as persistent as in the data and in our model. In addition, the PD ratio in Boldrin, Christiano, and Fisher (2001) displays negative comovement with hours worked and the investment-to-output ratio. We have encountered such negative correlations also in the rational expectations version of our model above. The reason for this negative correlation is, that capital prices are much less persistent than \( H \) and \( I/Y \) in Boldrin, Christiano, and Fisher (2001) and thus only mildly procyclical, but rental rates are similarly persistent as macro aggregates. This leads to a negative correlation of \( H \) \((I/Y) \) and the PD ratio, despite the fact that they both move in the same direction on impact in response to a technology shock.

Next, we compare the financial moments of our model to the baseline specification of the endowment-economy model of Adam, Marcet, and Beutel (2017) reported in their Table 3 (last column, labeled “diagonal \( \Sigma \) matrix”). As their data sample is almost identical to ours, we consider again the estimated model from Section 8. Table 11 reports the results. Not unexpected, our production economy matches the moments less well than the endowment economy studied in Adam, Marcet, and Beutel (2017). Yet, overall the performance of our model is relatively close to that of the Adam, Marcet, and Beutel (2017) model. Given that our model has at the same time realistic business cycle implications, we consider this a substantial achievement.

\footnote{We solve the Boldrin, Christiano, and Fisher (2001) model by solving the associated planner problem using standard Bellman-equation-based iteration methods. We evaluate the accuracy of our solution by computing Euler errors and replicating the model moments reported in Boldrin, Christiano, and Fisher (2001). Additional moments reported by us are computed using the same method as described in footnote a in all of their tables (based on 500 simulations of sample size 200).}

\footnote{This mirrors our estimation procedure for the subjective belief model.}
Table 11
Model comparison with Adam, Marcet, and Beutel (2017)

<table>
<thead>
<tr>
<th>Moment</th>
<th>Data</th>
<th>Model</th>
<th>AMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[P/D]$</td>
<td>152.3 (25.3)</td>
<td>149.95</td>
<td>115.2</td>
</tr>
<tr>
<td>$\sigma(P/D)$</td>
<td>63.39 (12.39)</td>
<td>44.96</td>
<td>88.20</td>
</tr>
<tr>
<td>$\rho(P/D)$</td>
<td>0.98 (0.003)</td>
<td>0.97</td>
<td>0.98</td>
</tr>
<tr>
<td>$E[r^e]$</td>
<td>1.87 (0.45)</td>
<td>1.25</td>
<td>1.82</td>
</tr>
<tr>
<td>$\sigma(r^e)$</td>
<td>7.98 (0.35)</td>
<td>7.07</td>
<td>7.74</td>
</tr>
<tr>
<td>$E[r^f]$</td>
<td>0.25 (0.13)</td>
<td>0.78</td>
<td>0.99</td>
</tr>
<tr>
<td>$\sigma(r^f)$</td>
<td>0.82 (0.12)</td>
<td>0.06</td>
<td>0.83</td>
</tr>
<tr>
<td>$\sigma(D_{t+1}/D_t)$</td>
<td>1.75 (0.38)</td>
<td>2.46</td>
<td>1.92</td>
</tr>
<tr>
<td>$\rho(E[P/r^e], P/D)$</td>
<td>0.79 (0.07)</td>
<td>0.52</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Note: Column “Model” refers to our model, column “AMB” refers to Adam, Marcet, and Beutel (2017).

10 Welfare Implications of Belief-Driven Stock Price Booms

This section studies to what extent the presence of belief-driven boom and bust cycles affects allocations and household welfare. To this end, it considers the estimated subjective belief model from Section 8 and the allocative and welfare implications associated with imposing fully rational expectations on households. In the absence of subjective belief distortions, the competitive allocation is efficient and thus represents a natural welfare benchmark against which one can judge the costs of belief-driven boom and bust cycles. When evaluating welfare in the subjective belief model, we consider ex-post realized welfare rather than ex-ante expected welfare, as the latter would depend also on the degree of subjective optimism.

Table 12 reports business cycle and financial moments for a setting with and without subjective beliefs.\(^{79}\) It shows that the elimination of subjective price beliefs decreases output volatility by about 13%. This is mainly driven by a substantial fall in the volatility of investment and a dramatic fall in the volatility of hours worked. These effects are partly compensated by an increase in consumption volatility, in both absolute and relative terms. The elimination of subjective beliefs has, however, its strongest effect on the volatility of financial variables. The standard deviation of the PD ratio decreases by more than three quarters and the standard deviation of stock returns falls even more dramatically. Dividend growth volatility also falls considerably.

The increase in consumption volatility and the fact that the volatility of hours is

\(^{79}\)The results for both models are based on the estimated parameters for the subjective belief model in Table 5.
Table 12
The effects of shutting down subjective price beliefs

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Subjective Belief Model</th>
<th>REE Implied by Subj. Belief Model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Business Cycle Moments</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma(Y)$</td>
<td>1.72 (0.25)</td>
<td>1.83</td>
<td>1.60</td>
</tr>
<tr>
<td>$\sigma(C)/\sigma(Y)$</td>
<td>0.61 (0.03)</td>
<td>0.67</td>
<td>0.89</td>
</tr>
<tr>
<td>$\sigma(I)/\sigma(Y)$</td>
<td>2.90 (0.35)</td>
<td>2.90</td>
<td>1.59</td>
</tr>
<tr>
<td>$\sigma(H)/\sigma(Y)$</td>
<td>1.08 (0.13)</td>
<td>1.06</td>
<td>0.12</td>
</tr>
<tr>
<td>$\rho(Y,C)$</td>
<td>0.88 (0.02)</td>
<td>0.84</td>
<td>0.96</td>
</tr>
<tr>
<td>$\rho(Y,I)$</td>
<td>0.86 (0.03)</td>
<td>0.89</td>
<td>0.91</td>
</tr>
<tr>
<td>$\rho(Y,H)$</td>
<td>0.75 (0.03)</td>
<td>0.70</td>
<td>0.70</td>
</tr>
<tr>
<td><strong>Financial Moments</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[P/D]$</td>
<td>152.3 (25.3)</td>
<td>150.0</td>
<td>199.7</td>
</tr>
<tr>
<td>$\sigma(P/D)$</td>
<td>63.39 (12.39)</td>
<td>44.96</td>
<td>8.99</td>
</tr>
<tr>
<td>$\rho(P/D)$</td>
<td>0.98 (0.003)</td>
<td>0.97</td>
<td>0.99</td>
</tr>
<tr>
<td>$E[r^e]$</td>
<td>1.87 (0.45)</td>
<td>1.25</td>
<td>0.68</td>
</tr>
<tr>
<td>$\sigma(r^e)$</td>
<td>7.98 (0.35)</td>
<td>7.07</td>
<td>0.19</td>
</tr>
<tr>
<td>$E[r^f]$</td>
<td>0.25 (0.13)</td>
<td>0.78</td>
<td>0.68</td>
</tr>
<tr>
<td>$\sigma(r^f)$</td>
<td>0.82 (0.12)</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>$\sigma(D_{t+1}/D_t)$</td>
<td>1.75 (0.38)</td>
<td>2.46</td>
<td>0.92</td>
</tr>
</tbody>
</table>

welfare neutral – given our linear specification for the disutility of work – suggests that
the welfare implications of eliminating subjective beliefs are likely to be small. Indeed,
we find that the utility gain associated with the elimination of subjective beliefs amounts
to 0.29% of consumption. This is of the same order of magnitude as the welfare gains
associated with the elimination of the business cycle.

The welfare gains associated with the elimination of subjective beliefs arise mainly
through changes in the mean levels of consumption and work. Table 13 depicts the
mean value of (detrended) consumption and labor under the two belief specifications.\textsuperscript{80}
It shows that average consumption is higher and average labor is lower in a setting with
fully rational expectations. This is the case because with rational expectations asset
prices correctly signal the value of investment opportunities. Instead, under subjective
beliefs, investment can be triggered by a belief-driven price boom, so that high levels
of hours worked and high levels of investment might occur during times in which new
capital is not particularly productive. As a result, agents can work on average more but
achieve lower average consumption levels.

\textsuperscript{80}The utility from the common consumption trend, which is a function of the productivity process
$\{Z_t\}$ only, enters utility separately.
Table 13
Mean values of (detrended) consumption and labor

<table>
<thead>
<tr>
<th></th>
<th>Subjective Belief Model</th>
<th>REE Implied by Subj. Belief Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[C_{det}]$</td>
<td>2.7719</td>
<td>2.7768</td>
</tr>
<tr>
<td>$E[L]$</td>
<td>0.7152</td>
<td>0.7141</td>
</tr>
</tbody>
</table>

We conclude this section with two remarks. First, one reason for the low welfare effects of fluctuations induced by price cycles is the low curvature in assumed preferences. As curvature increases, agents would be willing to give up more consumption to eliminate fluctuations. However, the welfare gains associated with the elimination of fluctuations induced by stock price cycles are bounded above by the welfare gains associated with the elimination of all fluctuations and thus the cost of business cycles. Unless higher curvature in preferences also increases the mean effects reported in Table 13, we expect our result to generalize that costs of belief-driven price cycles are small and of the same order of magnitude as the costs of business cycles.

Second, the representative agent assumption is critical for the result that welfare effects of stock price cycles are small. Adam, Beutel, Marcet, and Merkel (2016) show that a belief-driven boom-bust cycle can have substantial redistributive effects in the presence of belief heterogeneity. In their model agents trade with each other at prices that are – relative to a setting without belief distortions – too high or low. In the present model, prices only affect allocations through aggregate investment. Likely, our welfare assessment would change in the presence of heterogeneity that leads to trading among agents over the course of stock price cycles, if the welfare objective was to include distributional concerns.

11 Conclusions

We have presented a simple theory of the joint behavior of stock prices and business cycles. This theory is quantitatively consistent with key facts about the business cycle, stock price behavior and the interaction between the two. In particular, it can reconcile smooth business cycles and volatile stock prices. Relative to the previous literature, no labor market frictions or inseparable preferences are required to achieve this. Instead, in our model stock prices are volatile because agents hold subjective stock price expectations and extrapolate observed past capital gains into the future as in Adam, Marcet, and Beutel (2017). Extrapolation creates price-belief feedback loops that can occasionally disconnect prices from fundamentals and lead to large and persistence stock price cycles. In the model, such price cycles have allocative implications, because they lead to cyclical variation in investment demand. Our theory suggests that this variation is an important
source of the business cycle fluctuations of investment and hours worked, but has only minor effects on consumption variability.

We also assess the welfare implications of belief distortions in stock markets. In the presence of such distortions, prices are only imperfect signals of the relative scarcity of capital goods. Consequently, in times of optimism there can be over-investment, in times of pessimism under-investment. Inefficient timing of investment leads to a lower average level of consumption despite a higher average amount of hours worked. However, we find the welfare costs of belief-driven stock price cycles to be relatively moderate. Several considerations absent in our model might overturn this result. First, effects of price cycles could be much larger in the presence of heterogeneity and trading in a setting where the wealth distribution matters. Second, in a richer model with a large cross-section of sectors in which sectoral price cycles only partially comove fluctuations in relative firm valuations may lead to additional misallocation, absent in our model. Third, if asset prices play an important role for the balance sheet dynamics of a financially constrained sector,\textsuperscript{81} fluctuations in asset prices may have larger effects on allocations than in the model presented in this paper. Exploring these channels appears to be an interesting avenue for future research.

