The interacting effects of unsteady pressure gradients and static stability in the atmospheric boundary layer

Running Head: The effects of unsteady pressure gradients and stability in the ABL

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Abstract

This study investigates the response of the Atmospheric Boundary Layer (ABL) under steady surface heating to changes in the horizontal pressure gradients. Both stabilizing and destabilizing surface heat fluxes are considered. Using large-eddy simulations (LES), the maxima of the wind speeds in the ABL (e.g. low-level jets in stable conditions) after increasing the pressure gradient are shown to be related to the strength of the pre-existing turbulent vertical coupling in the system, with the velocity overshoots decreasing when surface heat flux increases from the most stable to the most unstable cases. The profiles of temperature, heat flux and turbulent kinetic energy are found to be much more sensitive to unsteady pressure forcing in stable ABLs, compared to their unstable counterparts where the turbulence is dominant and filters the pressure gradient variations.

To better understand the fundamental physics that control these responses, a reduced analytical model for mean wind is further developed in this paper to be applicable under all stabilities. The model is directly derived from the Reynolds-averaged Navier-Stokes equations and validated against LES. The dynamics of the system according to this model are analogous to a damped oscillator where the inertial, Coriolis, and friction forces correspond to the mass, spring and damper, respectively. The generalization of the model to the non-neutral ABL requires considering the effects of buoyancy on the model, and we show that this primarily impacts the damping term corresponding to friction forces and vertical coupling.
1. Introduction

Atmospheric and oceanic flows continuously vary with time. The winds in the atmosphere fluctuate over a spectrum ranging from the synoptic (~ days), to the meso (~ hours), to the micro (~ seconds) time scales. Many interacting and complex physical mechanisms underlie this variability, which makes its observational study quite challenging. Models on the other hand can start with one simple feature, and then sequentially add interacting and complicating features (potentially by aggregating many individual studies) to the problem to better understand the interacting dynamics of multiple processes. In the context of the velocity fields in the atmospheric boundary layer (ABL), the timeline of this gradual build-up of complexity is illustrated in Table 1. This table lists some previous efforts to model the ABL over flat homogenous terrains under steady-state conditions, as well as under variable static stability and variable driving mean horizontal pressure gradient (topography and heterogeneity add further layers of complexity that are out of the scope of this paper). Here, we focus on studies that aimed to develop simplified models of the dynamics. For example, multiple studies investigated a steady ABL that is governed by the equilibrium of the Coriolis, a constant friction, and the pressure gradient forces without accounting for stability effects (i.e. the neural Ekman boundary layer, see rows 1, 3, 10, and 11 of Table 1), although some also considered the baroclinic change of the pressure gradient with height.

However, the real-world winds vary in time due to the changes in the static stability of the ABL and/or due to a variable pressure gradient. The diurnal variability of the surface buoyancy flux can lead to the formation of the nocturnal low-level jets (see e.g. Banta et al. 2006; Bain et al. 2010; Mahrt 2013) that was first modelled by Blackadar (1957) using the inertial oscillation concept. Some extensions to this mechanism were later suggested by many other researchers (rows 6, 13, 14, 15, and 16 of Table 1). The fluctuations in the geostrophic forcing (which is simply
representing the large-scale horizontal pressure gradients driving the flow) cause variations in the
flow velocity as well (see large-eddy simulation studies of atmospheric and oceanic cases by Hsu
et al., 2000; Lohmann et al., 2006; Radhakrishnan and Piomelli, 2008; Gayen et al., 2010), and
various reduced models were proposed to explain and predict such variations (see rows 4, 12, and
16 of Table 1).

Nevertheless, as Table 1 illustrates, the previous analytical solutions in the unsteady
framework have some limitations including i) constraining assumptions about the structure of the
eddy-viscosity (thus limiting their applicability range), ii) intricate solutions of the governing
PDEs that are not always straightforward to acquire or interpret physically, or iii) lack of validation
against more accurate benchmarks such as large-eddy simulations (LES). Furthermore, no
previous work analyzes the unsteady pressure forcing cases under the full range of static stabilities
observed in the ABL. This paper aims to bridge this gap and propose a reduced model that is
applicable in a very wide array of conditions, including broad ranges of ABL stabilities and eddy
viscosity with time and height variabilities, by providing thorough LES validations.

In Momen and Bou-Zeid (2016), we developed a model for the unsteady pressure gradient
effects on neutral ABL mean wind velocities using an analogy to a damped-oscillator where the
inertial, Coriolis, and friction forces reflect the mass, spring and damper respectively. Next, in
Momen and Bou-zeid (2017a), we generalized the model by introducing a time-dependent
damping coefficient that was shown to be necessary to capture the gradual decrease in the surface
heat–flux in neutral to stable ABL transition (under steady pressure forcing). In both papers, we
validated the solutions with accurate LES results. Nonetheless, while we previously investigated
variations in the surface buoyancy flux and the pressure gradients, the two were considered
separately. Therefore, the influence of static stability, even when steady, on the response of the
ABL to an unsteady pressure gradient was not considered despite being the analogue of many real-world conditions.

In this paper, we add this layer of complexity, brought about by a steady surface buoyancy, to the investigation of ABLs under unsteady pressure gradients. We also extend the damped oscillator modelling framework to parametrize the stability effects on turbulent friction in the system, thus allowing the model to be applied without requiring LES. We also examine the turbulent kinetic energy (TKE) budget dynamics in the diabatic unsteady ABLs, which have been rarely studied before.

In particular, our research questions that address the physics and the mathematical modelling of the diabatic ABLs (stable and unstable) are respectively as follows:

1) How does the interaction of buoyancy and variable pressure forcing influence mean and turbulence in the ABL?

2) What modifications are required to the reduced damped-oscillator model of unsteady ABLs to account for the effect of static stability?

We briefly describe the details of our LES in the next section. In Section 3, we investigate the mean and turbulence dynamics of the unsteady diabatic ABLs to answer question 1. In Section 4, we model wind variations using the damped-oscillator concept to answer question 2. It will be shown that a time-invariant damping coefficient (though a function of height) can capture the LES results well when the surface heat fluxes are steady. We also suggest a parametrization for the damping term that is a function of the stability in the ABL. Finally, a summary of our findings is given in Section 5.
Table 1. A summary of previous studies of atmospheric boundary layer winds over homogenous terrain that proposed reduced analytical models of the dynamics.

