The problem of characterizing the maximum of a set of random variables, also known as extreme-value statistics (EVS), has attracted considerable interest for several decades now. EVS has found a remarkable number of applications in several scientific fields, including engineering [1], finance [2], and biology [3]. Moreover, EVS has raised particular interest in physics [4], describing the phenomenology of several systems of general interest, such as dynamical [5], and disordered [6] systems. In particular, EVS has been recently shown to play a role in the critical regime of spin glasses, i.e. disordered uniaxial magnetic materials like Fe_{0.5}Mn_{0.5}TiO_{3} and Eu_{0.5}Ba_{0.5}MnO_{3} [7, 8]. In this Letter, we establish a novel connection between EVS and a completely different class of solid materials. These systems, known as structural glasses, are liquids–like o-Terphenyl and Glycerol–that have been cooled fast enough to avoid crystallisation [9].

Since its very first development, the theory of structural glasses has continuously drawn the attention of the scientific community: Understanding the low-temperature behavior of these systems and the nature of their glassy phase is still one of the deepest unsolved questions in condensed-matter theory [10]. In particular, the existence of a freezing transition in structural glasses has been the subject of an ongoing debate for the last few decades [11]. The development of exactly solvable models [12, 13] mimicking the phenomenology of structural glasses showed that such a transition does exist on a mean-field level and, later on, further studies suggested [11, 14] that a freezing transition might occur also for realistic, non-mean-field [15] systems.

We consider two well-established [11, 16] mean-field models of the freezing transition of structural glasses, the random energy model (REM) [12] and the p-spin model (PSM) [13], and we study the disorder-induced fluctuations of the critical temperature arising when the system size is large but finite. We show that for the REM the fluctuations of the critical temperature are described by a EVS distribution of independent variables: The Gumbel distribution [1]. For the PSM, the finite-size fluctuations of the critical point are described by a EVS distribution of correlated variables, the Tracy-Widom (TW) distribution [17]. The TW distribution has been recently discovered in the theory of random matrices, and it describes random fluctuations in a variety of physical systems [18, 19].

**Random energy model** - Let us start by considering the simplest model of a structural glass exhibiting a freezing transition: The random energy model (REM) [12]. Here, the REM will serve as an illustrative model to show the role played by EVS in the structural-glass freezing transition. The REM is defined as a system of $N$ Ising spins $S_{i}=±1$: An energy $E[S]$ is assigned to every spin configuration $S$, and the energies $E \equiv \{E[S]\}$ are independent and identically distributed (IID) Gaussian random variables with zero mean and variance $N/2$. In the thermodynamic limit, the REM has a freezing phase transition: There is a critical value of the energy $e_{c}$, which is is the lowest value of $e$ such that the number of states $S$ with energy $E[S]=Ne$ is exponentially large in the system size. The inverse critical temperature $\beta_{c}$ is determined from the threshold energy $e_{c}$ by the temperature-energy relation $\beta_{c}=-\partial_{e}E_{c}$, which can be obtained by computing the Legendre transform of the partition function [12]. Let us now introduce a critical temperature for a REM with a finite number of spins and let us study its sample-to-sample fluctuations. If the system size $N$ is sufficiently large, the threshold energy $e_{c,E}$ of an energy sample $E$ coincides with the lowest energy value, i.e. $Ne_{c,E}=\min_{\hat{S}}E[\hat{S}]$. It follows [20] that for large $N$ the threshold energy is $e_{c,E}=e_{c}-\frac{\chi}{2N\log 2}$, where $\chi$ is a random variable distributed according to the Gumbel distribution: $P(\chi \leq x) = \exp(-\exp(-x))$. It is easy to show that the Gumbel distribution describes not only the statistics of the ground state [20], but also the fluctuations of the critical temperature. A natural way to introduce a finite-size critical temperature is to extend the temperature-energy relation $\beta_{c}=-2e_{c}$ to systems with finite sizes [7, 8, 21]; We set $\beta_{c,E}=-2e_{c,E}$, where $\beta_{c,E}$ is the finite-size critical temperature of sample $E$. Putting this definition together with the above expression for $e_{c,E}$, we obtain the expression for the finite-size critical temperature $\beta_{c,E}=\beta_{c}+\frac{\chi}{N\log 2}$. This shows that the finite-size fluctuations of the critical temperature of the REM are described by a EVS distribution, the Gum-
given two independent replicas to have a mixed first/second order phase transition: number of disorder samples studied systems with sizes and numbers of disorder samples. Specifically, we ing method to use an e-tem size as though the CPU time for the PSM increases with the sys-