Conceptually, we show how to specify expectations that combine subjective beliefs for some model variables with rational expectations along other dimensions. The proposed expectation concept that is “partially rational” with respect to a subset of model-endogenous variables can be generalized beyond our specific setting. In particular, it may prove fruitful in other macroeconomic applications where beliefs can be disciplined empirically along some dimensions, but one wishes to maintain the rational expectations hypothesis along others.

References


\textsuperscript{81}Arguably, this is less relevant for stocks than for other asset classes such as real estate. House prices also display large and persistent cycles in many advanced economies. To the extent that such cycles are also driven by the same belief dynamics analyzed here, one could easily extend the analysis to include housing.


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A Appendix

A.1 Data Sources

**Data on Macro Aggregates** Data series on macro aggregates and related variables have all been downloaded from the Federal Reserve Economic Data (FRED) database maintained by the federal reserve bank of St. Louis (https://fred.stlouisfed.org). All time series refer to the United States, are at quarterly frequency and cover the sample period Q1:1955 to Q4:2014. Data on output, consumption and investment is from the National Income and Product Accounts of the BEA. We measure nominal output by gross domestic product (FRED Code “GDP”), nominal consumption by personal consumption expenditures in nondurable goods (“PCND”) and services (“PCESV”) and nominal investment by fixed private investment (“FPI”). Nominal values of investment subcomponents for Table 4 are private residential fixed investment (“PRFI”) and private nonresidential fixed investment in structures (“B009RC1Q027SBEA”), equipment (“Y033RC1Q027SBEA”) and intellectual property products (“Y001RC1Q027SBEA”). These nominal series have been deflated by the consumer price index for all urban consumers (“CPIAUCSL”) from the BLS, which is consistent with the deflating procedure used for stock prices and interest rates by Adam, Marcet, and Beutel (2017) and us. Hours worked are based on an index of nonfarm business sector hours (“HOANBS”) published by the BLS. Working age population is based on data published by the OECD (“LFWA64TTUSQ647N”).

**Stock Prices, Interest Rates and Investor Expectations** We use identical data sources as in Adam, Marcet, and Beutel (2017) and refer to their data appendix for details. They use ‘The Global Financial Database’ to obtain data on stock prices and interest rates until Q1:2012. We extend their stock price and interest rate data to Q4:2014 using identical data sources as they do.

We compute dividends based on price and total return index data of the SP 500 index by the procedure outlines there, resulting in a dividend series \( \{D_t\} \), where \( t \) runs through all quarters from Q1:1955 to Q4:2014. Given the price series from the price index \( \{P_t\} \), where \( P_t \) is the closing price of the last trading day in quarter \( t \), we define the price-dividend ratio as the ratio \( P_t/D_t \) of the end-of-quarter closing price \( P_t \) and the within quarter dividend \( D_t \).

A.2 Details of the Projection Facility

Following Adam, Marcet, and Nicolini (2016), we modify the belief updating equation (14) to

\[
\log m_{s,t} = w_{s,t} \left( \log m_{s,t-1} + g \left( \log Q_{s,t-1} - \log Q_{s,t} - \log m_{s,t-1} \right) + g \log \varepsilon_{s,t} \right)
\]
where $w_{s,t}(\cdot)$ is a differentiable function satisfying $w_{s,t}(x) = x$ for $x \leq m_{s,t}$ and $w_{s,t}(x) \leq m_{s,t}$ for all $x$, with $m_{s,t} > m_{s,t}$. Beliefs are thus bounded below $m_{s,t}$, but evolve as described in the main text as long as they remain below $m_{s,t}$. Following Adam, Marcet, and Nicolini (2016) we consider the function

$$w_{s,t}(x) = \begin{cases} m_{s,t} + \frac{x - m_{s,t}}{x + m_{s,t} - 2m_{s,t}}(m_{s,t} - m_{s,t}) & \text{if } x \leq m_{s,t} \\ \frac{x}{m_{s,t}} & \text{if } m_{s,t} < x. \end{cases}$$

(40)

and calibrate the critical values $(m_{s,t}, m_{s,t})$ in both sectors $s = c, i$ such that $m_{s,t}$ is the degree of optimism that implies a quarterly PD ratio of 250 and $m_{s,t}$ is the degree of optimism implying a PD ratio of 500. The critical PD values of 250 and 500 are taken from Adam, Marcet, and Nicolini (2016).

We now explain how these critical values can be computed. The Euler equation for capital in sector $s$ is

$$Q_{s,t} = \beta (1 - \delta_s) E_t^P \left[ \frac{W_t}{W_{t+1}} Q_{s,t+1} \right] + \beta E_t^P \left[ \frac{W_t}{W_{t+1}} R_{s,t+1} \right]$$

$$= \beta (1 - \delta_s) E_t^P \left[ \frac{W_t}{W_{t+1}} \right] m_{s,t} Q_{s,t} + \beta E_t^P \left[ \frac{W_t}{W_{t+1}} R_{s,t+1} \right]$$

implying

$$Q_{s,t} = \frac{\beta E_t^P \left[ \frac{W_t}{W_{t+1}} R_{s,t+1} \right]}{1 - \beta (1 - \delta_s) E_t^P \left[ \frac{W_t}{W_{t+1}} \right] m_{s,t}}$$

and thus the price-dividend ratio is

$$\frac{P_{s,t}}{D_{s,t}} = \frac{1 - \delta Q_{s,t}}{p R_{s,t}} + \frac{1 - p}{p} = \frac{1 - (1 - \delta) \beta E_t^P \left[ \frac{W_t}{W_{t+1}} \frac{R_{s,t+1}}{R_{s,t}} \right]}{p \left[ 1 - \beta (1 - \delta) E_t^P \left[ \frac{W_t}{W_{t+1}} \right] m_{s,t} \right]} + \frac{1 - p}{p}. \quad (41)$$

The value $m_{s,t}$ is the value for $m_{s,t}$ in the preceding equation that causes the PD to be equal to 250; likewise, $m_{s,t}$ is the value that causes the PD ratio to be equal to 500.\(^{82}\) Since the expectations of $W_{t+1}$ and $R_{s,t+1}$ both depend on $K_{c,t+1}$ (this follows from equations (21) and (22) and the fact that $Y_{c,t+1} = K_{c,t+1}^{\alpha_c} (Z_{t+1} (1 - \alpha_c))^{1-\alpha_c}$), solving for the values $(m_{c,t}, m_{c,t})$ thus requires simultaneously solving for the optimal investment $Y_{i,t}$ in period $t$. Since the capital stock dynamics in the investment sector are exogenous, $m_{i,t}, m_{i,t}$ can be computed once the new belief $m_{c,t}$ that incorporates the projection facility has been determined.

\(^{82}\)In addition, we do not allow $m_{c,t}$ to exceed the theoretical upper bound on beliefs for which uniqueness has been proven in Appendix A.3.3 (see Lemma 4), irrespective of the implied PD ratio. This is done to insure equilibrium uniqueness, but is a purely theoretical concern. In practice, we encountered not a single case in which this upper bound was binding in our numerical simulations.
A.3 Details on the Representation of Beliefs and Equilibrium Existence

A.3.1 Belief Representations on $\Omega$ and $\tilde{\Omega}$

We provide here details on the remarks made in footnote 39. To ease the discussion, we first introduce some notation that will also be used in Sections A.3.2 and A.3.3.

Notation For each random (upper case) symbol $A$, denote elements of $\Omega_A$ by $a$ (these are real sequences) and their $t$-th component by $a_t$ (these are real numbers). For each random sequence $\{A_t\}_{t=0}^\infty$, write shorter just $A$. On the domain $\tilde{\Omega} = \Omega_Z \times \Omega_{Q,c} \times \Omega_{Q,i}$ we define the random variables (sequences) $\tilde{Z}$, $\tilde{Q}_c$ and $\tilde{Q}_i$ as projections on the first, second and third component, respectively,

$$\tilde{Z}(z, q_c, q_i) = z, \quad \tilde{Q}_c(z, q_c, q_i) = q_c, \quad \tilde{Q}_i(z, q_c, q_i) = q_i.$$ 

Similarly, we define the random variables $Z$, $X$, $W$, $R_c$, $R_i$, $Q_c$, $Q_i$ as projections from the domain $\Omega = \Omega_Z \times \Omega_X \times \Omega_W \times \Omega_{R,c} \times \Omega_{R,i} \times \Omega_{Q,c} \times \Omega_{Q,i}$ to the respective factor. We make the difference between $\tilde{Z}$ (defined on $\tilde{\Omega}$ consisting of typical elements $\tilde{\omega} = (z, q_c, q_i)$) and $Z$ (defined on $\Omega$ consisting of typical elements $\omega = (z, x, w, r_c, r_i, q_c, q_i)$) explicit to avoid any ambiguity and confusion arising in the arguments below. In the main text, we regularly do not make these distinctions and use the same symbols, whenever a variable has the same interpretation, no matter on which space it is defined and whether it is a random variable or a realization.

We regularly have to work with the big vector $(Z, X, W, R_c, R_i, Q_c, Q_i)$ of random sequences on $\Omega$. For space reasons, we denote this vector just by $O$ (“observables vector”).

Furthermore, we use the following two conventions for the outcome mapping $F : s_t \mapsto (w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)$ (where $s_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, m_{i,t}, q_{c,t-1}, q_{i,t-1})$) introduced at the end of Section 5:

- We denote by $F_W$, $F_{R,c}$, $F_{R,i}$ etc. the components of that function.
- By a slight abuse of notation, we denote by $F$ also the mapping from the full state sequence $s = \{s_t\}_{t=0}^\infty$ into the full outcome sequence $(w, r_c, r_i, y, c, i, h) = \{(w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)\}_{t=0}^\infty$, similarly we use the component functions $F_W$, $F_{R,c}$, $F_{R,i}$ etc.

---

83 Because the variables $m_c$ and $m_i$ are lower case in the model, those symbols can denote both random variables and realizations. This ambiguity should not lead to any confusion below.

84 E.g. $Z(z, x, w, r_c, r_i, q_c, q_i) = z$.

85 As a mapping, this vector just agrees with the identity $\text{id}_\Omega$, but this notation conceals the economic interpretation of the random variables.
Mapping a Recursive Evolution into a Sequence Representation  Consider a fixed initial state $s_0 = (z_0, k_{c,0}, k_{i,0}, m_{c,0}, m_{i,0}, q_{c,-1}, q_{i,-1})$ and a measurable function $G$ describing a recursive state evolution as in the main text. For any sequence $z \in \Omega_Z$ such that $z_0$ is consistent with $s_0$, the recursion

$$s_{t+1} = G(s_t, z_{t+1}) \quad t = 0, 1, 2, \ldots$$

defines then a unique sequence $s = \{s_t\}_{t=0}^{\infty}$ in $\Sigma := \prod_{t=0}^{\infty} \mathbb{R}^7$ ($\Sigma$ is the space of all state sequences). The procedure just outlined defines therefore a function $H : \Omega_Z \rightarrow \Sigma$ which maps technology sequences $z$ into state sequences $s$. Obviously, if $G$ is measurable and $\Sigma$ is endowed with the usual product $\sigma$-field, then $H$ is also a measurable function. Call $H$ the sequence representation associated with $G$ (and the initial state $s_0$).

Consistency of Beliefs with an Evolution  The following definition clarifies the notion in the main text that beliefs about wages and rental rates be consistent with the equilibrium mappings $G$ and $F$. The definition is formulated for arbitrary mappings $G$ and $F$ that do not necessarily need to correspond to the equilibrium mappings of the model.