<table>
<thead>
<tr>
<th>Pressure gradient</th>
<th>Buoyancy flux</th>
<th>Validation</th>
<th>Comments</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Steady</td>
<td>None</td>
<td>The first model with height- and time-constant viscosity</td>
<td>Ekman (1905)</td>
</tr>
<tr>
<td>2</td>
<td>Steady</td>
<td>None</td>
<td>Inertial oscillations of low-level jets with no frictional effects</td>
<td>Blackadar (1957)</td>
</tr>
<tr>
<td>3</td>
<td>Steady</td>
<td>None</td>
<td>Ekman solutions for exponential thermal winds</td>
<td>Mahrt and Schwertfeger (1970)</td>
</tr>
<tr>
<td>4</td>
<td>Unsteady</td>
<td>None</td>
<td>Analytical solutions for diurnally periodic winds</td>
<td>Bonner and Paegle (1970)</td>
</tr>
<tr>
<td>5</td>
<td>Steady</td>
<td>None</td>
<td>Suddenly applied uniform shear stress</td>
<td>Madsen (1977)</td>
</tr>
<tr>
<td>6</td>
<td>Steady</td>
<td>Wangara observations</td>
<td>A two-layer slab model based on the idea of Blackadar (1957)</td>
<td>Thorpe and Guymer (1977)</td>
</tr>
<tr>
<td>7</td>
<td>Steady</td>
<td>Observation (Leipzig profile)</td>
<td>Incorporated a surface layer at the bottom of the Ekman layer</td>
<td>Yordanov et al. (1983)</td>
</tr>
<tr>
<td>8</td>
<td>Steady</td>
<td>None</td>
<td>Oscillatory character of mixed layers with a zero-th-order model; only numerical simulations were validated with Cabauw observations</td>
<td>Byun and Arya (1986)</td>
</tr>
<tr>
<td>9</td>
<td>Steady</td>
<td>Observation (Leipzig profile)</td>
<td>A gradually varying eddy-viscosity in height</td>
<td>Miles (1994)</td>
</tr>
<tr>
<td>10</td>
<td>Steady</td>
<td>None</td>
<td>A gradually varying eddy-viscosity in height</td>
<td>Grisogono (1995)</td>
</tr>
<tr>
<td>11</td>
<td>Steady</td>
<td>None</td>
<td>A height-varying eddy-viscosity and a baroclinic ABL</td>
<td>Tan (2001)</td>
</tr>
<tr>
<td>12</td>
<td>Unsteady</td>
<td>None</td>
<td>Provides exact solutions subject to a prescribed eddy-viscosity profile</td>
<td>Lewis and Belcher (2004)</td>
</tr>
<tr>
<td>13</td>
<td>Steady</td>
<td>Cabauw observations</td>
<td>Included the frictional effects in the Blackadar mechanism</td>
<td>Van de Wiel et al. (2010)</td>
</tr>
<tr>
<td>14</td>
<td>Steady</td>
<td>None</td>
<td>Impulsively varying eddy-viscosity</td>
<td>Shapiro and Fedorovich (2010)</td>
</tr>
<tr>
<td>15</td>
<td>Steady</td>
<td>None</td>
<td>Used the concept of inertial oscillations</td>
<td>Schröter et al. (2013)</td>
</tr>
<tr>
<td>16</td>
<td>Unsteady</td>
<td>None</td>
<td>Both Holton and Blackadar mechanisms with 2 eddy-viscosities</td>
<td>Du and Rotunno (2014)</td>
</tr>
<tr>
<td>17</td>
<td>Unsteady</td>
<td>3D LES</td>
<td>Steady but height variable frictional effects, proposed neutral parametrization</td>
<td>Momen and Bou-Zeid (2016)</td>
</tr>
<tr>
<td>18</td>
<td>Unsteady</td>
<td>3D LES</td>
<td>Height and time-variable frictional effects parametrization</td>
<td>Momen and Bou-Zeid (2017a)</td>
</tr>
<tr>
<td>19</td>
<td>Unsteady</td>
<td>3D LES</td>
<td>Height and time-variable eddy-viscosity, proposing neutral and non-neutral parametrization</td>
<td>Current work</td>
</tr>
</tbody>
</table>
2. Details of the large-eddy simulations

2.1. Governing equations

The basic premise in LES is that the largest eddies contain most of the energy and are responsible for
the majority of the transport of momentum and scalars and consequently they are the most
dynamically important scales. Hence, the LES technique consists of directly resolving only the large
high-energy scales, while parametrizing eddies smaller than a cut-off (filter or grid) scale, to reduce
the computational cost of high Reynolds number (Re) simulations significantly. The mean pressure
gradient in our simulations is represented as a horizontal geostrophic wind with components $U_g$
and $V_g$ as:

$$
(U_g(z,t), V_g(z,t)) = \frac{1}{G} \left( -\frac{1}{f \rho} \frac{\partial p}{\partial y}(z,t), \frac{1}{f \rho} \frac{\partial p}{\partial x}(z,t) \right),
$$

where $f$ is the Coriolis parameter ($1.394 \times 10^{-4}$ Hz here), and $G$ is the magnitude of the geostrophic
wind vector (here the numerical value of $G$ is taken as 8 m s$^{-1}$) which, along with the ABL depth $z_i$, is used to normalize the governing equations and all the results of the paper.

The governing equations of the code are: the resolved mass conservation equation (an
equivalent Poisson equation for pressure is actually solved), the resolved momentum equation
(neglecting the molecular terms due to a very high $Re$ regime) written in rotational form to ensure
conservation of mass and kinetic energy (Orszag and Pao, 1975), and the resolved thermal energy
conservation equation written in terms of the potential temperature, which are respectively given
as

$$
\frac{\partial \bar{u}_i}{\partial x_i} = 0,
$$

(2)
\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \left( \frac{\partial \mathbf{u}}{\partial x_j} - \frac{\partial \mathbf{u}}{\partial x_i} \right) = -\frac{1}{\rho} \frac{\partial p^*}{\partial x_i} + \frac{\partial \mathbf{\tau}_{ij}}{\partial x_j} + g \frac{\partial \theta^*}{\partial x_j} \delta_{ij} + f (U_g - \tilde{u}_i) \delta_{i2} - f (V_g - \tilde{u}_2) \delta_{i1},
\]

(3)

\[
\frac{\partial \tilde{\theta}}{\partial t} + \mathbf{u}_j \frac{\partial \tilde{\theta}}{\partial x_j} = -\frac{\partial \pi_j}{\partial x_j},
\]

(4)

where \( x_i \) is the position vector; \( \mathbf{u}_i = (\tilde{u}, \tilde{v}, \tilde{w}) = (\tilde{u}_i, \tilde{u}_2, \tilde{u}_3) \) is the resolved velocity vector; \( \mathbf{\tau}_{ij} = u_iu_j - \bar{u}_i \bar{u}_j \) is the deviatoric part of the sub-grid scale (SGS) stress tensor; \( p^* \) is a modified pressure; \( \theta \) is potential temperature with its reference value \( \theta_r \); \( \theta' \) is the deviation of the local \( \theta \) from its horizontal average; \( g \) represents the gravitational acceleration; and \( \pi_j = \bar{\theta} u_j - \bar{\theta} \bar{u}_j \) is the SGS heat flux vector. Since the focus here is not on the LES, the details of the code such as the imposed boundary and initial conditions and the grid configuration are discussed in Appendix A.

2.2. Suite of large-eddy simulations

We impose seven surface heat fluxes and change the pressure gradient for each simulation in two ways: one by sinusoidally varying it and the other by instantly doubling it (step change). Table 2 shows all the performed large-eddy simulations. To study the ABL behaviour under the sinusoidal pressure-gradients, we impose a frequency for the forcing variability, \( \tau_f \), that is equal to the natural frequency of the inertial oscillations in the ABL, which is the Coriolis parameter characterizing the response time of the mean flow as \( \tau_{ABL} = 2\pi f \approx 12.5 \text{ hr} \) (Tennekes and Lumley, 1972). The pressure forcing of these resonant frequency cases, expressed as \( (U_g, V_g) \), is thus given as a sinusoidally-varying function of time

\[
U_g(t) = \cos \left( \frac{2\pi t}{\tau_f} \right), \quad V_g(t) = \sin \left( \frac{2\pi t}{\tau_f} \right),
\]

(5)

where \( \tau_f \) is the forcing time scale. The pressure forcing of the step changes is given as
Recall that these are the values normalized by the dimensional geostrophic wind velocity $G = 8 \text{ m s}^{-1}$. We also note that these very long times of constant stability may not occur regularly in the real ABLs due to the diurnal cycle (perhaps they happen only in the poles or in some engineering applications). In addition, such a long-lived consistent variability in the pressure forcing is unlikely. However, the long simulations are aimed at ensuring convergence of the computed statistics, which will be applicable to shorter occurrences of these conditions when they actually arise in nature (e.g. a step change in the pressure gradient, which represents a front, could happen under a stable or an unstable ABL for several hours).

Table 2. Suite of the large-eddy simulations. $L$ is the Obukhov length and $z_i$ is the initial (steady-state) inverse height that is listed in Table 3.

