Investment, Bargaining, and Efficiency*

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Abstract
In some contexts of bilateral trade it is natural to think of the valuations as emerging from investment decisions which are not fully observed. We augment the bilateral trade problem of Myerson and Satterthwaite (1983, J. Econ. Theo. 29(2):265-281) to allow the agents to make hidden investments in the value of the good. At the bargaining stage, strategic uncertainty about investments will look like asymmetric information. Contrary to the inefficiency that emerges when the asymmetric information is exogenous, in our setting the results lean strongly to efficiency although they are not fully Coasian. First, we find that there are never equilibria having the inefficiency that emerges in Myerson and Satterthwaite; equilibrium drives players to invest in ways that escape that result. Second there is always an assignment of transferable property rights that fully implement the first-best outcome. However, unless one player has a dramatically less efficient investment technology, an initial allocation of property rights to the wrong player results in equilibrium multiplicity; there are equilibria in which the less efficient player invests optimally and obtains the good. Certain types of options can solve this problem.

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1 Introduction

Consider the canonical problem of bilateral trade with asymmetric information with one buyer, one seller, and a joint distribution over the valuations each player assigns to one indivisible item. Many exchange games have been considered in this context. Chatterjee and Samuelson (1983), for example, characterize an equilibrium to the double auction in this setting showing that buyers will have an incentive to understate their value of the item and that sellers will overstate it. In this equilibrium, not all efficient trades are reached. More generally, Myerson and Satterthwaite (1983) show that when an overlap condition is satisfied, no efficient trading rules satisfy an interim participation constraint and incentive compatibility. Participation and incentive constraints require that some efficient trades are missed.

In many settings, it is natural to think that the valuations for the item will be influenced by prior investment decisions. This creates two new possible forms of inefficiency in addition to the one identified by Myerson and Satterthwaite: first, equilibrium pressures may drive players to under or over invest, and if asymmetric information is present at the bargaining stage then the inefficiencies found in Myerson and Satterthwaite might also cause allocational losses. Second, if the traders also differ in how effectively they can convert investments into valuations, then the player with the less efficient investment technology may wind up getting the good. This would constitute an inefficient outcome even if her investment is optimal, given the conjectures that (i) she will obtain the good with probability 1 and (ii) the price paid will not depend on this investment.

To see when these three potential inefficiencies exist, we augment the canonical description of bilateral trade with asymmetric information to allow for the asym-
metric information to emerge endogenously. Previous work, including Myerson and Satterthwaite, tends to consider the informational environment as exogenous. Our approach captures the incentives for investment prior to bargaining, as in the literature on the hold-up problem or moral hazard in principal agent models. When players cannot observe each other’s investment decisions, asymmetric information at the bargaining stage can result from strategic uncertainty if players use mixed strategies in the investment stage. By characterizing equilibrium investment strategies, we can derive conditions under which the types of information environments that result in inefficiencies at the bargaining stage are to be expected.

Unexpectedly, we find that the possibility of several forms of inefficiency is actually efficiency-enhancing. If players anticipate the use of a trading rule which is second-best given the equilibrium beliefs about valuations and participation constraints, then the strategic uncertainty that emerges will not lead to allocation inefficiencies. The intuition behind this conclusion can be seen in the optimal Myerson Satterthwaite mechanism for the case in which both the buyer’s and seller’s valuations are independent draws from the same uniform distribution. In the second-best rule, certain types of buyers do not trade with any seller. If valuations are themselves the result of strategic decisions, we also would not expect buyers to be willing to expend resources to obtain these particular valuations. This then suggests that the emergence of uniformly distributed valuations is not possible given an expectation that this is what is occurring in equilibrium. This sort of unraveling is quite pervasive. Although it is possible to support lotteries over valuations which have overlapping supports, these distributions will not actually satisfy the conditions in Myerson and Satterthwaite, and efficient allocations will be possible in the bargaining problem.
The starker conclusion is that knowledge that the bargaining mechanism is chosen optimally, given the relevant constraints and equilibrium beliefs about the investment strategies, implies that the form of allocation inefficiency that emerges in Myerson and Satterthwaite is not consistent with equilibrium play. Moreover, the investment decisions will typically be in pure strategies and will be optimal, given the identity of the player getting the item on the equilibrium path. These two features of the equilibrium support a Coasian view where trade generally is efficient.