Definition 2. For a given measurable state evolution $G$ and a given measurable outcome function $F$, we say that a measure $P$ on $\Omega$ implies beliefs about $(W, R_c, R_i)$ consistent with mappings $G$ and $F$, if

$$W = F_W \circ H \circ Z, \quad R_c = F_{R,c} \circ H \circ Z, \quad R_i = F_{R,i} \circ H \circ Z$$

$P$-a.s. Here, $H$ is the sequence representation associated with $G$.

Constructing Consistent Beliefs from $G$, $F$ and $\tilde{P}$  The following lemma is the main result of this section and provides the justification why it is sufficient to work with the smaller probability space $\tilde{\Omega}$ instead of $\Omega$.

Lemma 1. For any given measure $\tilde{P}$ on $\tilde{\Omega}$ and measurable $G$ and $F$, there is a unique measure $P$ on $\Omega$ with the following properties:

1. The distribution of $(Z, Q_c, Q_i)$ under $P$ equals $\tilde{P}$;
2. The joint distribution of \((Z, X)\) under \(P\) is consistent with the exogenous relationship between \(Z\) and \(X\);

3. \(P\) implies beliefs about \((W, R_c, R_i)\) that are consistent with the mappings \(G\) and \(F\).

\begin{proof}
We give an explicit construction of the measure \(P\) as the distribution of a set of suitable random variables on \(\tilde{\Omega}\) under \(\tilde{P}\). First, the capital accumulation equation (28) for investment-sector capital and the assumption \(K_{i,t+1} \propto Z_t\) imply

\[
\bar{k}_i Z_t = (1 - \delta) \bar{k}_i Z_{t-1} + X_t \Rightarrow X_t = \bar{k}_i (Z_t - (1 - \delta_i) Z_{t-1}),
\]

where \(\bar{k}_i\) is the (fixed) proportionality constant.\(^{89}\) The second requirement that the joint distribution of \((Z, X)\) under \(P\) be consistent with the exogenous relationship between those variables means that equation (42) has to hold \(P\)-a.s. for all \(t\). We thus define a random variable \(\tilde{X}_t: \tilde{\Omega} \to \Omega\) in a way that is consistent with the analog equation (42) on the \(\tilde{\Omega}\) domain:\(^{90}\)

\[
\tilde{X}_t := \bar{k}_i \left( \tilde{Z}_t - (1 - \delta_i) \tilde{Z}_{t-1} \right).
\]

Next, let the function \(H: \Omega \to \Sigma\) be the sequence representation associated with \(G\). As for \(X\), we simply define random variables for wages, \(\tilde{W}: \tilde{\Omega} \to \Omega\), and rental rates, \(\tilde{R}_c: \tilde{\Omega} \to \Omega_{R,c}\), \(\tilde{R}_i: \tilde{\Omega} \to \Omega_{R,i}\), on the domain \(\tilde{\Omega}\) in a way that they satisfy the analog of the consistency condition for the probability space \((\tilde{\Omega}, \tilde{S}, \tilde{P})\), namely

\[
\tilde{W} = F_W \circ H \circ \tilde{Z}, \quad \tilde{R}_c = F_{R,c} \circ H \circ \tilde{Z}, \quad \tilde{R}_i = F_{R,i} \circ H \circ \tilde{Z}.
\]

With these definitions the random (observables) vector

\[
\tilde{O} := (\tilde{Z}, \tilde{X}, \tilde{W}, \tilde{R}_c, \tilde{R}_i, \tilde{Q}_c, \tilde{Q}_i)
\]

is a measurable mapping from \(\tilde{\Omega}\) to \(\Omega\) and thus its distribution defines a measure \(P\) on \(\Omega\). We claim that this measure satisfies the three conditions in the assertion and is the only measure to do so:

1. \(Z\) is the projection defined by \(Z(z, x, w, r_c, r_i, q_c, q_i) = z\), so \(Z(\tilde{O}) = \tilde{Z}\) and a similar argument shows \(Q_c(\tilde{O}) = \tilde{Q}_c\) and \(Q_i(\tilde{O}) = \tilde{Q}_i\). So we get

\[
(Z, Q_c, Q_i)(\tilde{O}) = (\tilde{Z}, \tilde{Q}_c, \tilde{Q}_i) = \text{id}_{\tilde{\Omega}}.
\]

Because \(P\) is the distribution of \(\tilde{O}\) under \(\tilde{P}\), this equation implies that the distribution of \((Z, Q_c, Q_i)\) under \(P\) and the distribution of \(\text{id}_{\tilde{\Omega}}\) under \(\tilde{P}\) must be identical.

As the latter distribution is \(\tilde{P}\) itself, this proves the first property.
\end{proof}

\(^{89}\)Strictly speaking, \(\bar{k}_i\) is a model parameter. However, the choice of this parameter is not discussed in Section 8, because its value does not matter for any results reported in the main text. Intuitively, changing \(\bar{k}_i\) just scales up or down the production of consumption-sector capital and thereby changes the units in which this capital is measured, which has no economic relevance.

\(^{90}\)For \(t = 0\) one must back out \(\tilde{Z}_{t-1} = z_{-1}\) from the entry \(k_{i,0}\) of the initial state: \(z_{-1} = \frac{k_{i,0}}{k_i}\).
2. The following equation holds by definition of the random variables $X$, $Z$ and $\tilde{X}$ (for all $\tilde{\omega} \in \tilde{\Omega}$)

\[
X_t(\tilde{\Omega}) = \tilde{X}_t = \tilde{k}_i \left( \tilde{Z}_t - (1 - \delta_i) \tilde{Z}_{t-1} \right)
= \tilde{k}_i \left( Z_t(\tilde{\Omega}) - (1 - \delta_i) Z_{t-1}(\tilde{\Omega}) \right).
= [\tilde{k}_i (Z_t - (1 - \delta_i) Z_{t-1})](\tilde{\Omega})
\]

As this equation holds on $\tilde{\Omega}$, it must in particular hold $\tilde{\mathbb{P}}$-a.s., so $X_t$ and $\tilde{k}_i (Z_t - (1 - \delta_i) Z_{t-1})$ must coincide a.s. with respect to the distribution of $\tilde{\Omega}$ under $\tilde{\mathbb{P}}$, which is exactly $\mathbb{P}$. Hence, equation (42) holds $\mathbb{P}$-a.s.

3. The consistency proof works along the same lines as the proof that (42) has to hold $\mathbb{P}$-a.s. by reducing it to the analogous consistency condition in the tilde space for the tilde variables. The argument is omitted for this reason.

For uniqueness, suppose that $\mathbb{P}'$ is another (arbitrary) measure on $\Omega$ such that properties 1-3 are satisfied. Then in particular equation (42) holds $\mathbb{P}'$-a.s. for all $t$ and by the definition of $\tilde{X}$ and $\tilde{Z}$ we obtain for all $t$

\[
\tilde{X}_t(Z, Q_c, Q_i) = \tilde{k}_i \left( \tilde{Z}_t - (1 - \delta_i) \tilde{Z}_{t-1} \right) (Z, Q_c, Q_i)
= \tilde{k}_i (Z_t - (1 - \delta_i) Z_{t-1})
= X_t
\]

(here, all equalities except for the last hold even $\omega$-by-$\omega$ and the last is just equation (42)). Similarly, from $W = F_W \circ H \circ Z$ $\mathbb{P}'$-a.s. (the consistency condition for wage beliefs under $\mathbb{P}'$) and the definition of $\tilde{W}$ and $\tilde{Z}$ we can conclude

\[
\tilde{W}(Z, Q_c, Q_i) = F_W \circ H \circ \tilde{Z}(Z, Q_c, Q_i) = F_W \circ H \circ Z = W \quad \mathbb{P}'$-a.s.
\]

Identical arguments also yield

\[
\tilde{R}_c(Z, Q_c, Q_i) = R_c \quad \mathbb{P}'$-a.s.
\]
\[
\tilde{R}_i(Z, Q_c, Q_i) = R_i \quad \mathbb{P}'$-a.s.
\]

Combining those results with the obvious equations $\tilde{Z}(Z, Q_c, Q_i) = Z, \tilde{Q}_c(Z, Q_c, Q_i) = Q_c, \tilde{Q}_i(Z, Q_c, Q_i) = Q_i$ implies

\[
id_{\Omega} = O = \tilde{O}(Z, Q_c, Q_i) \quad \mathbb{P}'$-a.s.
\]

But by property 1, the distribution of $(Z, Q_c, Q_i)$ under $\mathbb{P}'$ must be $\tilde{\mathbb{P}}$ and thus the equation shows that the distribution of $id_{\Omega}$ under $\mathbb{P}'$ must be equal to the distribution of $\tilde{O}$ under $\tilde{\mathbb{P}}$, which is by definition $\mathbb{P}$. Hence, $\mathbb{P}' = \mathbb{P}$. \qed
A.3.2 Construction and Uniqueness of $\tilde{P}$

We first construct a measure and show that it has all the properties that any candidate for $\tilde{P}$ has to have. We then argue why it is the only such measure. First, the following two auxiliary constructions are required. As always, we assume implicitly, that an initial state $s_0 = (z_0, k_{c,0}, k_{i,0}, m_{c,0}, m_{i,0}, q_{c,-1}, q_{i,-1})$ is fixed.

1. Let $P_Z$ be a measure on $\Omega_Z$ that describes the exogenous evolution of $Z$, i.e. under $P_Z$ for all $t \geq 1$

$$\log \varepsilon_t := \log \left( \frac{Z_t}{\gamma Z_{t-1}} \right)$$

is iid normal with mean $-\frac{\sigma^2}{2}$ and variance $\sigma^2$ and $Z_0 = z_0 P_Z$-a.s. Clearly, a unique measure with this property exists.

2. For $s \in \{c, i\}$ let $P_{Q,s}$ be a measure on $\Omega_{Q,s}$ that describes the subjective evolution of $Q_s$ under learning, i.e. under $P_{Q,s}$ for all $t \geq 1$

$$\log \varepsilon_{t}^{Q,s} := \log \left( \frac{Q_{s,t}}{m_{s,t-1}Q_{s,t-1}} \right)$$

is iid normal with mean $-\frac{\sigma^2_Q}{2}$ and variance $\sigma^2_Q$ and $\sigma^2_Q = \sigma^2_{\varepsilon}$. Here $m_{s,t}$ is recursively defined by equation (14), for $t \geq 1$\footnote{Note that we here ignore the additional observable innovation present in that equation in line with footnote 38.}

$$m_{s,t} = m_{s,t-1} \left( \frac{Q_{s,t-1}}{m_{s,t-1}Q_{s,t-2}} \right)^g.$$ 

In addition, $Q_{s,-1} = q_{s,-1} P_{Q,s}$-a.s. Also, the measure $P_{Q,c}$ is uniquely defined by these properties.