temsize as numerically by means of Monte Carlo simulations com-


dom variables equal to $H$ follows, we will focus on the PSM with $glasses that are absent in the REM [11, 16], and thus they develop a nonzero mutual overlap: $Q_i(h) > 0$, $\langle Q^2 \rangle - \langle Q \rangle^2 > 0$. Let us now introduce a finite-size critical temperature for the PSM. In the first place, the above discussion of the phase transition of the PSM shows that the infinite-volume critical point $\beta_c$ is the value of the temperature at which the order-parameter fluctuations (OPF) arise: $\langle Q^2 \rangle - \langle Q \rangle^2 > 0$ if $\beta \leq \beta_c$, $\langle Q^2 \rangle - \langle Q \rangle^2 > 0$ if $\beta \geq \beta_c$. To introduce a critical temperature $\beta_{c,J}$ of a finite-size PSM with couplings $J$, we recall that the Binder cumulant $B \equiv 1/2(3\langle Q^2 \rangle - \langle Q \rangle^2)$ of a finite-size PSM has a minimum at a given temperature: This temperature, which we will denote by $\beta_{c,J}$, is the temperature at which critical OPF arise [22], and it is shown in Fig. 1. It follows that the quantity $\langle Q^2 \rangle(\beta_{c,J}) - \langle Q \rangle(\beta_{c,J})^2$ represents the average OPF at the critical point. Given these average critical OPF, it is natural to identify the finite-size critical temperature $\beta_{c,J}$ of sample $J$ as the value of $\beta$ for which the OPF of the sample $\langle Q^2 \rangle_J(\beta) - \langle Q \rangle_J(\beta)^2$ are equal to the above average critical value

$$\langle Q^2 \rangle_J(\beta_{c,J}) - \langle Q \rangle_J(\beta_{c,J})^2 = \frac{\langle Q^2 \rangle_{\beta_{c,N}} - \langle Q \rangle_{\beta_{c,N}}^2}{\langle Q \rangle_{\beta_{c,N}}^2} (1)$$

The definition (1) of finite-size critical temperature is depicted in Fig. 2. As shown in the top inset of Fig. 3, the variance $\sigma^2_J \equiv \langle Q^2 \rangle_J - \langle Q \rangle_J^2$ of the critical-temperature distribution is a decreasing function of the system size $N$,
in particular $\sigma_{N} \sim N^{-\phi}$, with $\phi = 0.45 \pm 0.04$. Indeed, $\sigma_{N}$ represents the width of the critical region of a system with size $N$: As the system size gets larger and larger, the width $\sigma_{N}$ shrinks, and the finite-size critical temperature converges to the infinite-size value $\beta_{c}$ (see Supplemental Material) [7, 8, 21, 25]. Given that that the distribution of $\beta_{c}$ obeys the above scaling with the system size $N$, one can expect [8, 21] that the distribution of the normalized critical temperature $x_{J} \equiv (\beta_{c} - \beta_{c,J})/\sigma_{N}$, converges to a finite limiting shape as the system size goes to infinity. This claim is supported by the numerical data shown in the main panel and bottom inset of Fig. 3, showing that the distribution of $x_{J}$ appears to converge to a limiting distribution for large $N$. In contrast to the REM, the shape of this limiting distribution does not seem to be the Gumbel distribution, but another EVS distribution recently discovered in the theory of random matrices, the Tracy-Widom (TW) distribution [17], as shown in the main panel and bottom inset of Fig. 3 and as suggested by statistical-hypothesis tests (see Supplemental Material). Interestingly, the TW distribution has been recently shown to play a role in describing the critical behavior of disordered systems: For example, the TW distribution characterizes the average number of minima in a simple model of a random-energy landscape close to its freezing transition [25], and the finite-size fluctuations of the critical temperature in mean-field spin glasses [7, 8].