The third form of inefficiency, however, is harder to escape. Here the findings are only partially Coasian. For any assignment of transferable property rights, there are equilibria in which the first-best occurs, i.e., the player with the more efficient investment technology (producing higher valuation given optimal investments) ends up with the item. But, if the good is initially allocated to the wrong player and unless one player has a substantially better investment technology, there are also equilibria in which trade does not occur. The ability of the seller to buy an option to sell at a high price (but one the buyer would be willing to play following some investment levels) can resolve this problem and implement the first-best.

2 Related Literature

The idea that incomplete information between potential trading partners can emerge from hidden investments is not new. This perspective emerges in the literature on the hold-up problem. In the bilateral bargaining context, Gul (2001) shows that if the buyer’s investment is a hidden action instead of observed then even when the seller has all the bargaining power the hold-up problem can be resolved if repeated offers
are allowed. Gul also considers the case of two-sided investments, but he assumes that the seller’s investment is observed prior to bargaining.\footnote{Incidentally, Gul finds that the seller will have an incentive to underinvest, and points out the challenges to applying his arguments to the case of a continuum of types.} Gul’s model is close to ours in an important dimension; both only allow asymmetric information to emerge from strategic uncertainty caused by mixed strategies and hidden actions. However, it is quite different in an important respect. In Gul, bargaining only involves one-sided asymmetric information, and thus the form of inefficiency that comes out of Myerson-Satterthwaite cannot emerge in his model. Our primary interest is understanding how this potential form of inefficiency interacts with the incentives for pre-bargaining investments.

In thinking about the literature on pre-bargaining investment, four central distinctions appear. In some work, all relevant fundamentals are assumed to be observable at the time of bargaining (Grossman and Hart, 1986; Hart and Moore, 1988). Gul’s two-sided extension allows one player’s investment to be unobserved. More recently, González (2004) and Lau (2008) also consider investments as hidden actions, but these papers consider the case where only one player can invest. Lau (2011) considers the case of one-sided hidden investments and exogenous asymmetric information, and captures some of the relevant tradeoffs but in her paper the asymmetric information is not directly attributed to a choice by the players. Perhaps closest is Rogerson (1992), who provides a quite general treatment of the case where multiple players can invest and there are no externalities. His case of completely private information is closest in spirit to the environments we consider. The key distinction is that Rogerson assumes that there is a random component connecting each player’s investment to its type. In particular, by assuming that investment decisions always admit unique
optima, he excludes the case where investments completely determine a player’s type (as in Gul (2001) and our paper). Rogerson also does not impose the individual rationality constraint imposed by Myerson and Satterthwaite and thus, in principle is free to work with a larger set of mechanisms (he does require budget balance and incentive compatibility). Finally, Rogerson assumes that the mechanism is committed to prior to investment decisions, and shows that d’Aspremont and Gérard-Varet (1979) and Cremer and Riordan (1985) mechanisms also create incentives for optimal investment. We are interested in the same participation constraints as Myerson and Satterthwaite, and allow for mechanisms that are optimal given equilibrium beliefs about investments. Thus, we do not necessarily endow the designer with the ability to solve a time-consistency problem.

In terms of the taxonomy described above, we know of no papers that consider unobserved investment by the buyer and seller under the Myerson-Satterthwaite contractual constraints. To connect most clearly with extant work, we maintain the assumption that investments only affect the payoff of the investor under the condition that he ends up with the good. In order to facilitate an endogenous account of asymmetric information, we proceed as in Gul instead of Rogerson—assuming that a player’s investment determines its valuation with no additional randomness.

As Gul (2001) notes, a common feature of his result and the literature on moral hazard and renegotiation (Che and Chung, 1999; Che and Hausch, 1999; Fudenberg and Tirole, 1990; Hermalin and Katz, 1991; Ma, 1991, 1994; Matthews, 1995) is that pure strategies by the agent generate strong reactions from the principal and thus, in equilibrium, the agents’ randomization generates asymmetric information. Our

\footnote{See also Hart and Moore (1988) for a similar observation in case of two-players and an indivisible item—as in our model}
analysis offers a counter-point to this result. The presence of randomization by both traders is typically hard to support, and with one type of exception, impossible to support if the traders anticipate that a designer is using a second-best trading rule (as in Myerson and Satterthwaite), given rational expectations about the randomization over investment employed by the traders. The exceptional case involves an investment technology (the cost functions) that supports an equilibrium in which the buyer and seller mix over a small interval and disconnected atoms, with most of the probability being allocated to the atoms. In cases like this, the inefficiency result in Myerson and Satterthwaite breaks down even though there is overlap in the supports and, in fact, first-best trading rules exist. Therefore, this form of strategic uncertainty is also not consistent with the inefficiency that emerges in Myerson and Satterthwaite, as first-best becomes possible with this uncertainty.3

3 Model

Consider a trading problem with a seller who owns an indivisible object and a buyer who may wish to acquire it. The seller, player 1, and the buyer, player 2, value the object at \( v_1 \geq 0 \) and \( v_2 \geq 0 \), respectively. In the standard trading model, these two valuations are treated as independent random variables with cumulative distribution functions \( F_1(\cdot) \) and \( F_2(\cdot) \).