It is obvious, that $P_Z$ indeed describes the exogenous evolution of $Z$ as defined in equation (2) and that $P_{Q,c}$ and $P_{Q,i}$ represent the marginal distribution of $Q_c$ and $Q_i$, respectively, under agents’ beliefs, if they are to be consistent with the Bayesian learning formulation in Section 5. So any measure $\tilde{P}$ on $\tilde{\Omega}$ that correctly represents subjective beliefs as defined in Section 5 must imply the marginal measures $P_Z$, $P_{Q,c}$ and $P_{Q,i}$ on $\Omega_Z$, $\Omega_{Q,c}$ and $\Omega_{Q,i}$, respectively. The only issue left to discuss is thus which assumptions about the dependence structure of the processes $Z$, $Q_c$ and $Q_i$ lead to a valid belief measure $\tilde{P}$ in line with the assumptions made in Section 5. We claim that only independence does and thus $\tilde{P}$ must be given by $\tilde{P} = P_Z \otimes P_{Q,c} \otimes P_{Q,i}$. 
To see this, note that on the extended probability space that includes latent variables in households’ filtering problem, agents must think that the four equations

\[
\begin{align*}
\log Q_{c,t} &= \log Q_{c,t-1} + \log \beta_{c,t} + \log \varepsilon_{c,t} \\
\log \beta_{c,t} &= \log \beta_{c,t-1} + \log \nu_{c,t} \\
\log Q_{i,t} &= \log Q_{i,t-1} + \log \beta_{i,t} + \log \varepsilon_{i,t} \\
\log \beta_{i,t} &= \log \beta_{i,t-1} + \log \nu_{i,t}
\end{align*}
\]

hold with probability 1 for all \( t \geq 1 \). In addition, as stated in Section 5, \( \{\varepsilon_{c,t}\}_{t=1}^{\infty}, \{\nu_{c,t}\}_{t=1}^{\infty}, \{\varepsilon_{i,t}\}_{t=1}^{\infty}, \{\nu_{i,t}\}_{t=1}^{\infty} \) are independent stochastic processes and independent of all other model variables, including the process \( Z \). But those two facts can only be simultaneously true, if the three processes \( Z, Q_c \) and \( Q_i \) are independent.\(^{92}\)

### A.3.3 Existence and Uniqueness of \( G \) and \( F \)

The goal of this section is to prove Proposition 1. We start with a result that collects important equations and gives an explicit characterization of the function \( F \) – up to the presence of the argument \( Q_{c,t} \) which is not part of the state \( S_t \).

**Lemma 2.** In any equilibrium, irrespective of beliefs, the following equations have to hold

\[
\begin{align*}
W_t &= K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} (1 - \alpha_c)^{1-\alpha_c} \\
R_{c,t} &= \alpha_c K_{c,t}^{\alpha_c-1} Z_t^{1-\alpha_c} (1 - \alpha_c)^{1-\alpha_c} \\
R_{i,t} &= \alpha_i \left( 1 - \frac{\alpha_i}{W_t} \right) Z_t^{\frac{1-\alpha_i}{\alpha_i}} Q_{c,t}^{\frac{1}{\alpha_i}} \\
Y_t &= K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} (1 - \alpha_c)^{1-\alpha_c} + Q_{c}^{ss} K_{i,t} Z_t^{\frac{1}{\alpha_i}} \left( \frac{1 - \alpha_i}{W_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} \\
C_t &= K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} (1 - \alpha_c)^{1-\alpha_c} \\
I_t &= Q_{c}^{ss} K_{i,t} Z_t^{\frac{1}{\alpha_i}} \left( \frac{1 - \alpha_i}{W_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} \\
H_t &= 1 - \alpha_c + K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left( \frac{1 - \alpha_i}{W_t} \right)^{\frac{1}{\alpha_i}}
\end{align*}
\]

for all \( t \geq 0 \). In particular, \((W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t)\) is a deterministic function of \((Z_t, K_{c,t}, K_{i,t}, Q_{c,t})\).

Conversely, if these equations hold, and \( H_{c,t}, H_{i,t} \) are given by equations (30) and (32), then allocations \((C, H, H_c, H_i, K_c, K_i)\) and prices \((Q_c, Q_i, R_c, R_i, W)\) are consistent.

\(^{92}\)Formally, this would require a simple induction proof over time, which is not explicitly spelled out here.
with all equilibrium conditions (equations (17)-(28)), except for the two Euler equations (equations (18) and (19)) and the two capital accumulation equations (equations (27) and (28)).

Proof. In any competitive equilibrium with subjective beliefs, the equilibrium equations (17)-(28) stated in Section 6 have to hold. Based on these equations and definitions in the model description, the expressions in the assertion can be computed. The wage is given by

\[ W_t = C_t \]

\[ = Y_{c,t} \]

\[ = K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} H_{c,t}^{1-\alpha_c} \]

\[ = (1 - \alpha_c)^{1-\alpha_c} K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c}. \]

Similarly, the rental rate in the consumption sector is

\[ R_{c,t} = \]

\[ = \frac{\alpha_c Y_{c,t}}{K_{c,t}} \]

\[ = \alpha_c (1 - \alpha_c)^{1-\alpha_c} K_{c,t}^{\alpha_c-1} Z_t^{1-\alpha_c}. \]

Substituting \( H_{i,t} \) as given by equation (32) into the capital first-order condition of investment firms (24) yields for the rental rate in the investment sector

\[ R_{i,t} = \]

\[ = \alpha_i Q_{c,t} K_{i,t}^{\alpha_i-1} Z_t^{1-\alpha_i} \]

\[ = \alpha_i \left( \frac{1 - \alpha_i}{W_t} \right)^{1-\alpha_i} \]

\[ = \frac{1}{W_t} \]

\[ = \frac{1}{Q_{c,t}} \]

\[ = H_{i,t}. \]

Output is defined as \( Y_t = C_t + I_t \), so the asserted output equation follows from the equations for \( C_t \) and \( I_t \). The equation for \( C_t \) follows from \( C_t = W_t \) (17) and the equation for \( W_t \) already proven. The equation for \( I_t = Q_{c,t} Y_{i,t} \) is obtained by substituting \( H_{i,t} \) stated in (32) into the investment-sector production function (1). Finally, the expression for \( H_t \) just combines hours in the two sectors (given by (30) and (32)) with labor market clearing (26).

For the second part of the lemma, we just remark that all the equations (17), (21), (22), (23), (24), (25) and (26) have been used in deriving the equations in the first part of the lemma and inverting the arguments used there shows that these equations also necessarily need to hold, if the equations in the lemma and (30) and (32) hold.

The functional relationships in Lemma 2 contain \( Q_{c,t} \) as the sole argument that is not contained in the state of time \( t \). \( Q_{c,t} \) and \( K_{c,t+1} \) are simultaneously determined by the consumption-sector Euler equation (18) and the accumulation equation of consumption-sector capital (27). It has been shown in the main text how the former equation can be solved for \( Q_{c,t} \) under the assumption of subjective price beliefs, compare equation
(35). In this equation, still a conditional expectation $E_t\left[\frac{1}{W_{t+1}}\right]$ appears. However, given the representation of $W$ in Lemma 2, this conditional expectations can be easily solved explicitly

$$E_t\left[\frac{1}{W_{t+1}}\right] = E_t\left[\frac{1}{K^\alpha c_{t+1} Z_{t+1}^{1-\alpha c}(1-\alpha_c)^{1-\alpha_c}}\right] = \frac{1}{(1-\alpha_c)^{1-\alpha_c} K^\alpha c_{t+1}} E_t\left[(\gamma Z_t \varepsilon_{t+1})^{\alpha_c-1}\right] = \frac{1}{(1-\alpha_c)^{1-\alpha_c} K^\alpha c_{t+1} \gamma^{1-\alpha_c} Z_t^{1-\alpha_c}},$$

where it has been used that $K_{c,t+1}$ is known at the end of period $t$ and $Z_{t+1} = \gamma Z_t \varepsilon_{t+1}$. The expectation $E_t\left[\varepsilon_{t+1}^{\alpha_c-1}\right]$ of the log-normal variable $\varepsilon_{t+1}^{\alpha_c-1}$ is simply given by $e^{(\alpha_c-1)(\alpha_c-2)\frac{\sigma^2}{2}}$. Combining this result with equation (35), the price of consumption capital must be given by equation (43) in the next lemma, which is key for the argument to follow. We first formulate a version under the assumption of predetermined beliefs $m_{c,t}$ (no projection facility applied). As in Appendix A.3.1, we use in the following lower case letters to denote realizations of random variables.

**Lemma 3.** For any given $(z_t, k_{c,t}, k_{i,t}, m_{c,t})$ with $z_t, k_{c,t}, k_{i,t}, m_{c,t} > 0$, the equations

$$q_{c,t} = \frac{\alpha_c \beta}{1 - \beta (1 - \delta_c) \frac{w_t}{k^\alpha c_{t+1}(1-\alpha_c)^{1-\alpha_c} e^{(\alpha_c-1)(\alpha_c-2)\frac{\sigma^2}{2}}} m_{c,t}} \frac{w_t}{k^\alpha c_{t+1}} \tag{43}$$

$$k_{c,t+1} = (1 - \delta_c) k_{c,t} + \frac{1}{\sigma^2} k^\alpha c_{t+1} \frac{(1 - \alpha_i) q_{c,t}}{w_t} \left(\frac{1-\alpha_i}{\alpha_i}\right) \tag{44}$$

have a unique solution $(q_{c,t}, k_{c,t+1})$ with $q_{c,t}, k_{c,t+1} > 0$. Here, $w_t = k^\alpha c_{c,t} (1 - \alpha) z_t^{1-\alpha}$ is a function of $k_{c,t}$ and $z_t$.

**Proof.** Equation (43) expresses $q_{c,t}$ as a function of $k_{c,t+1}$, $q_{c,t} = f(k_{c,t+1})$, equation (44) expresses $k_{c,t+1}$ as a function of $q_{c,t}$, $k_{c,t+1} = g(q_{c,t})$. Clearly, due to $w_t, z_t > 0$ and $\alpha_i < 1$ the function $g$ is strictly increasing on the domain $(0, \infty)$. Furthermore, as long as the denominator on the left of equation (43) is positive, i.e. for $k_{c,t+1} \in (\bar{K}, \infty)$ with

$$\bar{K} := \left(\beta (1 - \delta_c) \frac{w_t}{(1 - \alpha_c)^{1-\alpha_c} e^{(\alpha_c-1)(\alpha_c-2)\frac{\sigma^2}{2}}} m_{c,t}\right)^{\frac{1}{\alpha_c}},$$

Note also, that the second equation (44) is just a combination of the capital accumulation equation (27) with $Y_{c,t} = \frac{\delta_c}{\alpha_c}$ and the expression for $I_t$ from Lemma 2, so this equation has to hold along any equilibrium path as well.
\( f(k_{c,t+1}) \) is strictly decreasing in \( k_{c,t+1} \).\(^{94}\) Hence, also the function

\[
 h : (K, \infty) \rightarrow ((1 - \delta_c) k_{c,t}, \infty), k_{c,t+1} \mapsto g(f(k_{c,t+1}))
\]

must be strictly decreasing and thus there is at most one fixed point \( k_{c,t+1}^* \) (satisfying \( k_{c,t+1}^* = h(k_{c,t+1}^*) \)). As any solution \((q_{c,t}, k_{c,t+1})\) to (43) and (44) must satisfy \( q_{c,t} = f(k_{c,t+1}) \) and \( k_{c,t+1} = g(q_{c,t}) \), any such \( k_{c,t+1} \) must necessarily be a fixed point of \( h \). Thus, there can be at most one (positive) solution to (43) and (44).

Conversely, for any fixed point \( k_{c,t+1}^* \) of \( h \), the pair \((q_{c,t}^*, k_{c,t+1}^*) = (f(k_{c,t+1}^*), k_{c,t+1}^*)\) is obviously a (positive) solution to (43) and (44). It is thus left to show that a fixed point always exists. The function \( h \) is continuous and the following limit considerations show the existence of a fixed point by the intermediate value theorem:

- \( f(k) \to 0 \) as \( k \to \infty \), so \( h(k) \to g(0) = (1 - \delta_c) k_{c,t} \) as \( k \to \infty \), hence for large \( k \), \( h(k) - k \) is negative
- \( f(k) \to \infty \) as \( k \downarrow K \) and \( g(q) \to \infty \) as \( q \to \infty \), so \( h(k) \to \infty \) as \( k \downarrow K \), hence for small \( k \) close to \( K \), \( h(k) - k \) is positive.