<table>
<thead>
<tr>
<th>Surface heat flux</th>
<th>#</th>
<th>Acronym</th>
<th>Run Time</th>
<th>$dt$ (s)</th>
<th>$N_{x,y,z}$</th>
<th>$z_i/L$</th>
<th>$L_z$</th>
<th>Imposed Inversion base</th>
<th>Sponge Layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>–40 w/m²</td>
<td>1</td>
<td>Step–40</td>
<td>40 hr</td>
<td>0.2</td>
<td>160$^3$</td>
<td>7.2</td>
<td>1600m</td>
<td>1200m at 1200m</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>Sin–40</td>
<td>25 hr</td>
<td>0.08</td>
<td>160$^3$</td>
<td>7.2</td>
<td>1600m</td>
<td>1200m at 1200m</td>
<td></td>
</tr>
<tr>
<td>–20 w/m²</td>
<td>3</td>
<td>Step–20</td>
<td>40 hr</td>
<td>0.2</td>
<td>160$^3$</td>
<td>2.8</td>
<td>1600m</td>
<td>1200m at 1200m</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>Sin–20</td>
<td>25 hr</td>
<td>0.08</td>
<td>160$^3$</td>
<td>2.8</td>
<td>1600m</td>
<td>1200m at 1200m</td>
<td></td>
</tr>
<tr>
<td>–10 w/m²</td>
<td>5</td>
<td>Step–10</td>
<td>40 hr</td>
<td>0.2</td>
<td>160$^3$</td>
<td>1.5</td>
<td>1600m</td>
<td>1200m at 1200m</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>Sin–10</td>
<td>25 hr</td>
<td>0.08</td>
<td>160$^3$</td>
<td>1.5</td>
<td>1600m</td>
<td>1200m at 1200m</td>
<td></td>
</tr>
<tr>
<td>0 w/m²</td>
<td>7</td>
<td>Step 0</td>
<td>40 hr</td>
<td>0.2</td>
<td>128$^3$</td>
<td>0</td>
<td>1000m</td>
<td>NO NO</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>Sin 0</td>
<td>40 hr</td>
<td>0.1</td>
<td>128$^3$</td>
<td>0</td>
<td>1000m</td>
<td>NO NO</td>
<td></td>
</tr>
<tr>
<td>+25 w/m²</td>
<td>9</td>
<td>Step+25</td>
<td>40 hr</td>
<td>0.2</td>
<td>128$^3$</td>
<td>–7.0</td>
<td>1700m</td>
<td>1250m NO</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>Sin+25</td>
<td>40 hr</td>
<td>0.1</td>
<td>128$^3$</td>
<td>–7.0</td>
<td>1700m</td>
<td>1250m NO</td>
<td></td>
</tr>
<tr>
<td>+100 w/m²</td>
<td>11</td>
<td>Step+100</td>
<td>40 hr</td>
<td>0.2</td>
<td>128$^3$</td>
<td>–20.4</td>
<td>1700m</td>
<td>1250m NO</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>Sin+100</td>
<td>40 hr</td>
<td>0.1</td>
<td>128$^3$</td>
<td>–20.4</td>
<td>1700m</td>
<td>1250m NO</td>
<td></td>
</tr>
<tr>
<td>+400 w/m²</td>
<td>13</td>
<td>Step+400</td>
<td>40 hr</td>
<td>0.2</td>
<td>128$^3$</td>
<td>–61.2</td>
<td>1700m</td>
<td>1250m NO</td>
<td></td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>Sin+400</td>
<td>40 hr</td>
<td>0.1</td>
<td>128$^3$</td>
<td>–61.2</td>
<td>1700m</td>
<td>1250m NO</td>
<td></td>
</tr>
</tbody>
</table>
3. Mean and turbulence evolution

In this section, we aim to answer our first question: how does the diabatic ABL respond to an unsteady pressure forcing? To that end, we first compare the mean wind and TKE dynamics under various stability conditions. Then, we investigate the implications on the temperature and heat flux profiles. Finally, we examine the behaviour of the different terms of the TKE budget in such flows.

3.1. Wind velocity and TKE profiles

The evolution of the mean velocity field in the ABL for three step-change cases, the most stable, the neutral, and the most unstable cases, are exhibited as they illustrate the general trends. Figure 1(a) shows the $u$-velocity time series at $z/z_i = 0.65$. It is clear that the stable case overshoots more strongly and peaks earlier, while the velocity in the unstable case fluctuates less and peaks later than the other cases as the forcing varies. This response is a direct result of the fact that before the jump (at $t < 20$ hr) the stabilizing heat fluxes, in the stable case, damp the turbulence and decrease the friction and vertical coupling in the system. Hence, the velocity increases more since the turbulent stresses that would hold it back are weaker and take a longer time to increase, after the shear increases, to balance the newly imposed pressure gradient. On the other hand, in the unstable case, the destabilizing heat fluxes produce higher turbulence and friction and hence lower vertical gradients are required to balance the pressure gradient. Moreover, the stresses develop faster due to the shorter eddy turn over times, which leads to a slightly delayed peak. Consequently, the overshoot in the velocity is lower than other cases. The neutral case stands in between these two limits.

We investigate the space-time responses of the ABLs under the step-change pressure gradient to fully unravel their dynamics. Figure 1(b), (c), and (d) display these graphs respectively in the most stable (see also Video S1), neutral, and most unstable (see also Video S2) conditions.
In addition to the observation that the velocity is usually higher in the stable case and lower in the unstable one as illustrated in Figure 1(a), one can also note here that in the unstable case (Figure 1(d)) the profiles are more uniform with height due to strong vertical mixing. In the stable ABLs, the atmospheric strata are more decoupled resulting in stronger vertical variability as expected. As a result, there exists a layer near the surface where the time variation of the velocity is stronger in the unstable case than stable one due to strong vertical coupling causing a high downward flux of momentum (compare 1(b) and 1(d) close to the surface); this layer was found to extend up to the height of the turbulent stable boundary before the jump (e.g. ≈ 160 m in Step–40 case).

Table 3 lists the friction velocity $u^*$ and the height of the ABL $z_i$ before and after the pressure-gradient jump in the step-change simulations. For the stable cases, we calculated $z_i$ by locating the height where the TKE decreases to 5% of its magnitude near the surface. In the unstable ABL, the imposed inversion fixes $z_i$, which remains constant. One can see that the $u^*$ after the jump increases in all the cases, which is expected since the pressure gradient, velocity, and hence the shear rise after the jump. Furthermore, the friction velocities increase monotonically with increasing surface heat flux. The height of the ABL, $z_i$, also grows after the jump in the stable conditions, and as expected it decreases as the stabilizing surface heat flux strengthens.
Figure 1. Evolution of $u$ under various stabilities: (a) times series in the middle of the ABL; and space-time graphs for cases (b) Step–40, (c) Step 0, and (d) Step+400.
Table 3. Friction velocity and ABL heights under two pressure-gradient values and different stability conditions.

<table>
<thead>
<tr>
<th>Surface heat flux</th>
<th>#</th>
<th>Name</th>
<th>$G = 8$ m/s</th>
<th>$G = 16$ m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$u^*$ (m/s)</td>
<td>$z_i$ (m)</td>
</tr>
<tr>
<td>–40 w/m²</td>
<td>1</td>
<td>Step–40</td>
<td>0.21</td>
<td>160</td>
</tr>
<tr>
<td>–20 w/m²</td>
<td>3</td>
<td>Step–20</td>
<td>0.25</td>
<td>210</td>
</tr>
<tr>
<td>–10 w/m²</td>
<td>5</td>
<td>Step–10</td>
<td>0.27</td>
<td>280</td>
</tr>
<tr>
<td>0 w/m²</td>
<td>7</td>
<td>Step 0</td>
<td>0.31</td>
<td>1000</td>
</tr>
<tr>
<td>+25 w/m²</td>
<td>9</td>
<td>Step+25</td>
<td>0.36</td>
<td>1250</td>
</tr>
<tr>
<td>+100 w/m²</td>
<td>11</td>
<td>Step+100</td>
<td>0.4</td>
<td>1250</td>
</tr>
<tr>
<td>+400 w/m²</td>
<td>13</td>
<td>Step+400</td>
<td>0.43</td>
<td>1250</td>
</tr>
</tbody>
</table>

Inertial oscillations in the stable ABL occur near $z_i$, where the TKE is low. The horizontal winds oscillate with the Coriolis frequency and form the much-studied low-level jets (LLJ, see Blackadar, 1957; Shapiro and Fedorovich, 2010; Van de Wiel et al., 2010). The mean and TKE time series and profiles for the stable ABL simulations are shown in Figure 2. The left panels of Figure 2 exhibit the dimensional LES data, while the right panels show the same data normalized with $z_i$ for elevation and with the geostrophic velocity scale ($G = 8$ m/s) for velocities, as shown in Table 2. The formation of a supergeostrophic velocity can be observed in the stable cases (Figure 2(e) and 2(f)). As we increase the stabilizing heat flux, the height of the ABL and of the LLJ decrease, as illustrated in Figure 2(e). Furthermore, the friction in the system decreases with increasing stability; consequently, the increase in the velocities, i.e. the magnitude of the LLJ, is higher in the strongly stable cases. The level of the LLJ almost doubles after the jump. The resonance phenomenon (due to imposing a forcing with a time scale close to the inertial time scale)...
is clear from Figure 2(g) and 2(h) showing super-geostrophic velocities. The overshoots in the
Step–40 case are larger than the Step–20 and Step–10 since the turbulent friction and vertical
coupling are weaker in this case.