Our point of departure is to treat these valuations as endogenous; they emerge from investment decisions. For example, suppose that the object in question is exclusive ownership of a computing technology such as a search algorithm or mapping

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3This analysis requires extending the arguments in Myerson and Satterthwaite to the case of distributions with wholes and atoms. This extension (Theorem 3) involves standard arguments but might be of some independent interest.
software, and the potential owners are two competing search engines, such as Yahoo and Google. Each potential owner could make investments in the the ability to interface the new technology with its existing products. Each could also invest time or money in finding alternatives to the technology in question. These investments then influence the value of the trade to each player. To frame this setting as one with a buyer and seller, we could imagine that Google owns the technology and is considering the sale of it to Yahoo. If there is no trade, the seller can capitalize on his investment, but investment returns are lost to him if the object is sold. The opposite is true for the buyer; her investment generates value only when she also purchases the good. In particular, we assume that these investments do not generate value for the other player.

At the beginning of the game each player also faces a decision about how to invest in valuation of the item. Agent $i$ faces a cost function that determines the cost to obtaining a conditional valuation of $v_i$, $c_i(v_i)$, where $c_i(v_i)$ is strictly increasing, strictly convex and differentiable. We assume that $c_i(0) = 0$ for both buyer and seller. An investment strategy, possibly mixed, for player $i$ is a cumulative distribution function $F_i(\cdot)$ over valuations. Importantly, we treat the investment choices as hidden actions unobservable to the other player. We sometimes refer to these cost functions as the exogenous investment technologies.

After the investment stage, the players negotiate over the possible transfer of the good from the seller to the buyer, and what price the buyer will pay to receive the good. The result of such a game then is a pair, $(p, x)$, where $p$ is the probability of trade from the seller to the buyer, and $x$ is the expected transfer from the buyer to the seller. As there are many games that might be played after investment, we appeal
to the following revelation principle in our analysis.

**Revelation Principle** Fix the cost functions. If there exists a game with equilibrium investing decisions given by the mixtures $F_1$ and $F_2$ and the joint distribution $G(p, x)$ over outcomes, then there exists a direct mechanism satisfying the condition that, in the modified game in which the players invest and then play the direct mechanism there is a Bayesian Nash equilibrium in which the investment strategies are given by $F_1$ and $F_2$ and the message strategies are truthful $m_i(v_i) = v_i$, which induces the same distribution over the outcomes.

This revelation principle simply notes that for fixed investment strategies, the standard composition argument as found in Myerson (1979) applies. Because investment decisions are privately observed and reports are unverifiable, this first stage introduces no additional complications. For the moment, we take as given a pair of lotteries over investments and pin down the details of the direct mechanism. The approach then is to look at equilibrium incentives in direct mechanisms, and then focus on Bayesian Nash equilibria to the induced game of investing and playing the direct mechanism.

Formally, let the message space be $\mathbb{R}_+$ and define a mechanism by two mappings. The first is $p(m_1, m_2) : \mathbb{R}^2_+ \to [0, 1]$ that determines the probability of trade from the seller to the buyer, and the second is $x(m_1, m_2) : \mathbb{R}^2_+ \to \mathbb{R}$ that describes the transfer from the buyer to seller. The total payoffs for a profile of messages and valuations are given by:

$$W_1(v_1, m_1, v_2, m_2) = v_1(1 - p(m_1, m_2)) + x(m_1, m_2) - c_1(v_1)$$
$$W_2(v_2, m_2, v_1, m_1) = v_2p(m_1, m_2) - x(m_1, m_2) - c_2(v_2).$$
Given the direct mechanism, a strategy profile is then a pair of lotteries over investments and reports that may depend on the realization of investment actions. A clear consequence of the revelation principle is that we can focus just on strategies in which the reports are truthful. Thus, a strategy for player $i$ is a lottery $F_i(\cdot)$ with support $V_i \subset \mathbb{R}^+$. In this environment we define an equilibrium to our games as follows:

**Definition 1.** An equilibrium is a pair of investment lotteries, $(F_1, F_2)$ and a direct mechanism, $(p, x)$ s.t. given the lotteries $(F_1, F_2)$, the mechanism $(p, x)$ is Bayesian incentive compatible and, given the payoffs associated with the mechanism, the investment strategies $(F_1, F_2)$ form a Nash equilibrium.