We will now provide an analytical argument to give insight into the above finding that the finite-size critical temperature fluctuates according to the TW distribution. Let us consider a version of the PSM where spins $S_{i}$ are not binary, but continuous variables satisfying the spherical constraint $\sum_{i=1}^{N} S_{i}^{2} = N$ [26]. This model is known as the spherical PSM, and it has exactly the same behavior as the PSM with Ising spins above [22], but it is more suitable for analytical studies [16]. Let us now consider this PSM for any finite $p > 2$ close to the transition point: The system is in the high (low) temperature phase if the average internal energy $E$ is higher (lower) than the average energy $E_{c}$ of the local energy minima [16]. Now, consider a finite-size sample $J$ of the PSM. If the system’s internal energy $E$ is larger than the energy of the lowest local minimum, the system is in the high-temperature phase where it explores exponentially many local minima. Otherwise, the system is in the low-temperature phase where it explores the low-lying minima [16]. Hence, the critical energy $E_{c,J}$ of this sample is given by the energy of the lowest local minimum, as illustrated in Fig. 4. Since the spin variables are continuous, we can introduce the Hessian matrix of the Hamiltonian $\partial^{2}H/\partial S_{i}\partial S_{j}$ evaluated in a local minimum, and its smallest eigenvalue $\lambda_{\text{min}}$. The fluctuations of $E_{c,J}$ can be understood by considering the geometry of the local energy minima. Indeed, following [11] the deeper the minimum, the larger the curvature of $H^{S}$ vs. $S$ at the minimum, and so the larger $\lambda_{\text{min}}$, as illustrated in Fig. 4. Let us then assume [11] that the energy $E_{c,J}$ of the lowest-lying minimum is set by the smallest eigenvalue $\lambda_{\text{min}}$ of the Hessian in such minimum. Since for any $p > 2$ the Hessian is a random matrix belonging to the Gaussian orthogonal ensemble (GOE) [26, 27], its smallest eigenvalue $\lambda_{\text{min}}$ is distributed according to the TW distribution [17]. Given that $\lambda_{\text{min}}$ is TW distributed, the above geometrical argument shows that the fluctuations of the critical energy $E_{c,J}$ are described by the TW distribution. Importantly, this argument shows that the emergence of the TW distribution shown in the numerical simulations for $p = 3$ holds for any finite $p > 2$. Finally, let us discuss the large-$p$ limit: In this limit, the PSM is equivalent to the REM [12], and the distribution of the finite-size critical temperature converges to the Gumbel distribution. Indeed, for large $p$ the above local geometrical structure of the energy landscape (see Fig. 4) disappears, and the local minima become simply a set of IID Gaussian random variables. Given that the energy threshold $E_{c,J}$ is the minimum of these IID Gaussian random variables, $E_{c,J}$ is distributed according to the Gumbel distribution [28], and so is the critical temperature.

Conclusions - In this Letter we have studied the finite-size fluctuations of the freezing-transition temperature of two mean-field models of structural glasses: The random energy model (REM) [12] and the $p$-spin model (PSM) [13] with $p = 3$. We find that for both the REM and the PSM, the finite-size fluctuations of the critical temperature are described by extreme-value-statistics probability distributions. For the REM, the critical-temperature
fluctuations are described by the Gumbel distribution \([28]\), while for the PSM the critical temperature is distributed according to the Tracy-Widom (TW) distribution, which has been recently discovered in the theory of random matrices \([17]\). For the PSM, we have provided an analytical argument to understand the emergence of the TW distribution. Recent studies \([7,8]\) have shown that the TW distribution also emerges in spin glasses: The above analytical argument shows that the physics underlying the emergence of the TW distribution in structural glasses is completely different from spin glasses, because it involves a different mechanism related to the statistics of glassy metastable states. Taken together, the results provided in this Letter establish an unprecedented connection between the theory of extreme values and the freezing transition of structural glasses.

As a topic of future studies, it would be interesting to study the fluctuations of the critical temperature in structural-glass models with short-range interactions, such as the Edwards-Anderson (EA) model in an external magnetic field \([29]\). In this regard, previous studies have shown that EVS distributions play a role in short-range systems with quenched disorder, such as the EA model with no external field \([8]\). In fact, one could imagine short-range systems to behave as an ensemble of nearly independent subsystems, each subsystem having its own critical temperature: The finite-size critical temperature of the system as a whole is then given by the smallest of the subsystems' critical temperatures \([8]\). This raises the possibility that the critical temperature of the system as a whole could be distributed according one of the three EVS distributions of independent and identically distributed random variables: The Gumbel, Fréchet, or Weibull distribution, as predicted by the extreme-value theorem \([28]\).