The right panels of this Figure show that after normalizing the profiles and time series, they
collapse well (Figure 2(f) and 2(h)) except perhaps velocities of some cases near the wall. Since
$G$ is the same for all cases, this collapse is mainly due to the rescaling of the elevation with $z_i$. The
few deviations could be due to the fact that for these cases, immediately after the jump, turbulence
is unsteady while we are normalizing with the steady-state variables and several inertial
oscillations are required for the flow to reach a full (quasi) steady-state condition. We can account
for unsteadiness by rescaling based on a time-dependent variable. For example, a proper
characteristic velocity near the wall is friction velocity rather than the geostrophic velocity. In
supplementary Appendix S1, we normalize the velocity profiles in the stable step-change
simulations, and the profiles collapses improve near the wall (especially for $z/z_i < 0.15$).

The TKE in the stable step-change simulations, depicted in Figure 2(c) and 2(d), changes
along with the mean variations shown in Figure 2(a) and 2(b). Another interesting feature of the
TKE time series shown in Figure 2(c) and 2(d) is the lag between the two heights (compare solid
and dashed lines at $t \approx 20$ hr). As this Figure indicates, there is $\sim 1$ hr lag between the lower and
higher elevations at $t \approx 20$ hr. This lag is due to the fact that shear production first generates TKE
near the surface, and then that TKE is transported upward. The lag is more pronounced in the
stable ABLs than the unstable ones, indicating that the layers are more decoupled, with slower
TKE transport, under stabilizing heat fluxes.
Figure 2. $M$ and TKE time-series, and $M$ profiles for 6 stable cases under step (a, b, c, d, e) and sinusoidal change (g, h) for dimensional data (left panel) and normalized data with $G$ and $z_i$ (right panel).
In the unstable ABL simulations, the velocities are, in general, more uniform due to higher mixing (Figure 3(a), (b), (e), and (f)). The TKE increases with increasing destabilizing heat flux, in these cases as shown in Figure 3(c) and 3(d). As a result, the friction in the system is higher in the stronger unstable cases and thus after the jump the overshoot is slightly lower for the simulations with stronger heat fluxes (Figure 3(a)). This damping is more obvious in Figure 3(b) for the resonant-frequency cases, where the stronger unstable cases show a weaker increase in the velocity magnitude as time evolves.

In addition, since the velocity profiles are more uniform in the unstable step-change cases, we generally obtain better collapses of the velocity time series and profiles as Figure 3(a) and 3(e) indicate. Nevertheless, for the resonant frequency cases, which are always unsteady, the velocities in the three cases increasingly diverge for longer times (Figure 3(f)) if we use the constant characteristic velocity scale $G$ for non-dimensionalizing velocities. Although we do not investigate the formulation of improved characteristic velocities for this case, we suggest that a combination of $u^*$ and $w^*$ might yield better results. The weaker unstable case, which has lower turbulent friction, exhibits higher peaks than the other cases as expected (Figure 3(f)).
Figure 3. LES results of the step-change (left panels) and sinusoidal change (right panels) in the pressure gradient under three unstable cases for: (a,b) the magnitude of the velocity time series, (c,d) the TKE time series, and (e,f) the time evolution of the velocity profiles.
3.2. Temperature and heat-flux profiles

Since static stability was shown to influence the response of the ABL to a change in the pressure gradient, if that response results in changes in temperature and heat-flux profiles, feedbacks will set in and lead to an increased dynamical complexity. Figure 4(a) indicates that in the steady part of the *Step–40* simulation, before the jump in the pressure gradient (i.e. at \( t \leq 20 \) hr), the temperature decreases due to the negative surface heat flux but the gradients that control the static stability do not change markedly. However, in the transition period after the jump, the ABL warms up because of the increase in the shear and higher mixing that brings hotter air from aloft closer to the surface. After the ABL reaches the new equilibrium (e.g. \( \sim 35 \) hr as shown in Figure 4(a) and 4(b)), it starts cooling again and the vertical mixing decreases. The rate of the pre-jump cooling and post-jump heating of the ABL at each height could be inferred from the profiles of the heat flux depicted in Figure 4(b), where a positive vertical gradient of the flux indicates cooling at that height and a negative gradient heating. In the stably stratified ABL, the heat-flux profiles are often represented according to the similarity relationship:

\[
\frac{w' \theta'}{w' \theta'_{\text{surface}}} = \left( 1 - \frac{z}{h} \right)^\gamma
\]

(Stull 1988). We obtain a \( \gamma \approx 1.3 \) for *Step–40* case before the jump, which is consistent with the range indicated in Stull (1988) as \( \gamma = 1-3 \).

A similar trend for the strongly stable ABL under the sinusoidal change in the pressure gradient is found as Figure 4(c) and 4(d) exhibit. However, because the increase in the shear and consequently the mixing is continuously rising in this case, due to forcing at the resonant frequency, the ABL keeps heating until it reaches the point where the maximum magnitude of the velocity and the rate of mixing become constant.

Figure 4(e) and 4(f) show the temperature and heat-flux profiles for the strongly unstable case under step-change and sinusoidal forcing. It is clear that doubling the pressure gradient in this
case does not have a significant impact on the heating of the ABL or on the shape of the temperature profiles because the mixing is already very high in the unstable ABL. Furthermore, the heat-flux profiles and temperature gradients are similar under the step-change and sinusoidal-change forcings. The temperatures are almost the same at the beginning of Step+400 and Sin+400 (Figure 4(e)), and after about two inertial oscillations (~25 hr) the sinusoidal case becomes only slightly warmer, perhaps due to slightly higher shear turbulent kinetic energy production. In the convective ABL, before and after the jump, the heat fluxes are changing almost linearly in height, consistent with the similarity relationship:

\[
\frac{w' \theta'(z)}{w' \theta'_{\text{surface}}} = 1 - \frac{\lambda z}{z_i}
\] (Stull, 1988) where a value of \(\lambda \approx 1.2-1.5\) is reported. In our LES, we obtain a value of 1.4 for Step+400 based on Figure 4(f).

In summary, the static stability inferred from the temperature gradients or heat-flux profiles is strongly altered by the changes in the pressure gradients under stable conditions, which then feeds back to the response of the velocity and stress to these changes. On the other hand, the static stability profiles in the unstable ABL are more resilient and remain unchanged when the pressure gradients vary (Figure 4(e) and 4(f)).
Figure 4. Potential temperature and heat-flux profile at different times for 3 cases: (a,b) Step–40, (c,d) Sin–40, (e,f) Step+400 and Sin+400. In the stable sinusoidal forcing (c,d), the times 4 hr and 9 hr correspond to the local minima in the wind-speed oscillations, while times 14 and 19 hr are near local maxima in the time series. The times in the step change simulations are before and after the jump in the forcing that occurs at $t = 20$ hr. We also selected the same times in the unstable sinusoidal simulation as those for the step-change to show the effect of the type of variability of the pressure-gradient in the dynamics. The differences between the two variabilities are minor indicating that in unstable cases the scalar quantities are not influenced significantly by the change in the pressure gradient.
3.3. TKE budget