Two other restrictions on the set of possible bargaining games and strategies will be considered. First, we will require that participation in the game is voluntary. In the bilateral trade setting with incomplete information, Myerson and Satterthwaite (1983) restrict attention to games that satisfy an *interim* participation constraint, where each type's expected net payoff from participating in the game is non-negative. In what follows, we will require that the equilibrium to a trading game also satisfies this condition after investments are realized.

Due to the ex ante investment choice, we will also require a second form of a participation constraint. For an investment level to occur in equilibrium, the player must also have a non-negative expected payoff.

**Definition 2.** We say that an equilibrium satisfies the *interim participation constraint* (Condition IP) if each player’s expected gains from trade is non-negative for almost every valuation in the support of its mixed strategy, $v_i \in V_i$. 
This condition implies a weaker ex ante condition, in that each player’s expected utility from the equilibrium is non-negative (when we do not condition on the investment).

Second, we are interested in the relevance of time-consistency and pre-commitment to a mechanism or trading scheme that is suboptimal given rational expectations about investing behavior.

**Definition 3.** We say that an equilibrium is interim optimal (Condition O) if, given the investment lotteries \((F_1, F_2)\), the mechanism \((p, x)\) maximizes the sum of players’ payoffs within the class of mechanisms that are incentive compatible, and satisfy the interim participation constraint given the lotteries, \(F_1(\cdot), F_2(\cdot)\).

### 4 Strategic Uncertainty and the Standard Trading Mechanism

Let \(F_i\) be player \(i\)’s mixed-strategy equilibrium distribution over the hidden action. Recall that our direct mechanism is a pair of functions \(x(m_1, m_2)\) that describes the report-contingent transfer to the seller and a function \(p(m_1, m_2)\) that determines the probability of trade. Expected gains from trade (net of investment costs) to the seller of reporting \(m_1\) in this direct mechanism, given investment \(v_1\), can then be written as

\[
U_1(v_1' | v_1) = \int_{v_2} [x(v_1', v_2) - p(v_1', v_2)v_1]dF_2(v_2) .
\] (1)
Similarly, for the buyer we have:

\[ U_2(v'_2|v_2) = \int_{V_1} [p(v_1, v'_2)v_2 - x(v_1, v'_2)]dF_1(v_1) : . \]  

(2)

In a slight abuse of notation, let \( U_i(v_i) = U_i(v_i|v_i) \).

We note a convenient feature of the supports of investment strategies. Since \( c_i(0) = 0 \), if

\[ c_i(\hat{v}_i) > \hat{v}_i, \]

then the investment \( \hat{v}_i \) is strictly dominated by \( v_i = 0 \). Let \( \overline{v}_i \) be the investment that makes \( c_i(v_i) = v_i \). In an equilibrium, investments must have support contained in the interval \([0, \overline{v}_i]\). We can then conclude that equilibrium investment strategies always have compact support.

We now turn to the study of what types of investment strategies are possible in an equilibrium. We find that the equilibrium conditions from strategic investment pin down a number of characteristics of the bargaining problem.

**Theorem 1. (Mixing theorem)** *In any equilibrium, if \( v_i \) is an accumulation point of the support of \( i \)'s mixed strategy, then*

\[ 1 + U'_1(v_1) = c'_1(v_1), \]  

(3)

\[ U'_2(v_2) = c'_2(v_2). \]  

(4)

**Proof.** Suppose there is an equilibrium to a bilateral trading mechanism with a corresponding direct mechanism \((x, p)\). To begin, consider the case of the seller. Take
any two investments $v_1, v'_1$ in the support of $F_1$. Then, because the seller is mixing
over these values

$$\int_{v_2} (1 - p(v_1, v_2))v_1 + x(v_1, v_2)dF_2(v_2) - c_1(v_1) = \int_{v_2} (1 - p(v'_1, v_2))v'_1 + x(v'_1, v_2)dF_2(v_2) - c_1(v_1).$$