Unfortunately, the above result treats \( m_{c,t} \) as fixed, but due to the projection facility described in Appendix A.2, \( m_{c,t} \) is not fully predetermined in period \( t \), instead might be projected downward, if it implies a too high PD ratio. Hence, the simple result above cannot always be applied. The following technical lemma deals with the more general case.\(^{95}\)

**Lemma 4.** For any given positive values of \( z_t, k_{c,t}, k_{s,t}, m_{c,t} \), consider the three equations

\( ^{94}\)When looking for a positive solution \((q_{c,t}, k_{c,t+1})\) to (43) and (44), we can restrict attention to \( k_{c,t+1} > K \), because otherwise \( q_{c,t} = f(k_{c,t+1}) \) becomes negative.

\( ^{95}\)The additional upper bound on beliefs stated in equation (48) is of little practical relevance. In all calibrations we consider, the upper bound is approximately equal to \( \frac{1}{1-\alpha_c} \approx 1.5 \), which implies an expected appreciation in the capital price of 50% within the next quarter and is much larger than any value \( m_{c} \) ever attained in our numerical simulations. We nevertheless tighten the projection upper bound \( \overline{m}_{c,t} \) to be always consistent with equation (48), see footnote 82. This additional modification does not invalidate the proof of the lemma.
(again, $w_t = (1 - \alpha_c)^{1-\alpha_c} k_c,t z_t^{1-\alpha_c}$)

\[
q_{c,t} = \frac{\alpha_c \beta}{1 - \beta (1 - \delta_c)^{\frac{w_t}{k_{c,t+1} \left(1 - \alpha_c \right)^{1 - \alpha_c}}} \left(\frac{w_t}{k_{c,t+1}}\right)^{1 - \alpha_c}} \frac{w_t}{k_{c,t+1}}
\]

(45)

\[
k_{c,t+1} = (1 - \delta_c) k_{c,t} + \frac{1}{\alpha_i} k_{i,t} \left(\frac{1 - \alpha_i}{w_t}\right)^{1 - \alpha_i}
\]

(46)

\[
m_{c,t} = \begin{cases} m_{c,t}, & m_{c,t} \leq \overline{m}(k_{c,t+1}) \\ \frac{(\overline{m} - m)(k_{c,t+1})}{m_{c,t} + (\overline{m} - 2m)(k_{c,t+1})} (m_{c,t} - \overline{m}(k_{c,t+1})), & m_{c,t} \geq \overline{m}(k_{c,t+1}) \end{cases}
\]

(47)

where the functions $m$, $\overline{m}$ are the projection thresholds for $m_c$ as defined in Appendix A.2.\textsuperscript{96}

If

\[
m_{c,t} < \frac{1}{\beta (1 - \alpha_c)} \left(\frac{\gamma}{\alpha_i} \left(1 - \overline{m}(k_{c,t+1})\right) e^{\frac{(2 - \alpha_c)\gamma^2}{2}}\right)^{1 - \alpha_c}
\]

(48)

then the equation system has a unique solution $(q_{c,t}, k_{c,t+1}, m_{c,t})$.

**Proof.** Ignore the capital accumulation equation (46) and first consider equations (45) and (47). To transfer the proof from Lemma 3, we need to show that after substituting $m_{c,t}$ into the first equation, this still defines a decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$. As $\overline{m}(k_{c,t+1})$ is strictly increasing in $k_{c,t+1}$ (and approaching $-\infty$ as $k_{c,t+1} \to 0$), there is some threshold $\hat{k}$, such that $\overline{m}(k_{c,t+1}) \leq m_{c,t}$ for $k_{c,t+1} \leq \hat{k}$ and $\overline{m}(k_{c,t+1}) \geq m_{c,t}$ for $k_{c,t+1} \geq \hat{k}$. We consider the two cases separately:

1. If $k_{c,t+1} \leq \hat{k}$, then the projection is actually used, so we have

\[
m_{c,t} = \frac{m_c + \overline{m} - m}{m_{c,t} + \overline{m} - 2m} (m_{c,t} - \overline{m})
\]

\textsuperscript{96}$m$, $\overline{m}$ also depend on $z_t$, $w_t$ and $k_{c,t}$, which is suppressed in the notation.
from the third equation. For notational convenience define the constants

\[
A = \frac{1}{\beta (1 - \delta_c) w_t \exp \left( (1 - \alpha_c) (2 - \alpha_c) \frac{z^2}{2} \right)}
\]

\[
B = \frac{\beta (1 - \delta_c) k_{e,t}}{(p (1 + PD) - 1) \beta (1 - \delta_c) w_t \exp \left( (1 - \alpha_c) (2 - \alpha_c) \frac{z^2}{2} \right)} = \frac{\beta (1 - \delta_c) k_{e,t}}{p (1 + PD) - 1} A
\]

\[
\mathcal{B} = \frac{\beta (1 - \delta_c) k_{e,t}}{(p (1 + PD) - 1) \beta (1 - \delta_c) w_t \exp \left( (1 - \alpha_c) (2 - \alpha_c) \frac{z^2}{2} \right)} = \frac{\beta (1 - \delta_c) k_{e,t}}{p (1 + PD) - 1} A
\]

\[
C = \beta (1 - \delta_c) \frac{w_t \exp \left( (1 - \alpha_c) (2 - \alpha_c) \frac{z^2}{2} \right)}{\gamma (1 - \alpha_c) (1 - \alpha_c) z^{1 - \alpha_c}} = A^{-1}
\]

and drop all subscripts for the following argument (k refers to \(k_{e,t+1}\), q to \(q_{e,t}\) and m to \(m_{e,t}\)).

Then we have (this follows from equations (40) and (41) for \(s = c\))

\[
m = Ak^{\alpha_c} - B k^{\alpha_c - 1}, \quad \bar{m} = Ak^{\alpha_c} - \mathcal{B}k^{\alpha_c - 1}
\]

and (from equation (45))

\[
q = \text{const} \frac{1}{1 - \mathcal{C}k^{\alpha_c} m^p k}.
\]

Using \(m^p = m + \frac{\bar{m} - \bar{m}}{m + \bar{m} - 2m} (m - m)\), we obtain

\[
\text{const} \frac{q}{k} = k - Ck^{1 - \alpha_c} m^p
\]

\[
= k - Ck^{1 - \alpha_c} \left( m + \frac{\bar{m} - m}{m + \bar{m} - 2m} (m - m) \right)
\]

\[
= k - Ck^{1 - \alpha_c} \left( Ak^{\alpha_c} - B k^{\alpha_c - 1} \right)
\]

\[
+ \frac{Ak^{\alpha_c} - \mathcal{B}k^{\alpha_c - 1} - Ak^{\alpha_c} + B k^{\alpha_c - 1}}{m + Ak^{\alpha_c} - \mathcal{B}k^{\alpha_c - 1} - 2Ak^{\alpha_c} + 2B k^{\alpha_c - 1}} \left( m - Ak^{\alpha_c} + B k^{\alpha_c - 1} \right)
\]

\[
= CB - C (\bar{B} - \mathcal{B}) \frac{m - Ak^{\alpha_c} + B k^{\alpha_c - 1}}{m - Ak^{\alpha_c} + (2B - \mathcal{B}) k^{\alpha_c - 1}}
\]

\(CB\) and \(C (\bar{B} - \mathcal{B})\) are positive constants, so \(q\) is decreasing in \(k\), if and only if the expression \(\frac{m - Ak^{\alpha_c} + B k^{\alpha_c - 1}}{m - Ak^{\alpha_c} + (2B - \mathcal{B}) k^{\alpha_c - 1}}\) is. Using that the derivative of \(x \mapsto \frac{u(x)}{u(x) + v(x)}\) is
given by $\frac{u'(x)v(x)-v'(x)u(x)}{(u(x)+v(x))^2}$, we find that $\frac{m-Ak^{\alpha_c}+Bk^{\alpha_c-1}}{m-Ak^{\alpha_c}+(2B-B)k^{\alpha_c-1}}$ is (strictly) decreasing in $k$, if and only if\footnote{Here, $u(x) = m-Ax^{\alpha_c}+Bx^{\alpha_c-1}$, $\Rightarrow u'(x) = -A\alpha_c x^{\alpha_c-1} + (\alpha_c - 1) B x^{\alpha_c-2}$ and $v(x) = (B - B) x^{\alpha_c-1}$, $\Rightarrow v'(x) = (\alpha_c - 1) (B - B) x^{\alpha_c-2}$.}

\[
(-A\alpha_c k^{\alpha_c-1} + (\alpha_c - 1) B k^{\alpha_c-2}) (B - B) k^{\alpha_c-1} < (\alpha_c - 1) (B - B) k^{\alpha_c-2} (m - Ak^{\alpha_c} + Bk^{\alpha_c-1})
\]

After expanding the products on both sides and canceling common terms, this inequality simplifies to

\[
0 < (\alpha_c - 1) (B - B) mk^{\alpha_c-2} + A (B - B) k^{2\alpha_c-2}
\]

\[
\Leftrightarrow m < \frac{Ak^{\alpha_c}}{1 - \alpha_c}
\]

Finally, using the definition of $A$, we obtain the condition (from now on subscripts are added back again for clarity about the timing of variables)

\[
m_{c,t} < \frac{\gamma^{1-\alpha_c} (1 - \alpha_c) z_t^{1-\alpha_c} k_{c,t+1}^{\alpha_c}}{\beta (1 - \delta_c) w_t \exp \left( (1 - \alpha_c) (2 - \alpha_c) z_t^2 \right) (1 - \alpha_c)}
\]

\[
= \frac{\gamma^{1-\alpha_c} (1 - \alpha_c) z_t^{1-\alpha_c} k_{c,t+1}^{\alpha_c}}{\beta (1 - \delta_c) (1 - \alpha_c) \exp \left( (1 - \alpha_c) (2 - \alpha_c) z_t^2 \right) (1 - \alpha_c)}
\]

This condition is tighter, the smaller is $k_{c,t+1}$. The smallest possible value of $k_{c,t+1}$ given $k_{c,t}$ is $k_{c,t+1} = (1 - \delta_c)k_{c,t}$ (otherwise equation (46) is inconsistent with a positive $q_{c,t}$), so the above condition on $m_{c,t}$ is certainly satisfied, if

\[
m_{c,t} < \frac{1}{\beta (1 - \alpha_c) \left( \frac{\gamma}{(1 - \delta_c) e^{(2-\alpha_c)z_t^2}} \right)^{1-\alpha_c}},
\]

which is exactly the condition required in the assertion. So as long as $k_{c,t+1} \leq \hat{k}$, the third and first equation define a strictly decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$.

2. If $k_{c,t+1} \geq \hat{k}$, then $m_{c,t}^p = m_{c,t}$ does not depend on the level on $k_{c,t+1}$ anymore and thus the third and first equation define a strictly decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$ by arguments made in the proof of Lemma 3.
After substituting the third equation into the first, we have as before two functional relationships $q_{c,t} = f(k_{c,t+1})$ and $k_{c,t+1} = g(q_{c,t})$ with $f$ strictly decreasing and $g$ strictly increasing. The same arguments made in the proof of Lemma 3 guarantee a unique solution. The associated level of $m_{c,t}^p$ can then be computed from the third equation.

We have now all the tools available to prove Proposition 1.