We now investigate the response of the different terms of the TKE budget to variable pressure
gradients. We neglect the SGS kinetic energy since it only makes a small contribution relative to
its resolved counterpart over most of the domain (see Momen and Bou-zeid 2017b). Moreover, we
assume no subsidence \( w=0 \) and statistical homogeneity in both horizontal directions due to domain
periodicity. This allows us to simplify the budget equations to the following conventional form for
the resolved TKE \( \tilde{e} = 0.5 \overline{u_i' u_i'} \):

\[
\frac{\partial \tilde{e}}{\partial t} = -u' w' \frac{\partial \tilde{u}}{\partial z} - v' w' \frac{\partial \tilde{v}}{\partial z} + g \frac{\tilde{w}' \theta'}{\theta} - \frac{\partial \tilde{w}' \rho'}{\partial z} - \frac{\partial \tilde{w}' e}{\partial w} - \frac{\partial u_i' \tau_i'}{\partial \tilde{w}} + \tilde{\tau}_{ij}' \tilde{S}_{ij}', \tag{7}
\]

where the overbar denotes the ensemble average, and the prime denotes the turbulent fluctuations
from that average. We compute the TKE budget in terms of the resolved variables and define the
following expressions for the terms in Eq. (7) as

\[
P = \overline{u' w'} \frac{\partial \tilde{u}}{\partial z} - v' \frac{\partial \tilde{v}}{\partial z} = \left( \overline{u w - u w} \right) \frac{\partial \tilde{u}}{\partial z} + \left( \overline{v w - v w} \right) \frac{\partial \tilde{v}}{\partial z},
\]

\[
BP = g \frac{\tilde{w}' \theta'}{\theta} = g \frac{\left( \overline{w \theta - w \theta} \right)}{\theta}, \tag{8}
\]

\[
RT = -\frac{\partial \tilde{w}' e}{\partial \tilde{w}} = -\frac{1}{2} \frac{\partial}{\partial \tilde{w}} \left( \overline{u^2 w - 2u uu w + v^2 w - 2v vv w + w^3} \right),
\]

\[
\varepsilon = \tau_{ij}' \tilde{S}_{ij}' = \tau_{ij}' \tilde{S}_{ij}' - \tau_{ij}' \tilde{S}_{ij}'
\]

where \( P, BP, RT, \) and \( \varepsilon \) are respectively the shear production, buoyant production/destruction,
resolved turbulent transport, and SGS dissipation (which is a TKE cascade to the SGS scales).
Here, we only calculate and show the resolved turbulent transport term and neglect other transport
terms (SGS and pressure transport) since they make smaller contributions than the resolved
turbulent transport term. Furthermore, we focus primarily on the production and dissipation terms,
both from shear and buoyancy. Please see supplementary Appendix S2, or Momen and Bou-Zeid (2017b) for further details about the calculation of these terms and the TKE budget closure in the LES code.

After the jump in the pressure forcing, the shear production increases in Step–40 case as Figure 5(a) shows. This rise in the shear happens in other cases (Figure 5(b)) as well when the magnitude of the velocity increases. The SGS dissipation increases accordingly to adapt to the higher produced TKE in the flow.

Figures 5(c) and 5(d) display the TKE budget terms in the unstable conditions. Lenschow et al. (1980) measured these terms in the convective ABL. Our results before the jump in Figure 5(c) are in a good qualitative agreement with their TKE budget measurements in the unstable ABL. While again the changes in the stable case are more pronounced than in the unstable case, the results in the unstable case do not collapse well at various times particularly near the wall. Although one might expect them to collapse if normalization with $u^*$ is carried out.

The buoyancy term destroys the turbulence in the stable ABL and produces TKE in the unstable ABL as Figure 5 confirms. The magnitude of this term is prescribed at the surface through the boundary conditions and thus the surface value is not affected by the change in the pressure gradient. This is the physically correct boundary condition to set since in both stable and unstable boundary layer the surface energy budget is controlled by the radiative balance of the surface. This net radiative balance $R_n$, along with the small contribution of the ground heat flux $G$, set the available energy $R_n - G$ that is to be distributed between the sensible ($H$) and the latent ($LE$) heat fluxes. Given a value of $R_n - G$, which to first order will remain constant (despite small changes in outgoing longwave radiation due to changes in the surface temperature if the wind speed increases), the turbulent transfer efficiencies for both sensible and latent heat will increase about
equally and thus the distribution of the available energy between $H$ and $LE$ will also, to first order, remain unchanged. Therefore, $H$ would remain approximately constant at the surface in the real world ABL. Nevertheless, at higher elevations in the stable ABL (Figure 5(a) and 5(b)), the heat flux responds to the variations of the pressure gradient and increases to destroy the additional produced turbulence. However, in the unstable ABL (Figure 5(c) and 5(d)), buoyancy production is not remarkably influenced by the increase of the shear since the buoyant production under strong unstable conditions is always dominant over the shear production except very close to the wall. This is in agreement with the observation in the previous subsection that the heat flux is not sensitive to the pressure gradient fluctuations.

As Figure 5 shows, the resolved transport term does not play a significant role in the stable ABL since the buoyant destruction reduces the turbulence at each level, damps the vertical velocity, and does not allow strong vertical transport of the TKE produced near the surface to higher elevations, which is another indication of vertical decoupling in such flows. However, it becomes quite important in the convective ABLs where it adjusts itself to the increase in the shear production while the buoyant production remains almost constant.

Furthermore, based on Figure 5(a), we can see certain heights in the stable boundary layer where the production term becomes negative. This negative shear production is called global backscatter, denoting energy transfer from the turbulence to the mean flow and is also observed when the flow undergoes abrupt transition in space or time (Bou-Zeid et al. 2009). The height of this negative production increases after the jump. If we compare the location of these negative production layers before and after the step change in the pressure gradient, we observe that it occurs close to the LLJ height.
Figure 5. TKE budget profile for 4 cases. In step-change simulations, \( t_1 \) is when the flow is steady right before the jump, and \( t_2 \) represents a time after the jump when the mean flow is unsteady and has not reached the full equilibrium. In the sinusoidal changes, both times indicate unsteady conditions in which \( t_1 \) is close to a trough and \( t_2 \) is close to a peak in the velocity time series. The first node is not shown here because it is highly influenced by the wall model. Note that all the profiles are time-averaged over \( \tau_{ABL}/4 \) around the times shown in the Figure.

The time series analyses of the TKE budgets also show that the variability in the pressure gradients strongly influences the TKE budget terms in the stable ABLs, while its effects are minor in unstable conditions, particularly at higher elevations \((z/z_i > 0.3)\). Figure 6 displays the TKE budget time series for all stable (top panel) and unstable (bottom panel) simulations. It is clear
again that the resolved transport terms are small in stable cases (since the layers are decoupled), but significant in unstable conditions as expected.

Furthermore, the buoyant production variability is not influenced by doubling the pressure gradient in all unstable ABLs regardless of the amount of the imposed heat flux. However, in stable cases buoyant destruction increases up to 5-12 times at their peak after the step-change in the pressure gradient, compared to their steady-state value before the change. The variability of buoyant production in the most stable case is more than the weakest stable simulation.

Figure 6. TKE time series for stable conditions (top panel) and unstable conditions (bottom panel). The stable cases are more sensitive to changing the pressure gradients than the unstable simulations. In unstable conditions, the budget unsteadiness decreased at higher elevations and the level of the variability before and after the change in the pressure gradient was almost the same at $z/z_i > 0.3$. 
4. Reduced model for the emerging dynamics

In this section, we investigate how we can model the mean velocity under variable pressure gradients under different stabilities. To that end, we will extend a simplified model, which was previously developed by Momen and Bou-Zeid (2016) for neutral ABLs, by parametrizing the effect of constant stabilizing/destabilizing heat fluxes on the frictional effects that modulate the mean velocity variations.

4.1. The damped-oscillator model

We start with the Reynolds-averaged Navier-Stokes (RANS) equations:

\[
\frac{\partial \bar{u}}{\partial t} = -f_c (V_g - \bar{v}) - \frac{\partial}{\partial z} \left( \bar{u}' w' \right), \quad \frac{\partial \bar{v}}{\partial t} = f_c (U_g - \bar{u}) - \frac{\partial}{\partial z} \left( \bar{v}' w' \right),
\]

where \( \bar{u} \) and \( \bar{v} \) are the mean horizontal velocity components at height \( z \) and time \( t \). The non-linear advective terms are negligible due to the horizontal homogeneity and small subsidence \( \bar{w} \).