The left-hand side equals

$$U_1(v_1) - c_1(v_1) + v_1,$$

and the right-hand side equals

$$U_1(v'_1) - c_1(v'_1) + v'_1,$$

and we can rewrite the equation above as

$$U_1(v_1) - c_1(v_1) + v_1 = U_1(v'_1) - c_1(v'_1) + v'_1$$

$$1 + \frac{U_1(v_1) - U_1(v'_1)}{v_1 - v'_1} = \frac{c_1(v_1) - c_1(v'_1)}{v_1 - v'_1}.$$  \hspace{1cm} (5) \hspace{1cm} (6)

At an accumulation point of the support of $F_1$, we can take the limits as $v'_i \to v_i$

$$1 + U'_1(v_1) = c'_1(v_1).$$ \hspace{1cm} (7)

This is the first equation in the theorem. Similar calculations give the identity for
the seller. \hfill \blacksquare

Thus, with investment in mixed strategies, it must be the case that for every point
in the support of the investment actions, either the derivative of the cost function and
the utility are equal (if player 2), or differ by exactly 1 (if player 1). The derivative of the utility for the trading game is pinned down by incentive compatibility so there must be a close connection between investment strategies, their implied trading probabilities, and the marginal cost of investment for the traders. Below, we show that this connection precludes equilibria with investment decisions that lead to Myerson-Satterthwaite inefficiencies.

4.1 Mechanisms with connected investment strategies

We start by considering the classical bilateral trading case investigated by Myerson and Satterthwaite, where both the buyer’s and seller’s valuations are distributed continuously over a connected domain. Myerson and Satterthwaite’s classical result is that no efficient mechanism exists that is both incentive compatible and individually rational, as long as the distributions of the players overlap. The theorem below, on the other hand, shows that such distributions cannot emerge from a mixed-strategy investment equilibrium, if the mechanism designer is choosing second-best mechanisms that maximize aggregate gains from trade.

**Theorem 2.** (No Connected Supports with IC, O, IP) Assume the cost function is strictly increasing. When the designer chooses an optimal IC and IP mechanism that maximizes aggregate gains from trade given the investment strategies (condition O) there is no mixed-strategy equilibrium with connected and overlapping supports and no atoms.

**Proof.** Suppose the seller and the buyer are following mixed strategies with positive probability densities over \([a_1, b_1]\) and \([a_2, b_2]\), respectively, and that the interiors of
the supports have a non-empty intersection. Myerson and Satterthwaite show that no efficient mechanism is possible under these assumptions, so the aggregate gains from trade will be maximized by a second-best mechanism characterized by Theorem 2 of Myerson and Satterthwaite, which states that the optimal second-best mechanism will ensure \( U_1(b_1) = U_2(a_2) = 0 \). It immediately follows that any \( a_2 > 0 \) is strictly dominated by investing 0 and not paying a cost; hence \( a_2 = 0 \). Now, from Theorem 1 and the envelope theorem of Myerson and Satterthwaite, we have for any incentive compatible mechanism:

\[
\phi'_2(v_2) = \bar{p}_2(v_2) 
\]  

(8)

Again, by Theorem 2 of Myerson and Satterthwaite, we know that the optimal second-best mechanism prescribes trade when

\[
v_2 - v_1 \geq \alpha \left( \frac{F_1(v_1)}{f_1(v_1)} + \frac{1 - F_2(v_2)}{f_2(v_2)} \right),
\]

(9)

where \( \alpha \in [0, 1] \) and \( F_i(\cdot) \) and \( f_i(\cdot) \) are the cumulative and probability density functions for the players. Note that \( \frac{1 - F_2(0)}{f_2(0)} > 0 \); hence, the right-hand side of (9) is strictly positive. This means that there exists an \( \epsilon > 0 \), for which \( \frac{1 - F_2(\epsilon)}{f_2(\epsilon)} > \epsilon \); and consequently, \( \bar{p}_2(\epsilon) = 0 \). In other words, even if the lower bound of the seller’s mixed strategy is at 0, the \( \epsilon \)-type buyer will not be able to trade with any seller because of the “wedge” required by the IC and interim participation (IP) constraints. However, since \( \phi'_2(\epsilon) > 0 \) by assumption, it follows that the buyer strictly prefers a lower investment to the \( \epsilon \)-investment, meaning that \( \epsilon \) cannot be part of the equilibrium support, a contradiction. 

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5 Myerson-Satterthwaite with atoms and gaps

The wedge introduced by the IP conditions is the reason that we cannot support a nice distribution in an equilibrium in which a second-best mechanism is played. From Myerson and Satterthwaite, we know that the lowest type buyers cannot trade with anyone. But then these types cannot be supported by equilibrium investment decisions.