Proof of Proposition 1. Existence and uniqueness of the measure $\tilde{P}$ has already been discussed in Section A.3.2 of this appendix. Here, it is left to construct the mappings $G$ and $F$. Throughout the construction only necessary equilibrium conditions are used and no construction step admits several choices. For this reason, the construction will also yield uniqueness of the mappings $G$ and $F$.

Suppose $S_t = s_t$ for an arbitrary (fixed) state $s_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, m_{i,t}, q_{c,t-1}, q_{i,t-1})$ such that $m_{c,t}$ respects the upper bound of Lemma 4. For the following argument it is also useful to define the “reduced state” $\hat{s}_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, q_{c,t-1})$ that does not include $m_{i,t}$ and $q_{i,t-1}$. We first show the existence of a unique state transition function $\hat{G}$ for this reduced state. In any equilibrium, the wage is a function of $K_{c,t}$ and $Z_t$ only, compare Lemma 2. Hence, conditional on $\hat{S}_t = \hat{s}_t$, $W_t = F_W(\hat{s}_t)$ with some deterministic function $F_W$ (whose explicit form is given in Lemma 2). Furthermore, in any equilibrium (with beliefs as specified in Section 5) the equations (45), (46), (47) have to hold along any equilibrium path. Lemma 4 thus implies the existence of a unique vector $(q_{c,t}, k_{c,t+1}, m_{c,t}^p)$, given $z_t, k_{c,t}, k_{i,t}, m_{c,t}$ and $w_t = F_W(\hat{s}_t)$, which are all uniquely determined by the reduced state $\hat{s}_t$, such that all three equations hold. The equations thus implicitly define three functions

$$\hat{G}_{Q,c}(\hat{s}_t) = q_{c,t}, \quad \hat{G}_{K,c}(\hat{s}_t) = k_{c,t+1}, \quad \hat{G}_{m^p,c}(\hat{s}_t) = m_{c,t}^p$$

and the argument given so far implies that along any equilibrium path $q_{c,t}, k_{c,t+1}$ and $m_{c,t}^p$ (the value of $m_{c,t}$ after projection) must necessarily be related to $\hat{s}_t$ as described by these three equations.

Next, $m_{c,t+1}$ must satisfy the belief updating equation (compare (14) for the consumption sector\(^{98}\))

$$\log m_{c,t+1} = \log m_{c,t}^p + g \left( \log q_{c,t} - \log q_{c,t-1} - \log m_{c,t}^p \right) =: \log \hat{G}_{m,c}(\hat{s}_t),$$

where the right-hand side is a function of $\hat{s}_t$, because $q_{c,t} = \hat{G}_{Q,c}(\hat{s}_t)$ and $m_{c,t}^p = \hat{G}_{m^p,c}(\hat{s}_t)$ are and $q_{c,t-1}$ is a component of $\hat{s}_t$.

In addition, $K_{i,t+1} \propto Z_t$ by definition. Let $\bar{k}_i$ be the proportionality constant, i.e. $K_{i,t+1} = \bar{k}_i Z_t$. This implies for realizations conditional on $\hat{S}_t = \hat{s}_t$ that $k_{i,t+1} = \bar{k}_i z_t =: \hat{G}_{K,i}(\hat{s}_t)$.

\(^{98}\)As remarked in footnote 38, we set the additional shock to agents’ information set to 0 in all periods.
In total, we obtain from the discussion so far that \( \hat{s}_{t+1} = (z_{t+1}, k_{c,t+1}, k_{i,t+1}, m_{c,t+1}, q_{c,t}) \) must necessarily satisfy

\[
\hat{s}_{t+1} = (z_{t+1}, \tilde{G}_{K,c}(\hat{s}_t), \tilde{G}_{K,i}(\hat{s}_t), \tilde{G}_{m,c}(\hat{s}_t), \tilde{G}_{Q,c}(\hat{s}_t)) =: \hat{G}(\hat{s}_t, z_{t+1}).
\]

Thus, in any equilibrium, the evolution of the reduced state \( \hat{s}_{t+1} \) must be governed by the transition function \( \hat{G} \). From our derivation it is also clear that the evolution of \( \hat{G} \) is consistent with the equations (45), (46), (47), the belief updating equation (14) in the consumption sector and the exogenous evolution of \( K_i \). In particular, \( \hat{G} \) is then consistent with the consumption-sector Euler equation (18) and with the capital accumulation equations (27), (28) in both sectors.

Next, we let \( \hat{F} \) be the mapping from reduced states \( \hat{s}_t \) to outcomes \( (w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t) \) defined by the formulas given in Lemma 2, if \( q_{c,t} \) is everywhere replaced by \( \tilde{G}_{Q,c}(\hat{s}_t) \).

Combining this with the obvious state reduction mapping \( s_t \mapsto \hat{s}_t \), we can define \( F(\hat{s}_t) := \hat{F}(\hat{s}_t) \). Lemma 2 tells us then that this choice of \( F \) is the only possible choice consistent with equilibrium and in turn this \( F \) is consistent with all equilibrium equations other than the two Euler equations and the two capital accumulation equations. Consequently, \( F \) and \( \hat{G} \) together are consistent with all the equilibrium conditions (17)-(28) except for the investment-sector Euler equation (19). In addition, \( \hat{G} \) is also consistent with the belief updating equation for consumption-sector capital prices.

To complete the existence proof, it is left to show that \( \hat{G} \) can be extended to a full state transition mapping \( G \) that is in addition consistent with the investment-sector Euler equation and the belief updating equation for investment-sector capital prices. First, consider the conditional expectations in equation (39), which is a partially solved version of the investment-sector Euler equation (19) from the main text. These conditional expectations are given by:

\[
\hat{F}_{W}(\hat{s}_t) = k_{c,t}^{1-\alpha_c}(1 - \alpha_c)^{1-\alpha_c},
\]

\[
\hat{F}_{R,c}(\hat{s}_t) = \alpha_c k_{c,t}^{1-\alpha_c}(1 - \alpha_c)^{1-\alpha_c},
\]

\[
\hat{F}_{R,i}(\hat{s}_t) = \alpha_i \left( \frac{1 - \alpha_i}{\hat{F}_{W}(\hat{s}_t)} \right)^{\frac{1-\alpha_i}{\alpha_i}} \tilde{G}_{Q,c}(\hat{s}_t),
\]

and so on.

This means,

\[
\hat{F}_{W}(\hat{s}_t) = k_{c,t}^{1-\alpha_c}(1 - \alpha_c)^{1-\alpha_c},
\]

\[
\hat{F}_{R,c}(\hat{s}_t) = \alpha_c k_{c,t}^{1-\alpha_c}(1 - \alpha_c)^{1-\alpha_c},
\]

\[
\hat{F}_{R,i}(\hat{s}_t) = \alpha_i \left( \frac{1 - \alpha_i}{\hat{F}_{W}(\hat{s}_t)} \right)^{\frac{1-\alpha_i}{\alpha_i}} \tilde{G}_{Q,c}(\hat{s}_t),
\]

and so on.

100 We keep the separate mapping \( \hat{F} \) as an auxiliary device for the extension of \( \hat{G} \) to \( G \) below.
expectations can be written as
\[
E_t \left[ \frac{W_t}{W_{t+1}} R_{i,t+1} \right] = E \left[ \frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W(\hat{S}_{t+1})} \hat{F}_{R,i}(\hat{S}_{t+1}) \bigg| S_t, S_{t-1}, \ldots \right]
\]
\[
= E \left[ \frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W(\hat{G}(\hat{S}_t, Z_{t+1}))} \hat{F}_{R,i}(\hat{G}(\hat{S}_t, Z_{t+1})) \bigg| \hat{S}_t \right],
\]
\[
E_t \left[ \frac{W_t}{W_{t+1}} \right] = E \left[ \frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W(\hat{S}_{t+1})} \bigg| S_t, S_{t-1}, \ldots \right]
\]
\[
= E \left[ \frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W(\hat{G}(\hat{S}_t, Z_{t+1}))} \bigg| \hat{S}_t \right],
\]
where in each case the second equality follows from the fact that all information in the history \( S_t \) not already contained in \( \hat{S}_t \) is redundant for predicting \( Z_{t+1} \) and \( \hat{S}_t \). Hence, both conditional expectations are deterministic functions of the current reduced state \( \hat{S}_t \).

As these two conditional expectations and \( R_{i,t} = \hat{F}_{R,i}(\hat{S}_t) \) are the only relevant variables to compute the projection bounds in the investment sector, compare Appendix A.2, conditional on \( \hat{S}_t = \hat{s}_t \), the projected belief in the investment sector is given by
\[
m_{i,t} = \tilde{G}_{m,i}(\hat{s}_t, m_{i,t})
\]
with some deterministic function \( \tilde{G}_{m,i} \). By equation (39), \( Q_{i,t} \) must then assume in equilibrium the value (conditional on \( \hat{S}_t = \hat{s}_t \) and \( m_{i,t} \))
\[
q_{i,t} = \frac{\beta E_t \left[ \frac{W_t}{W_{t+1}} R_{i,t+1} \bigg| \hat{S}_t = \hat{s}_t \right]}{1 - \beta E_t \left[ \frac{W_t}{W_{t+1}} \bigg| \hat{S}_t = \hat{s}_t \right] (1 - \delta) \tilde{G}_{m,i}(\hat{s}_t, m_{i,t})}.
\]
The right-hand side is a function of \( \hat{s}_t \) and \( m_{i,t} \) and therefore of the full state \( s_t \). Denote it by \( \tilde{G}_{Q,i}(s_t) \). Finally, the belief updating equation for the investment sector defines \( m_{i,t+1} \) as a function of the current state,
\[
m_{i,t+1} = \tilde{G}_{m,i}(s_t) := \tilde{G}_{m,p,i}(\hat{s}_t, m_{i,t}) \left( \frac{\tilde{G}_{Q,i}(s_t)}{\tilde{G}_{m,p,i}(\hat{s}_t, m_{i,t}) q_{i,t-1}} \right)^g.
\]
Define thus
\[
G(s_t, z_{t+1}) := \left( \hat{G}_Z(\hat{s}_t, z_{t+1}), \hat{G}_{K,e}(\hat{s}_t, z_{t+1}), \hat{G}_{K,i}(\hat{s}_t, z_{t+1}), \hat{G}_{m,e}(\hat{s}_t, z_{t+1}), \hat{G}_{m,i}(\hat{s}_t), \tilde{G}_{Q,e}(\hat{s}_t, z_{t+1}), \tilde{G}_{Q,i}(s_t) \right).
\]
By construction, \( G \) is then (together with \( F \)) consistent with all equations that \( \hat{G} \) is and in addition with the investment-sector Euler equation (19) and with the belief updating equation for \( m_i \). This finishes the proof of the proposition.

\[ \square \]

### A.4 The Transversality Condition

In this appendix we prove the following lemma that states sufficient conditions on subjective beliefs for the transversality condition (20) to hold. We verify these conditions in our model.

**Lemma 5.** Suppose the following conditions are satisfied:

1. According to households beliefs, capital holding bounds \( \overline{K}_{c,t+1} \) and \( \overline{K}_{i,t+1} \) are functions of histories of technology shocks \( Z^t \) only, that is for each \( t \) there is a (deterministic) function \( \bar{f}_t \) such that \( (\overline{K}_{c,t+1}, \overline{K}_{i,t+1}) = \bar{f}_t(Z^t) \) \( \mathcal{P} \)-a.s.\[101\]

2. The expectations
\[
E^\mathcal{P} \left[ \frac{K_{c,t+1}}{W_t} \right], \quad E^\mathcal{P} \left[ \frac{K_{i,t+1}}{W_t} \right]
\]
exist and remain bounded as \( t \to \infty \).