Note that in this section all the velocity variables (except where a perturbation is explicitly denoted by a prime) are the averaged velocities and overbars will be henceforth omitted for notational simplicity.

The turbulent stress gradients can be modelled using the mean velocities as

\[
\frac{\partial}{\partial z} (\bar{u}' w') = \alpha(z,t)u(z,t),
\]

\[
\frac{\partial}{\partial z} (\bar{v}' w') = \alpha(z,t)v(z,t),
\]

where \( 1/\alpha \) is a time scale in our problem. The exact expression of \( \alpha \) depends on the best similarity theory that describes the flow dynamics (e.g. log-law, defect-law, or Rossby-number similarity), and it should be stressed that the model in Eq. (10) is only accurate if the turbulence is in (or at
least close to) quasi-equilibrium with the mean flow (i.e. there exists a similarity theory that relates
the stresses to the mean velocities). This parametrization of the stress terms is a very classic model
of their divergence as being proportional to the velocity at the same height (e.g. Haurwitz 1947;
Schmidt 1947; McNider 1982; Baker et al. 1987; Polavarapu 1995; Pu and Dickinson 2014; Du
and Rotunno 2014; Momen and Bou-zeid 2017a). Note that $\alpha(z,t)$ in its general form is both a
function of height and time. But here we will postulate that its time-averaged value $\overline{\alpha(z)}$ is
sufficient for the current model for cases where the buoyancy forcing is steady in time (an
assumption that will be validated later) and we will hereafter use that averaged value (omitting the
overbar and writing it as $\alpha(z)$).

Let us define the complex variable $A = u + i v$; substituting Eq. (10) into Eq. (9) and
differentiating once with respect to time we get:

$$\frac{d^2 A}{dt^2} + \alpha \frac{dA}{dt} + f_c \left( f_c - i \alpha \right) A = f_c^2 A_x + if_c \frac{dA_x}{dt}. \quad (11)$$

The dynamics of the system are therefore analogous to a damped oscillator (such as a mass-spring-
damper or MSD system) where the inertial, Coriolis, and friction forces are equivalent to the mass,
spring and damper, respectively.

When a steady buoyancy is superposed on the unsteady pressure gradient, the same model
structure can be maintained, but the damping term in the model, corresponding to friction forces,
now needs to account for stability since the change in the stability modifies the similarity theory
(used in Eq. 10) that best describes the equilibrium profiles and needs to be accounted for (we will
validate this assumption in Section 4.3). Therefore, adding a constant stability can only change the
damper or $\alpha$ of our equation of motion (11). However, for the reverse case with constant pressure
gradient under variable surface heat flux and stability, the analytical model needs to be extended
to allow a time-variable damper coefficient, which was done in Momen and Bou-zeid (2017a) who
solved this problem for time-varying coefficients $\alpha$. A linear decrease in time of the heat flux with
a constant pressure gradient was examined in that work, and time-variable $\alpha$ solutions were
validated against the large-eddy simulations and showed better agreement with LES, for the LLJ
attributes, compared to previous models of Blackadar (1957) and Van de Wiel et al. (2010). In this
paper, a time-constant $\alpha$ is sufficient, and this leads to the following homogeneous solution of the
ODE

$$A_h(t) = C_0 e^{-\alpha t} (\cos f_c t - i \sin f_c t),$$

where $C_0$ is determined from initial conditions. The full general solution of (11) also requires
particular solutions associated with the forcing term (see Momen and Bou-Zeid, 2016).

4.2. The time scales and vertical coupling in the diabatic ABL

The derived equations show that the two important time scales in our problem are: 1) the natural
inertial time scale of the ABL $\tau_{ABL} = 2\pi/\nu$, and 2) the turbulence time scale (e.g. the time scale of
the largest turbulent eddies in the ABL which is $\tau_t = z/u^*$. If the turbulence intensity (taken as its
exact definition of TKE normalized by the mean flow velocity) is high in the flow, the
characteristic velocity of the eddies will increase and their turn-over time will decrease; therefore,
the friction and the damper coefficient $\alpha$ will increase. Note that $\alpha$ in our model, based on Eq. (10),
indicates how strong the vertical coupling in the system is. Turbulence intensity and vertical
coupling are related: under unstable conditions the upward transport of TKE from the layers near
the surface reduces the turbulence intensity in these layers and simultaneously increases it further
aloft, in the layers that are a net recipient of TKE transport; on the other hand, the reverse applies
in the stable ABL. This leads to $\alpha_{unstable} > \alpha_{neutral} > \alpha_{stable}$ in the outer layer and
\( \alpha_{\text{unstable}} < \alpha_{\text{neutral}} < \alpha_{\text{stable}} \) in the surface layer. But since the turbulence intensity is generally higher under unstable conditions, the overall solutions are damped more in an unstable regime than a stable regime. Moreover, this damping in the unstable regimes increases as the heat flux increases, and in the stable regimes weakens as the negative heat fluxes decrease (increases in magnitude). Therefore, as illustrated in Figures 1, 2(a), and 3(a), after the jump occurs in the geostrophic forcing, the most stable cases show higher overshoot because of the weaker damping.

This argument becomes clearer when \( \alpha \) is directly calculated from LES for all different stability scenarios. To do so, we define the total Reynolds stresses as:

\[
\begin{align*}
R_{13} & \equiv - \left( \bar{uw} - \bar{uw} \right) - \tau_{xz} \\
R_{23} & \equiv - \left( \bar{vw} - \bar{vw} \right) - \tau_{yz}
\end{align*}
\]

where the first term on the right-hand side of Eq. (13) is the resolved component and the second term denotes the SGS part of the total corresponding Reynolds stress. Figure 7 shows the calculated dimensional \( \alpha \) for different stabilities, after the jump in pressure gradient, based on Eqs (10) and (13). As this figure indicates, the directly computed \( \alpha \) increases monotonically away from the surface when the surface heat flux rises from the most negative to the most positive, while near the wall the opposite trend is observed. The opposing trends at various heights in the variability of \( \alpha \) in the surface and outer layers are directly related to the trends of stress gradient normalized by the mean wind speed (as the direct expression of \( \alpha \) in Eq. 14 will later show). Furthermore, this figure shows that \( \alpha \) becomes less variable in height as the heat flux increases, also implying higher mixing and hence uniformity of profiles under unstable conditions. Note that we will non-dimensionalize \( \alpha \) in the next subsection based on \( G \) and the physical \( z_i \), but here we show the dimensional values for better comparison of all the cases with different \( z_i \).
4.3. A similarity model for the turbulent friction

In this subsection, we parametrize the $\alpha$ coefficient of our damped-oscillator model based on characteristic length and velocity scales that vary with the stability of the ABL. This will allow us to apply our model in any setup without requiring LES.

For the neutral ABL, we previously proposed a formulation for the eddy-viscosity and showed its relation with $\alpha$ in Momen and Bou-Zeid (2016). This analytical expression for eddy-viscosity consisted of a characteristic length scale, and a characteristic velocity scale (the geostrophic velocity). The model was in a good agreement with LES, with some deviation near the wall since the assumed Boussinesq analogy (implying $\alpha_x = \alpha_y$) is not an accurate approximation in that region (it is simple to relax this assumption; but for the purpose of illustrating the model skill it is not critical and thus we will invoke this analogy in the present paper as well); in addition, the geostrophic velocity was used in that paper, which is not an appropriate velocity scale near the wall.

Figure 7. Dimensional $\alpha$ directly calculated by LES for all step-change simulations.
Now, we intend to develop a model for $\alpha$ that accounts for the stability effects. We model this coefficient in two regions using different scales: first in the surface layer ($z/z_i < 0.25$) where the effect of the wall and friction velocity is more important and second in the outer layer ($z/z_i > 0.25$) where the free stream conditions are more significant. Considering the definition of $\alpha$, we can approximate it as:

$$
\alpha = \frac{1}{M} \frac{\partial}{\partial z} \left( u'w' + v'w' \right) - \frac{u_z^2}{M l_m}.
$$

(14)

where $l_m$ denotes a characteristic mixing length.