Convergence to the infinite-size critical temperature

In this Section we show that the finite-size critical temperature for the $p$-spin model defined in Eq. (1) converges to the infinite-size critical temperature $\beta_c = 1.535$ [1]. To do so, we compute the quantity $\Delta \beta_c \equiv [(\beta_{c,J} - \beta_c)^2]^{1/2}$, representing the deviation of the finite-size critical temperature $\beta_{c,J}$ from the infinite-size critical temperature $\beta_c$: The smaller $\Delta \beta_c$, the more peaked the distribution of $\beta_{c,J}$ around $\beta_c$. Fig. S1 shows that $\Delta \beta_c$ is a decreasing function of the system size $N$. Hence, the numerical data suggest that for large $N$ the finite-size critical temperature $\beta_{c,J}$ converges to its infinite-size value $\beta_c$.

![FIG. S1: Deviation $\Delta \beta_c$ of the finite-size critical temperature $\beta_{c,J}$ from the infinite-size critical temperature $\beta_c$ as a function of the system size $N$.](image)

Statistical-hypothesis test

In this Section we present a statistical test supporting the emergence of the Tracy-Widom (TW) distribution as the distribution of the finite-size critical temperature of the $p$-spin model. First, in Fig. S2 we show $p(x_J)$ for multiple system sizes $N$ with the relative error bars, which have been estimated with the bootstrap method [2]: This Figure shows how the distribution $p(x_J)$ compares to the TW distribution relatively to the numerical error bars. Second, the samples of normalized finite-size critical temperatures $x_J$ have been compared with the TW, Gumbel and Gaussian distribution with zero mean and unit variance by using the Kolmogorov-Smirnov (KS) test [3]. The KS statistic is $D = \sup_x |F_J(x) - F(x)|$, where $F_J$ is the cumulative distribution function (CDF) of $x_J$, $F$ is the CDF of the TW, Gumbel or Gaussian distribution with zero mean and unit variance. The $p$-value is $\Pr(x \geq \sqrt SD)$, where $x$ is a random variable distributed according to the Kolmogorov distribution [3] and $S$ is the number of samples. Table S1 shows the $p$-value for system sizes $N = 16, 32, 64, 128$ shown in Fig. S1. At a significance level of 0.1, the data do not indicate that the hypothesis that the finite-size critical temperature is distributed according to the TW distribution should be rejected for $N > 16$, while at the same significance level the hypothesis that $\beta_{c,J}$ is distributed according to the Gaussian or Gumbel distribution is rejected for all the values of $N$ shown in Table S1.

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FIG. S2: Distribution $p(x_J)$ of the normalized critical temperature $x_J$ with error bars, for different system sizes $N = 16, 32, 64, 128$ (in red, blue, brown, black respectively). Each system size is represented in a different panel, together with the Tracy-Widom, Gumbel and Gaussian distribution, all with zero mean and unit variance. Insets: Same plots as in the main panels in linear scale.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$S$</th>
<th>$p_{TW}$</th>
<th>$p_{Gumbel}$</th>
<th>$p_{Gaussian}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$1.3 \times 10^5$</td>
<td>$1.1 \times 10^{-5}$</td>
<td>&lt; $10^{-10}$</td>
<td>&lt; $10^{-10}$</td>
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<td>0.26</td>
<td>&lt; $10^{-10}$</td>
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<tr>
<td>64</td>
<td>$1.3 \times 10^5$</td>
<td>0.25</td>
<td>&lt; $10^{-10}$</td>
<td>&lt; $10^{-10}$</td>
</tr>
<tr>
<td>128</td>
<td>$6.9 \times 10^4$</td>
<td>0.88</td>
<td></td>
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</tbody>
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TABLE S1: Kolmogorov-Smirnov (KS) test for the distribution of the finite-size critical temperature: For system sizes $N = 16, 32, 64, 128$ we show the number of samples $S$ and the $p$-value of the KS test which compares the distribution of the normalized critical temperature with a reference probability distribution: The Tracy-Widom, Gaussian or Gumbel distribution, all with zero mean and unit variance. Small value of the $p$-value suggest that the distribution of the normalized critical temperature is significantly different from the reference distribution.