3. \( \eta_Q, \eta_\beta < 1 \)

Then the transversality condition (20) is satisfied.

**Proof.** Under internal rationality, beliefs about own choices must respect individual constraints, so the capital holding constraints \( K_{c,t+1} \leq \overline{K}_{c,t+1}, K_{i,t+1} \leq \overline{K}_{i,t+1} \) must hold \( \mathcal{P} \)-a.s. Using this and the fact that \( Q_{c,t}, Q_{i,t}, W_t \geq 0 \) \( \mathcal{P} \)-a.s., we can estimate
\[
E^\mathcal{P} \left[ \frac{1}{W_t} (K_{c,t+1}Q_{c,t} + K_{i,t+1}Q_{i,t}) \right] = E^\mathcal{P} \left[ \frac{K_{c,t+1}}{W_t} Q_{c,t} \right] + E^\mathcal{P} \left[ \frac{K_{i,t+1}}{W_t} Q_{i,t} \right] \\
\leq E^\mathcal{P} \left[ \frac{\overline{K}_{c,t+1}}{W_t} Q_{c,t} \right] + E^\mathcal{P} \left[ \frac{\overline{K}_{i,t+1}}{W_t} Q_{i,t} \right]. \quad (49)
\]

Under assumption 1, \( \overline{K}_{c,t+1} \) and \( \overline{K}_{i,t+1} \) are functions of \( Z^t \) and due to rational wage expectations, also \( W_t \) is a function of \( Z^t \). \( Z^t \) and \( Q_s^t \) are independent under \( \mathcal{P} \) for both \( s = c \) and \( s = i \). Consequently,
\[
E^\mathcal{P} \left[ \frac{\overline{K}_{c,t+1}}{W_t} Q_{c,t} \right] = E^\mathcal{P} \left[ \frac{\overline{K}_{c,t+1}}{W_t} \right] E^\mathcal{P} [Q_{c,t}], \quad E^\mathcal{P} \left[ \frac{\overline{K}_{i,t+1}}{W_t} Q_{i,t} \right] = E^\mathcal{P} \left[ \frac{\overline{K}_{i,t+1}}{W_t} \right] E^\mathcal{P} [Q_{i,t}]
\]

\[101\]This is a convenient sufficient condition on beliefs about \( \overline{K}_{c,t+1} \) and \( \overline{K}_{i,t+1} \) for the transversality condition to hold. A functional relationship between \( Z^t \) and \( (\overline{K}_{c,t+1}, \overline{K}_{i,t+1}) \) clearly exists in equilibrium, so this condition holds for example, if households have rational expectations about capital bounds. We have not discussed beliefs of households with respect to capital bound processes, because they only affect subjectively expected utility (both its value and whether or not it is well-defined), but are inessential for decision functions and thus any positive prediction of our model.  

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Substituting these equations into (49) and using that $E^P \left[ \frac{K_{c,t+1}}{W_t} \right]$, $E^P \left[ \frac{K_{i,t+1}}{W_t} \right]$ are bounded above by some constant $B < \infty$ (assumption ...), we obtain the inequality

$$
\beta^t E^P \left[ \frac{1}{W_t} (K_{c,t+1} Q_{c,t} + K_{i,t+1} Q_{i,t}) \right] \leq B \cdot \beta^t \left( E^P [Q_{c,t}] + E^P [Q_{i,t}] \right).
$$

Consequently, if $\beta^t \left( E^P [Q_{c,t}] + E^P [Q_{i,t}] \right) \to 0$ as $t \to \infty$, then the transversality condition (20) must hold. To conclude the proof, it is thus only left to show $\lim_{t \to \infty} \beta^t E^P [Q_{s,t}]$ for $s \in \{c, i\}$, which we do in the following.

Iterating the subjective law of motion for $Q_s$ (equation (10)) backwards yields

$$
\log Q_{s,t} = \log \beta_{s,t} + \log \epsilon_t + (1 - \eta_Q) \log \bar{Q}_s + \eta_Q \log Q_{s,t-1}
$$

$$
= \log \beta_{s,t} + \eta_Q \log \beta_{s,t-1} + \log \epsilon_{s,t} + \eta_Q \log \epsilon_{s,t-1} + (1 + \eta_Q) (1 - \eta_Q) \log \bar{Q}_s + \eta_Q^2 \log Q_{s,t-2}
$$

$$
= \cdots
$$

$$
= \eta_Q^{t+1} \log Q_{s,-1} + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \log \beta_{s,\tau} + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \log \epsilon_{s,\tau} + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} (1 - \eta_Q) \log \bar{Q}_s.
$$

Similarly, we obtain for $\beta_s$ (iterating equation (11) backwards)

$$
\log \beta_{s,t} = \eta_\beta \log \beta_{s,t-1} + \log \nu_{s,t}
$$

$$
= \eta_\beta^2 \log \beta_{s,t-2} + \log \nu_{s,t} + \eta_\beta \log \nu_{s,t-1}
$$

$$
= \cdots
$$

$$
= \eta_\beta^{t+1} \log \beta_{s,-1} + \sum_{\tau=0}^{t} \eta_\beta^{t-\tau} \log \nu_{s,\tau}.
$$
Combining the two equations implies

\[
\log Q_{s,t} = \eta_{Q}^{t+1} \log Q_{s,-1} + \sum_{\tau=0}^{t} \eta_{Q}^{t-\tau} \eta_{\beta} \log \beta_{s,-1} + \sum_{\tau=0}^{t} \eta_{Q}^{t-\tau} \eta_{\beta} \log \nu_{s,t} \\
+ \sum_{\tau=0}^{t} \eta_{Q}^{t-\tau} \log \varepsilon_{s,t} + (1 - \eta_{Q}^{t+1}) \log \bar{Q}_{s} \\
= \eta_{Q}^{t+1} \log Q_{s,-1} + \left( \frac{\eta_{Q}^{t+1} - \eta_{\beta}^{t+1}}{\eta_{Q} - \eta_{\beta}} \right) \eta_{\beta} \log \beta_{s,-1} + \eta_{Q}^{t+1} \left( \frac{\eta_{Q}^{t-\tau}}{\eta_{Q} - \eta_{\beta}} \right) \eta_{\beta} \log \nu_{s,t} \\
+ \sum_{\tau=0}^{t} \eta_{Q}^{t-\tau} \log \varepsilon_{s,t} + (1 - \eta_{Q}^{t+1}) \log \bar{Q}_{s} \\
= \eta_{Q}^{t+1} \log Q_{s,-1} + \frac{\eta_{Q}^{t+1} - \eta_{\beta}^{t+1}}{\eta_{Q} - \eta_{\beta}} \eta_{\beta} \log \beta_{s,-1} + \sum_{\tau=0}^{t} \frac{\eta_{Q}^{t-\tau} - \eta_{\beta}^{t-\tau}}{\eta_{Q} - \eta_{\beta}} \eta_{\beta} \log \nu_{s,t} \\
+ \sum_{\tau=0}^{t} \eta_{Q}^{t-\tau} \log \varepsilon_{s,t} + (1 - \eta_{Q}^{t+1}) \log \bar{Q}_{s}
\]

\(Q_{s,-1}\) is a constant (part of the initial state) and the prior for \(\log \beta_{s,-1}\) as well as all (subjective) shocks \(\log \nu_{s,t}\), \(\log \varepsilon_{s,t}\) are normally distributed, so also \(Q_{s,t}\) must be lognormally distributed under \(P\) and it is thus fully characterized by the mean and variance of \(\log Q_{s,t}\). The mean is given by

\[
E^{P} [\log Q_{s,t}] = \eta_{Q}^{t+1} \log Q_{s,-1} + \frac{\eta_{Q}^{t+1} - \eta_{\beta}^{t+1}}{\eta_{Q} - \eta_{\beta}} \eta_{\beta} E^{P} [\log \beta_{s,-1}] \\
+ \sum_{\tau=0}^{t} \frac{\eta_{Q}^{t-\tau} - \eta_{\beta}^{t-\tau}}{\eta_{Q} - \eta_{\beta}} \eta_{\beta} \left( \frac{-\sigma_{\nu}^{2}}{2} \right) + \sum_{\tau=0}^{t} \eta_{Q}^{t-\tau} \left( \frac{-\sigma_{\varepsilon}^{2}}{2} \right) + (1 - \eta_{Q}^{t+1}) \log \bar{Q}_{s} \\
= \eta_{Q}^{t+1} \log Q_{s,-1} + \frac{\eta_{Q}^{t+1} - \eta_{\beta}^{t+1}}{\eta_{Q} - \eta_{\beta}} \eta_{\beta} E^{P} [\log \beta_{s,-1}] \\
- \frac{\sigma_{\nu}^{2}}{2} \frac{\eta_{\beta}}{\eta_{Q} - \eta_{\beta}} \left( \frac{1 - \eta_{Q}^{t+1}}{1 - \eta_{Q}} - \frac{1 - \eta_{\beta}^{t+1}}{1 - \eta_{\beta}} \right) - \frac{\sigma_{\varepsilon}^{2}}{2} \frac{1 - \eta_{Q}^{t+1}}{1 - \eta_{Q}} + (1 - \eta_{Q}^{t+1}) \log \bar{Q}_{s}
\]

and taking the limit \(t \to \infty\) yields (due to \(\eta_{Q}, \eta_{\beta} < 1\) by assumption 3)

\[
\lim_{t \to \infty} E^{P} [\log Q_{s,t}] = -\frac{\sigma_{\nu}^{2}}{2} \frac{\eta_{\beta}}{\eta_{Q} - \eta_{\beta}} \left( \frac{1}{1 - \eta_{Q}} - \frac{1}{1 - \eta_{\beta}} \right) - \frac{\sigma_{\varepsilon}^{2}}{2} \frac{1}{1 - \eta_{Q}} + \log \bar{Q}_{s} \\
= -\frac{\sigma_{\nu}^{2}}{2} \frac{\eta_{\beta}}{(1 - \eta_{Q})(1 - \eta_{\beta})} - \frac{\sigma_{\varepsilon}^{2}}{2} \frac{1}{1 - \eta_{Q}} + \log \bar{Q}_{s} < \infty
\]
Because both \( \lim \) \( \varP(\log Q_t) \) exist and are finite we obtain

\[
\lim_{t \to \infty} \varP(\log Q_t) = \sigma_\nu^2 \left( \frac{\eta_\beta}{(1 - \eta_Q)(1 - \eta_\beta)} \right)^2 \frac{1}{(1 - \eta_Q)^2} + \frac{1}{(1 - \eta_\beta)^2} - \frac{1}{1 - \eta_Q \eta_\beta} + \sigma_\varepsilon^2 \frac{1}{1 - \eta_Q^2} < \infty
\]

This completes the proof.

\[ A.5 \] Details of the Phase Diagram in Figure 3

Shifting equation (14) one period forward and setting the information shock \( \log \epsilon^1_{s,t+1} \) to zero, we have from (37)

\[
\log m_{c,t+1} = \log m_{c,t} + g (\log Q_{c,t} - \log Q_{c,t-1} - \log m_{c,t}) \\
= \log m_{c,t} + g (-\log (1 - B \cdot m_{c,t}) + \log (1 - B \cdot m_{c,t-1}) - \log m_{c,t}) \\
= (1 - g) \log m_{c,t} - g \log (1 - B \cdot m_{c,t}) + g \log (1 - B \cdot m_{c,t-1}) \\
= (m_{c,t})^{1-g} \frac{(1 - B \cdot m_{c,t-1})^g}{(1 - B \cdot m_{c,t})^g}
\]
This nonlinear second-order difference equation can be written as a first order two-dimensional difference equation system

\[
\begin{pmatrix}
  m_{c,t+1} \\
  m_{c,t}
\end{pmatrix} = \begin{pmatrix}
  (m_{c,t})^{1-g} \left( \frac{(1-B \cdot m_{c,t-1})^g}{(1-B \cdot m_{c,t})^g} \right) \\
  x
\end{pmatrix},
\]

which maps the pair \((m_{c,t}, m_{c,t-1})\) into a new pair \((m_{c,t+1}, m_{c,t})\). Clearly, the second coordinate is stable over time for \(m_{c,t-1} = m_{c,t}\), as indicated in figure 3. The first coordinate is stable \((m_{c,t+1} = m_{c,t})\) if

\[
m_{c,t} = (m_{c,t})^{1-g} \left( \frac{1 - B \cdot m_{c,t-1}}{1 - B \cdot m_{c,t}} \right)^g \Leftrightarrow m_{c,t-1} = \frac{1}{B} - \left( \frac{1}{B} - m_{c,t} \right) m_{c,t},
\]

which is the other solid line in figure 3.