For the surface layer, $l_m$ (similar to Delage, 1974; Lacser and Arya, 1986; Estournel and Guedalia, 1987; Stull, 1988) can be written as:

$$
\frac{1}{l_{m,s}} = \frac{1}{\kappa z} + \frac{\beta}{\kappa L},
$$

(15)

where $l_{m,s}$ represents the mixing length in the surface layer, $\kappa$ is the Von-Karman constant (here 0.4), $\beta$ is an empirical constant, and $L$ is the Obukhov length:

$$
L = \frac{-u_z \bar{\theta}'}{\kappa g w' \bar{\theta}'}.
$$

(16)

Note that one can also add a term to Eq. (15) to account for the free stream conditions like geostrophic velocity and Coriolis effect as in other works; however, we do not add that term here since this part of the model is strictly for the surface layer where the effect of the free stream conditions and Coriolis forcing are weak. In order to apply the model of Eq. (14), we also need to use an analytical expression for the profile of $M$. We use the Monin-Obukhov similarity theory (Monin and Obukhov 1954), which is valid in the surface layer, to find the necessary relationship.
for the wind profile with stability effects. Substituting the Monin-Obukhov profile and the mixing
length expression into Eq. (14) yields:

$$\alpha = \frac{u_*}{\log(z/z_0) + \psi(z/L)} \left( \frac{1}{z} + \frac{\beta}{L} \right),$$  \hspace{1cm} (17)

where $$\psi(z/L)$$ is the conventional stability correction function based on the Obukhov length. Here
we use the Businger-Dyer relationship for this function as:

$$\psi\left(\frac{z}{L}\right) = \begin{cases} 
-2 \ln \left(\frac{1 + x}{2}\right) - \ln \left(1 + \frac{x^2}{2}\right) + 2 \tan^{-1}(x) - \pi/2 & \text{if } z/L < 0 \\
\frac{4.7 z/L}{z/L > 0} & \text{if } z/L > 0
\end{cases},$$  \hspace{1cm} (18)

where $$x=(1-15z/L)^{1/4}$$. The $$\psi(z/L)$$ expression for the unstable case in Eq. (18) is found by Paulson

In the outer layer, our LES results indicate that the turbulent stresses change almost linearly
(supplementary Appendix S3). Therefore, the local $$u^2$$ decreases linearly with height. In addition,
the variation of $$M$$ is usually less than the turbulent stresses in the outer layer; hence, a first-order
approximation that assumes $$M$$ is a constant would be sufficient; it yields a linearly decreasing $$\alpha$$
with $$z$$. To ensure that the surface and outer layer parametrizations match, we compute $$\alpha(z/z_i=0.25)$$
from the surface layer and then decrease it linearly in height as:

$$\alpha(z/z_i) = \alpha(0.25) \left[ 1 + \frac{(0.25 - z/z_i)}{l_{m,o}} \right],$$  \hspace{1cm} (19)

where now the mixing length in the outer layer ($$l_{m,o}$$) depends on the stability and an outer
turbulence mixing length $$l_\infty$$ that applies far from the surface following

$$\frac{1}{l_{m,o}} = \frac{1}{l_\infty} + \frac{\beta}{\kappa L}.$$  \hspace{1cm} (20)
We use $l_\infty = 7m$ based on the results of Huang et al. (2013). $\beta$ is an empirical constant that can be found from experiments or numerical simulations. Our LES results indicate that the values of $\beta = 47$ in the inner layer, and $\beta = 24$ for the outer layer yield good results for all the simulations conducted in the paper (the validations and model comparisons with LES will be shown in Section 4.4).

4.4. Validating the turbulent friction parametrization and the damped-oscillator model

We compare the parametrization of $\alpha$ in Eqs. (17) and (19) directly to the one computed from the LES via Eq. (14) to test the model a priori. In addition, we validate the predictions of our proposed analytical model in Eq. (12) a posteriori in two modes: 1) by optimizing $\alpha$ based on the LES results (in a least square sense) which gives us the best match between the velocity of our analytical model and the LES, and 2) by using the analytical parametrization proposed in Section 4.3, which would be useable without any LES results. We call the first solutions the Optimized MSD system (OMSD), and the second approach the Parametrized MSD (PMSD). Note that the OMSD is only used to check whether the current structure of our MSD model is able to capture the LES results or not, without embedding the influence of the possible errors due to the parametrization of $\alpha$. On the other hand, in the second approach, we use our analytical model independent of LES data by using the introduced parametrization for $\alpha$. These PMSD solutions include the error of our MSD model as well as the errors associated with the modelling of $a_{PMSD}$, hence indicating the benefit of doing a direct validation of the $\alpha$ expression against the one obtained directly from LES $a_{LES}$. We also compare these values to the ones that result from the optimization, $a_{OMSD}$.

We validated our $\alpha$ parametrization as well as our reduced model for six cases: Step–40, Sin–20, Step–10, Step+100, Step+400, Sin+400, but since the skill of the model was similar in all cases here we only show Step–40 and Step+100 cases. Figure 8(a) displays $a_{OMSD}$, $a_{PMSD}$ and a
range of $\alpha_{LES}$ for the whole 20 hours steady-state simulation and Figure 8(b) exhibits a similar plot after the jump in pressure gradient for Step–40 simulation. The range of the LES is found by obtaining the maximum and minimum values of $\alpha$ at each height throughout the indicated period. Figure 8(d) and 8(e) show these calculated $\alpha$ for the Step+100 case. The validations for the Step–10 and Step+25 simulations are depicted in the supplementary Appendix S4. The obtained $\alpha_{PMSD}$ and $\alpha_{OMSD}$ are mostly in the LES range, meaning that our developed parametrization $\alpha_{PMSD}$ and the optimized $\alpha_{PMSD}$, which we determine by matching the velocity variations at each height from our reduced model against LES, agree well with the direct LES calculations from the stress terms. The $\alpha_{OMSD}$ in some cases (e.g. in Figure 8(d)) falls slightly outside of the range near the wall, probably due to the fact the optimization is correcting the structural model errors related to the Boussinesq analogy in that region. This figure also indicates a good agreement between our developed $\alpha_{OMSD}$ and $\alpha_{PMSD}$, implying that our analytical parametrization matches the best $\alpha$ of our reduced model, which yields lowest prediction errors.

We now use these $\alpha_{OMSD}$ and $\alpha_{PMSD}$ to predict the wind variations using our damped-oscillator model. We add the particular solution of the step-change forcing (see Momen and Bou-Zeid, 2016) to the homogenous solution of Eq. (12) to generate our reduced model predictions as shown in Figure 8(c) and 8(f). The results exhibit three heights, both in the surface and the outer layers. Figure 8(c) and 8(f) display the validation of the damped-oscillator model against the LES in both the optimized and parametrized modes for stable and unstable conditions, respectively. Both $\alpha$ models agree well with LES, though $\alpha_{PMSD}$ is slightly less accurate than the $\alpha_{PMSD}$ in these a-posteriori tests.
Figure 8. \( \alpha \) from LES, PMSD (a-priori analysis), OMSD (a-posteriori analysis) for 2 cases in the stable (a,b) and unstable ABL (d,e). The validation of the reduced model for the stable (c) and unstable (f) ABL.
Figure 9. $M$ profiles and error analysis for OMSD and PMSD models in which (a) and (c) show the predictions of our reduced model in two modes against LES results after the jump for the stable and unstable ABL respectively, and (b) and (d) exhibit the relative root mean square error of our model predicting the velocity variations at $t = 20-30$ hours for stable and unstable conditions respectively.