However, Theorem 2 above does not rule out potential investment strategies that involve both atoms and gaps. For example, one might think that placing a probability mass of sellers at zero investment, and a gap between the zero-type buyers and the next highest type in the mixed strategy’s support might resolve the issue identified in the previous section. In this section, we consider this possibility by extending the Myerson-Satterthwaite analysis to distributions with gaps and atoms. We show that this extension might allow efficient mechanisms satisfying IC and IP in some cases. However, we also show that distributions that do not admit first-best efficiency in the trading stage cannot be equilibrium mixed strategies. As a result, the conclusion that Myerson-Satterthwaite inefficiencies cannot occur with endogenously determined investments, extends beyond the case of lotteries described in Theorem 2. This result holds generally when the valuations are equilibrium choices as modeled here. We now consider the case of probability atoms and gaps.

In particular, consider a distribution of shape given in Figure 1. We prove the following extension of Myerson and Satterthwaite’s main theorem.

Theorem 3. (Myerson & Satterthwaite with atoms and gaps). Given a distribution with upper and lower atoms at $a_i$ and $b_i$ respectively, and a continuous
To prove Theorem 3, we start by showing that the envelope theorem extends to this case: Given a mechanism with allocation function $p(v_1, v_2)$ and transfer function $x(v_1, v_2)$, the expected gain (relative to non-participation) for the seller from declaring $v'_1$ when his real investment is $v_1$, is:

$$U_1(v'_1, v_1) = \int_{v_2}^{v_2} [x(v'_1, v_2) - p(v'_1, v_2)v_1]dF_2 + q_2(x(v'_1, v_2) - p(v'_1, v_2)v_1) + \overline{q}_2(x(v'_1, v_2) - p(v'_1, v_2)v_1).$$

(11)

Similarly, for the buyer we have:

$$U_2(v'_2, v_2) = \int_{0}^{v_1} [p(v_1, v'_2)v_2 - x(v_1, v'_2)]dF_1 + q_1(p(v_1, v'_2)v_2 - x(v_1, v'_2)) + \overline{q}_1(p(v_1, v'_2)v_2 - x(v_1, v'_2)).$$

(12)

Now, the expected probability of trade of a $v_1$-type seller is given by $\overline{p}_1(v_1) = \int_{v_2}^{v_2} p(v_1, v_2)dF_2 + q_2p(v_1, v_2) + \overline{q}_2p(v_1, v_2)$; similarly, we have $\overline{p}_2(v_2) = \int_{v_1}^{v_1} p(v_1, v_2)dF_1 +
\( q_1 p(a_1, v_2) + q_2 p(a_1, v_2) \). Incentive compatibility then implies:

\[-\overline{p}_1(v_1)[v_1 - v'_1] \geq U_1(v_1, v_1) - U_1(v'_1, v'_1) \geq -\overline{p}_1(v'_1)[v_1 - v'_1] .\]

For \( v_1 \) in the continuous part of the seller’s distribution, we divide by \( v_1 - v'_1 \), and take the limit as \( v'_1 \to v_1 \) to obtain:

\[ U'_1(v_1) = -\overline{p}_1(v_1) . \tag{13} \]

Integrating equation (13), we obtain for \( v_1 \) in the continuous part of the seller’s range:

\[ U_1(v_1, v_1) = U_1(\overline{v}_1, \overline{v}_1) + \int_{v_1}^{\overline{v}_1} \overline{p}_1(t_1)dt_1 . \tag{14} \]

With the same method, and making similar assumptions about the probability of trade of the 0- and \( \overline{v}_1 \)-type sellers, we arrive at:

\[ U_2(v_2, v_2) = U_2(\overline{v}_2, \overline{v}_2) + \int_{\overline{v}_2}^{v_2} \overline{p}_2(t_2)dt_2 . \tag{15} \]

Now, consider the expression:

\[
\begin{align*}
&\int_{\overline{v}_2}^{v_2} \int_{\overline{v}_1}^{\overline{v}_1} (v_2 - v_1)p(v_1, v_2)f_1(v_1)f_2(v_2)dv_1dv_2 + \int_{\overline{v}_1}^{\overline{v}_1} (a_2 - v_1)q_2p(v_1, a_2)f_1(v_1)dv_1 \\
&+ \int_{\overline{v}_2}^{\overline{v}_2} \overline{q}_2(b_2 - v_1)p(b_1, b_2)f_1(v_1)dv_1 + \int_{\overline{v}_2}^{\overline{v}_2} q_1(v_2 - a_1)p(a_1, v_2)f_2(v_2)dv_2 \\
&+ \int_{\overline{v}_2}^{\overline{v}_2} \overline{q}_1(v_2 - b_1)p(b_1, v_2)f_2(v_2)dv_2 + q_1 q_2(a_2 - a_1)p(a_2, a_1) \\
&+ q_1 q_2(b_2 - a_1)p(a_1, b_2) + \overline{q}_1 q_2(a_2 - b_1)p(b_1, a_2) + \overline{q}_1 \overline{q}_2(b_2 - b_1)p(b_1, b_2) \
\end{align*}
\] (16)
Adding and subtracting the appropriate transfer functions $x(\cdot, \cdot)$ (and its integrals) weighted by the relevant probabilities to this expression, we obtain:

$$\int_{\mathbb{V}_1} U_1(v_1)f_1(v_1)dv_1 + \int_{\mathbb{V}_2} U_2(v_2)f_2(v_2)dv_2 + q_1U_1(a_1) + q_2U_2(a_2) + \bar{q}_1U_1(b_1) + \bar{q}_2U_2(b_2).$$

(17)

Evaluating the integrals in the first two terms by using (14) and (15), we obtain:

$$\int_{\mathbb{V}_1} U_1(v_1)f_1(v_1)dv_1 = (1 - \bar{q}_1 - q_1)U_1(v_1) + \int_{\mathbb{V}_1} (1 - \bar{q}_2 - q_2)\pi(v_1)
+ q_2p(v_1, a_2) + \bar{q}_2p(v_1, b_2) \mathcal{F}_1(v_1)dv_1$$

(18)

$$\int_{\mathbb{V}_2} U_2(v_2)f_2(v_2)dv_2 = (1 - \bar{q}_2 - q_2)U_2(v_2) + \int_{\mathbb{V}_2} (1 - \bar{q}_1 - q_1)\pi(v_2) + q_1p(a_1, v_2)
+ \bar{q}_1p(b_1, v_2) (1 - \bar{q}_2 - q_2 - \mathcal{F}_2(v_2))dv_2,$$

(19)

where $\mathcal{F}_1(v_1) \equiv \int_{\mathbb{V}_1} f_1(t_1)dt_1$ and $\mathcal{F}_2(v_2) \equiv \int_{\mathbb{V}_2} f_2(t_2)dt_2.$

Substituting (18) and (19) into (17) and equating it with (16), we obtain:

\footnote{Note that these are not the cumulative density functions of the entire distribution.}
(1 - \bar{q}_1 - q_1)U_1(\bar{v}_1) + (1 - \bar{q}_2 - q_2)U_2(\bar{v}_2) + q_1 U_1(a_1) + \bar{q}_1 U_1(b_1) + q_2 U_2(a_2) + \bar{q}_2 U_2(b_2) = \\
\int_{\bar{v}_2}^{v_2} \int_{\bar{v}_1}^{v_1} \left[ \left( v_2 - \frac{1 - \bar{q}_2 - q_2 - \mathcal{F}(v_2)}{f_2(v_2)} \right) - \left( \bar{v}_1 + \frac{\mathcal{F}(v_1)}{f_1(v_1)} \right) \right] p(v_1, v_2) f_1(v_1) f_2(v_2) dv_1 dv_2 \\
- \int_{\bar{v}_1}^{v_1} \left[ v_1 + \frac{\mathcal{F}(v_1)}{f_1(v_1)} \right] \left[ q_2 p(v_1, a_2) + \bar{q}_2 p(v_1, b_2) \right] f_1(v_1) dv_1 \\
+ \int_{\bar{v}_2}^{v_2} \left[ v_2 - \frac{1 - \bar{q}_2 - q_2 - \mathcal{F}(v_2)}{f_2(v_2)} \right] \left[ q_1 p(a_1, v_2) + \bar{q}_1 p(b_1, v_2) \right] f_2(v_2) dv_2 \\
+ q_1 q_2 (a_2 - a_1) p(a_1, a_2) + q_1 \bar{q}_2 (b_2 - a_1) p(a_1, b_2) + \bar{q}_1 q_2 (a_2 - b_1) p(b_1, a_2) + \bar{q}_1 \bar{q}_2 (b_2 - b_1) p(b_1, b_2) \\
= \int_{\bar{v}_2}^{v_2} \left( v_2 - \frac{1 - \bar{q}_2 - q_2 - \mathcal{F}(v_2)}{f_2(v_2)} \right) \bar{p}(v_2) f_2(v_2) dv_2 - \int_{\bar{v}_1}^{v_1} \left( v_1 + \frac{\mathcal{F}(v_1)}{f_1(v_1)} \right) \bar{p}(v_1) f_1(v_1) dv_1 \\
+ q_1 q_2 (a_2 - a_1) p(a_1, a_2) + q_1 \bar{q}_2 (b_2 - a_1) p(a_1, b_2) + \bar{q}_1 q_2 (a_2 - b_1) p(b_1, a_2) + \bar{q}_1 \bar{q}_2 (b_2 - b_1) p(b_1, b_2). \\
\text{(20)}