### A.6 Details on the IRF Decay Restrictions in Estimation

Capital prices in the subjective belief model show a cyclical pattern in response to a technology shock: as prices fall from a boom to steady state, agents may be already slightly pessimistic and prices undershoot, as prices recover from a bust back to steady state, agents may already be slightly optimistic, triggering another boom. If these dynamics are too strong, deterministic cycles can exist or the cyclical dynamics can even be self-amplifying (small shocks lead to a sequence of cycles of increasing magnitude). We impose a decay restriction in the estimation to rule out such dynamics. The IRF decay restriction does not impose a certain speed of decay within one cycle, but rather requires the decay of subsequent cycle peaks over time to be sufficiently fast.

Specifically, for each sector \(s \in \{c, i\}\) we consider a long deterministic impulse response path (400 quarters) for the capital price \(Q_s\) to a one-standard-deviation technology shock starting in the steady state. We identify all peaks (local maxima) of the resulting path \(\{Q_{s,t}\}_{t=0}^{400}\), where at \(t\) there is a “peak”, if \(Q_{s,t} > Q_{s,t-1}, Q_{s,t+1}\). Let \(T_p\) be the set of all peak times of the impulse response path. If \(|T_p| \leq 1\), we set the peak decay rate to \(\infty\), thereby always admitting such a parameter combination. If \(|T_p| \geq 2\), we fit an exponential function through the points \(\{Q_{s,t}\}_{t \in T_p}\), specifically we estimate the least-squares regression

\[
\log \left( \frac{Q_{s,t}}{Q_{ss}} - 1 \right) = a + bt + \varepsilon_t, \quad t \in T_p,
\]

where \(Q_{ss}\) is the steady-state value of \(Q_s\). We call \(-b\) the peak decay rate of the impulse response. Figure 12 illustrates the procedure graphically: the blue solid line is the impulse response path, the red circles mark the peaks, i.e. the points \((t, Q_{s,t}/Q_{ss}^s)\) for \(t \in T_p\), and the yellow dashed line represents the fitted exponential.
Figure 12. Illustration of the decay rate definition
On the parameter we impose the restriction $-b \geq 1.16\%$. This implies a half-life of at most 60 quarters or – given a typical distance of approximately 40 quarters between two peaks – a size reduction of at least one third from one peak to the next.

Parameter combinations are admitted in the estimation, if they satisfy this condition for both $Q_c$ and $Q_i$.

### A.7 A Model Specification with Capital Adjustment Costs

In this appendix we present a model version with capital adjustment costs that is isomorphic to the model in the main text. This model version separates the price of investment goods and the price of installed capital and is able to account for the fact that the relative price of investment is countercyclical in the data (Greenwood, Hercowitz, and Krusell, 2000; Fisher, 2006).

In this model version, only the production technology changes relative to our baseline model. Specifically, the production function (1) of the representative investment sector firm is replaced by a linear labor-only production function ($ig$ for investment goods)

$$Y_{ig,t} = Z_t H_{ig,t},$$

but the output goods produced by the investment sector are just raw investment goods that are not immediately productive.

In order to become productive in the consumption sector, investment goods must be transformed into capital goods by an additional capital goods sector. This sector consists of $K_{i,t}$ firms, each of which operates a decreasing-returns-to-scale production technology

$$y_{i,t} = \phi(y_{ig,t}),$$

that transforms $y_{ig,t}$ units of investment goods into $y_{i,t}$ units of capital goods. $\phi$ is a concave function that describes the degree of adjustment costs. The aggregate output $Y_{i,t} = y_{i,t} K_{i,t}$ enters the capital accumulation equation for $K_c$ (equations (3) and (27)) in exactly the same way as in our baseline model.

Within each period, a fraction $\delta_i$ of capital goods firms receive a shock that makes their transformation technology obsolete. Also, at the end of each period, new capital goods firms are created by households according to the exogenous process $X_t$ in the main text, such that the number of capital goods firms at the end of the period, $K_{i,t+1}$, always remains proportional to the current level of technology, $Z_t$. Household can trade shares of capital sector firms at a price $Q_{i,t}$ per $K_i$-unit.

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102 We choose to index the capital goods sector by $i$ instead of $k$ (and use $ig$ instead of $i$ for the investment sector) to make the mapping between variables in the main text and variables in this alternative specification more transparent. With this naming convention, all equations stated in the main text remain valid in this alternative model specification.
This concludes the description of the alternative model specification, all other model elements are as in sections 4 and 5. To make sense in this new setting, all equations there have to be adjusted by replacing $H_{i,t}$ with $H_{ig,t}$.

Investment good firms choose labor input $H_{ig,t}$ to maximize profits

$$Q_{ig}Z_t H_{ig,t} - W_t H_{ig,t},$$

where $Q_{ig}$ is the price of investment goods which is the closest counterpart to the relative price of investment measured in the data. In equilibrium, investment good firms make zero profits and their choices imply

$$Q_{ig,t} = \frac{W_t}{Z_t}. \quad (51)$$

This relationship implies that the relative price of investment will be countercyclical, whenever the wage is not both procyclical and varies more strongly over the cycle than productivity $Z_t$.\(^{103}\) The presence of investment-specific technology shocks is not required to generate this cyclical pattern.

Capital goods firms purchase raw investment goods at price $Q_{ig,t}$ and sell their output to households at price $Q_{c,t}$. In order to maximize profits

$$Q_{c,t} \phi(y_{ig,t}) - Q_{ig,t} y_{ig,t}$$

their quantity choice $y_{ig,t}$ has to satisfy the first-order condition

$$\frac{Q_{c,t}}{Q_{ig,t}} = \frac{1}{\phi\left(\frac{Y_{ig,t}}{K_{i,t}}\right)}, \quad (52)$$

where the firm-specific choice $y_{ig,t}$ is already replaced by its representation $\frac{Y_{ig,t}}{K_{i,t}}$ in terms of aggregates (from investment goods market clearing). Equation (52) is the familiar Tobin’s $Q$ condition for optimal investment.

Each period, profits of capital goods firms are paid out to households. We denote profits per firm by $R_{i,t}$, because they correspond to the investment sector rental rate in our baseline model. Using again $y_{ig,t} = Y_{ig,t}/K_{i,t}$, profits are given by

$$R_{i,t} = Q_{c,t} \phi\left(\frac{Y_{ig,t}}{K_{i,t}}\right) - Q_{ig,t} \frac{Y_{ig,t}}{K_{i,t}}. \quad (53)$$

Equations (51), (52) and (53) jointly replace equations (23) and (24) in the baseline model and the production function (50) replaces the investment-sector part of (1). All

\(^{103}\)This is satisfied in our parameterization analyzed in section 8 and seems to be a robust feature of the model for any reasonable alternative calibration.
other equations collected in section 6 remain valid and are – together with the three
conditions derived here and the alternative production function – sufficient to characterize
the equilibrium in this alternative specification. The model presented here is thus identical
to the baseline model, if (50), (51), (52) and (53) are equivalent to the $Y_i$ equation in
(1) and the equations (23) and (24)\textsuperscript{104}. This depends obviously on the choice of the
adjustment cost function $\phi$. We discuss in the following the functional form of $\phi$ that
leads to equivalence of the models.

Substituting equation (50), (51) into (52) and (53) yields after some minor rearrange-
ments

\begin{align*}
W_t &= Q_{c,t} \phi' \left( \frac{Z_t H_{i,t}}{K_{i,t}} \right) Z_t \\
R_{i,t} &= Q_{c,t} \phi \left( \frac{Z_t H_{i,t}}{K_{i,t}} \right) - W_t \frac{H_{i,t}}{K_{i,t}}
\end{align*}

Comparing the first equation with equation (23), the two are equivalent, if and only if\textsuperscript{105}

\[ \phi' \left( \frac{Z_t H_{i,t}}{K_{i,t}} \right) = (1 - \alpha_i) \left( \frac{Z_t H_{i,t}}{K_{i,t}} \right)^{-\alpha_i}. \]

Integrating this equation implies the functional form

\[ \phi(t) = t^{1-\alpha_i}. \tag{54} \]

If this holds, then it is easy to see that also the second equation above (for $R_i$) is identical
to equation (24). In addition, aggregate output of the capital goods sector in equilibrium is

\[ Y_{i,t} = \phi \left( \frac{Z_t H_{i,t}}{K_{i,t}} \right) K_{i,t} = (Z_t H_{i,t})^{1-\alpha_i} K_{i,t}^{\alpha_i} \]

which corresponds exactly to the investment-sector production function of the model in
the main text (compare (1)).

Consequently, under the specific adjustment cost specification (54) the two models
are isomorphic. $\alpha_i$ should then not be interpreted as a capital share in production, but
rather as a curvature parameter of the capital transformation function. In the literature,
a popular adjustment specification for $\phi$ is\textsuperscript{106}

\[ \phi(t) = \frac{a_1}{1 - 1/\xi} t^{1-1/\xi} + a_2. \]

\textsuperscript{104}More precisely: have equivalent implications for all variables other than $Q_{i,t}$ for which no counter-
part exists in the main text.

\textsuperscript{105}Here we use the identity $H_{i,t} = H_{i,t}$ when mapping the model of this appendix into the baseline
model.

\textsuperscript{106}Compare e.g. Jermann (1998), Boldrin, Christiano, and Fisher (2001) and Guvenen (2009).
where $\xi$ is a curvature parameter chosen in the model’s calibration and $a_1$ and $a_2$ are usually fixed by imposing that the balanced growth path of the economy should be identical to the one of an economy without adjustment costs (in particular, Tobin’s Q is 1 along the balanced growth path). Equation (54) is a special case of this general functional form used in the literature, but it differs from the literature by not satisfying this additional steady state restriction. The important parameter is, however, the curvature parameter $\xi$ and using the definition $\xi = 1/\alpha_i$, our adjustment cost specification here is a special case of the standard specification in the literature. With this reinterpretation of the capital share $\alpha_i$ in capital goods production, our estimated parameter value of $\alpha_i = 0.73$ translates into a $\xi$ parameter of $\xi = 1.37$ which does not seem implausibly low relative to the values typically chosen in the literature (e.g. $\xi = 0.23$ in Jermann (1998) and Boldrin, Christiano, and Fisher (2001), $\xi = 0.4$ in Guvenen (2009)).

\footnote{Another difference is that in typical models with only one capital type, the number of investment firms $K_{i,t}$ here is identified with the quantity of capital $K_t$ available (which is also the good produced by the capital goods sector).}