We now examine the performance of the model in predicting the velocity profiles after the sudden jump in the pressure gradient in one stable and one unstable case. Figure 9(a) and 9(c) show that our models capture the LES velocity profiles quite well despite the fact that the performance of the parametrized model is slightly inferior to the OMSD. This is more obvious in the error analysis that is done in the right panel of Figure 10. The relative errors of our reduced
models for the period of $t = 20-30$ hr, starting right after the jump, are less than 15% for all heights, which confirms the very satisfactory skill of our damped-oscillator models in predicting the mean ABL response to changes in the pressure forcing. The performance of the PMSD model is less or equal to the performance of the OMSD at different heights and it could be improved with better parametrizations of $\alpha$, which is related to the turbulence closure problem and is a very wide research topic we cannot delve deeply into in this paper. We also varied $\alpha_{OMSD}$ by $\pm 25\%$ to investigate the sensitivity of our solutions to this coefficient; the results are shown in supplementary Appendix S5. Generally, we found that underestimating $\alpha$ causes higher errors than overestimation because of higher overshoots of velocity that occur with a lower $\alpha$.

To sum up, our reduced damped-oscillator model can also be used in diabatic ABLs to predict the wind-velocity variations. The stability only influences the damper of our model and hence the suggested analytical parametrization agrees well with the LES results.

5. Conclusions

We studied the effects of an unsteady pressure gradient on diabatic ABLs using a suite of large-eddy simulations, and developed an analytical reduced model for the resulting wind-velocity variations. Fourteen simulations were conducted to study the response of unsteady ABLs under various stability conditions. We considered two scenarios: a step change and a sinusoidal variation in the pressure gradient. Then, for each scenario, seven cases were run with different constant surface heat fluxes, ranging from strongly unstable, to neutral, to strongly stable conditions. Our major findings include:

- The maximum wind speeds in the ABL after increasing the pressure gradient are related to the rate of pre-existing vertical coupling and TKE in the system, and thus to stability. The maximum overshoots decrease when the surface heat flux increases, i.e. under increasingly
unstable conditions. The height of the low-level jets in the wind profiles is found to be constant when non-dimensionalized by the height of the ABL, \( z_i \) (defined as the height at which the TKE reaches \(~5\%\) of its surface value in stable ABLs).

- The temperature profiles in the stable ABL vary significantly when the pressure gradient changes. In particular, right after the sudden jump in the pressure gradient, the surface warms because mixing in the stable ABL increases and hotter air parcels are transported from higher levels to lower elevations. After about one inertial time scale, the stable ABL starts cooling again and the vertical mixing reduces. On the other hand, in the unstable ABL the temperature keeps increasing with almost the same rate after doubling the pressure gradient since the mixing was already high in the convective ABLs. There are little differences in the unstable ABL under a step change versus a sinusoidal variation in the pressure gradients, underlining the dominant role of turbulence in modulating the convective boundary layer.

- In the stable ABL, the buoyancy destruction responds to the variations of the pressure gradient and increases to destroy the additional produced turbulence. However, in the unstable ABL, the buoyancy term is not significantly influenced by the increase of the shear since the buoyant production is higher than the shear production at all elevations, except perhaps near the wall, and it dominates the dynamics at all times under unsteady conditions.

- Using LES, we showed that, even in unsteady pressure-gradient ABLs, the resolved transport term does not play a vital role in the stably stratified flows since the buoyant destruction reduces the turbulence at each level, damps vertical velocity, and does not allow for strong transport of the excess TKE produced near the surface to higher elevations,
which is another facet of vertical decoupling in such flows. However, the transport becomes more important in convective ABLs where it adapts to the variability of the shear production (due to changes in the pressure gradient) while the buoyant production is almost constant.

- The production term becomes close to zero or negative in some regions of the stratified ABLs, near the low-level jet location, indicating the existence of the uncommonly observed global backscatter.

- The findings confirm that turbulence is a dominant mechanism in the unstable ABL that smoothes out other disturbances, including in this case with the effects of pressure-gradient variations. On the other hand, in the stable ABL, the turbulence is weaker and other mechanisms become more consequential for the dynamics (as e.g. suggested by Bou-Zeid et al., 2010).

- The extension of the damped-oscillator model introduced in Momen and Bou-Zeid (2016) for neutral conditions to diabatic ABLs successfully captures the present LES results. In that model, inertial, Coriolis, and friction forces represented the mass, spring and damper effects respectively. The extension to diabatic conditions only alters the damper coefficient, which characterizes how strong the vertical coupling is in the system.

- When the friction force (i.e. the gradient of the momentum flux) increases, the damper coefficient $\alpha$ will increase. Hence, it is expected that the damper coefficient results in $\alpha_{\text{unstable}} > \alpha_{\text{neutral}} > \alpha_{\text{stable}}$ away from the wall where the momentum flux divergence is higher in the unstable ABL, while in the surface layer, the reverse trend is observed. This is supported by the direct calculations from the LES.
The frictional effects were parametrized using the MOST theory in the surface and a linearly decreasing function via a mixing-length model in the outer layer. The damped-oscillator model was run using this parametrization independent of LES, and good agreements with less than 15% relative root mean square error were found compared to the LES results.

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Appendix A: Boundary and initial conditions and numerical grid configuration of the LES code

The boundary conditions for solving Eq. (4) are an imposed kinematic surface buoyancy flux $\bar{w}' \bar{\theta}'$ at the surface and a no flux boundary ($d\theta/dz=0$) at the top of the domain. Periodic lateral boundaries in the horizontal directions are used for all variables. For the momentum equation, a wall shear stress boundary condition is imposed at the surface using a local log-law formulation (Bou-Zeid et al., 2005), while the top boundary condition for momentum is zero stress. Both top and bottom boundaries are impermeable and thus the vertical velocity, $w$, is set to zero there.

For the stable simulations, a sponge layer with a Rayleigh damping method (Israeli and Orszag 1981) is used as the upper boundary condition to damp all the vertical perturbations and
avoid reflections of waves from the top of the domain. In these cases, the domain height is $L_z = 1600$ m with an imposed inversion base height of $z_i = 1200$ m, which is also the height where the sponge layer effect starts. This will fix the maximum attainable height of the stable ABL. While this might appear too conservative for stable conditions, the need for such a high domain is illustrated later in Section 3 (due to gradually varying and/or growing boundary layers in some cases). In the neutral conditions, since we do not have a real inversion we set the inversion height as $z_i = L_z = 1000$ m in the code. Furthermore, the code does not solve the scalar Eq. (5) for neutral simulations and neglects the buoyancy terms in the momentum equation. For the unstable simulations, we set $L_z = 1700$ m and the inversion height to $z_i = 1250$ m. The differences in $z_i$ will not be important since all the results and equations are normalized with that height.

The simulations are integrated for 40 hours of physical time, which are equivalent to about 30 large eddy turnover times ($ABL$ height / friction velocity), or more. For these long simulations, if we do not control the inversion layer under unstable conditions it will be warmed up by the surface heat flux and will be dissipated after some time. To avoid continuous growth of the ABL height in the convective ABL and maintain a steady $z_i$, we keep the $\theta$ profile in the capping inversion layer steady by compensation for any heating and cooling by entrainment. Although this setup might not reflect the real top boundary conditions in the ABL, similar approaches have been previously adopted to ensure the statistical convergence of the result (Sescu and Meneveau 2014; Shah and Bou-Zeid 2014) and key profiles have been shown to remain very similar to the real-world ABL. In this study, we conducted further tests that confirm that the top boundary conditions have a minor influence on the flow dynamics inside the ABL with unsteady pressure gradient and steady surface heat flux.
In all the simulations, we set $L_x = L_y = 2\pi L_z$, the number of the numerical grid points in the stable ABL is $N_x = N_y = N_z = 160$ (which is sufficient even for strong stabilities as illustrated in Huang and Bou-Zeid 2013), and the number of the points in the unstable and neutral ABLs are $N_x = N_y = N_z = 128$.

For each surface heat flux, a 50-hour, $64 \times 64 \times 64$ nodes, steady simulation is performed to start warm-up. Then, the outputs are linearly interpolated into a higher resolution domain ($128^3$ or $160^3$) and used as initial conditions for the corresponding cases. More details on the LES model, the used SGS model, and the unsteady simulation setup could be found respectively in Albertson and Parlange (1999), Bou-Zeid et al. (2005), and Momen and Bou-zeid (2017b).

**Supporting information**

The manuscript includes two supporting videos and four supplementary appendices.

**References:**


Tennekes H, Lumley J (1972) A First Course in Turbulence. MIT, Boston