Now, the IC condition at the upper gap in the buyer’s distribution implies:

$$U_2(b_2) \geq U_2(\bar{v}_2) + (b_2 - \bar{v}_2) \bar{p}_2(\bar{v}_2),$$

as otherwise, the $b_2$-type buyer would prefer to report her type as $\bar{v}_2$. Likewise, we have for the other gaps:

$$U_2(\bar{v}_2) \geq U_2(a_2) + (\bar{v}_2 - a_2) \bar{p}_2(a_2)$$

$$U_1(\bar{v}_1) \geq U_1(b_1) + (b_1 - \bar{v}_1) \bar{p}_1(b_1)$$

$$U_1(a_1) \geq U_1(\bar{v}_1) + (\bar{v}_1 - a_1) \bar{p}_1(\bar{v}_1). \quad \text{(21)}$$
Thus, we can write

\[(1 - \eta_1 - \eta_2)U_1(v_1) + (1 - \eta_2)U_2(v_2) + \eta_1 U_1(a_1) + \eta_2 U_2(b_2) + \eta_1 U_1(b_1) + \eta_2 U_2(a_2) + \eta_1 U_1(b_1) + \eta_2 U_2(a_2) \geq U_1(b_1) + U_2(a_2) + (1 - \eta_1)(b_1 - v_1)\mathbf{\bar{p}}_1(b_1) + \eta_1 \left[ \int_{v_1}^{b_1} \mathbf{\bar{p}}_1(v_1)dv_1 + (v_1 - a_1)\mathbf{\bar{p}}_1(v_1) \right] \]

\[+ (1 - \eta_2)(v_2 - a_2)\mathbf{\bar{p}}_2(a_2) + \eta_2 \left[ \int_{v_2}^{a_2} \mathbf{\bar{p}}_2(v_2)dv_2 + (b_2 - v_2)\mathbf{\bar{p}}_2(v_2) \right]. \quad (22)\]

Equating the last line above with the right-hand side of (20) and rearranging, we obtain:

\[0 \leq U_1(b_1) + U_2(a_2) \leq \int_{v_2}^{b_2} \left( v_2 - \frac{1 - \eta_2 - F_2(v_2)}{f_2(v_2)} \right) \mathbf{\bar{p}}(v_2)f_2(v_2)dv_2 - \int_{v_1}^{b_1} \left( v_1 + \frac{F_1(v_1) + \eta_1}{f_1(v_1)} \right) \mathbf{\bar{p}}(v_1)f_1(v_1)dv_1 \]

\[-(1 - \eta_1)(b_1 - v_1)\mathbf{\bar{p}}_1(b_1) - \eta_1 (v_1 - a_1)\mathbf{\bar{p}}_1(v_1) - (1 - \eta_2)(v_2 - a_2)\mathbf{\bar{p}}_2(a_2) - \eta_2 (b_2 - v_2)\mathbf{\bar{p}}_2(v_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

\[+ \eta_1 \eta_2 (a_2 - a_1)p(a_1, a_2) + \eta_1 \eta_2 (b_2 - a_1)p(a_1, b_2) + \eta_1 \eta_2 (a_2 - b_1)p(b_1, a_2) + \eta_1 \eta_2 (b_2 - b_1)p(b_1, b_2) \]

The last thing we need to show is that assuming \( p(\cdot, \cdot) \) satisfying (23), and that \( \mathbf{\bar{p}}_1(\cdot) \) and \( \mathbf{\bar{p}}_2(\cdot) \) are weakly decreasing and increasing, respectively, a payment function exists that makes the mechanism satisfy IC and IP. To that end, consider the following functional form

\[x(v_1, v_2) = \chi_2(v_2) - \chi_1(v_1) + \text{constant}, \quad (24)\]
where \( \chi_1(\cdot) \) and \( \chi_2(\cdot) \) are given by:

\[
\chi_2(v_2) = \begin{cases} 
0 & \text{for } v_2 = a_2 \\
\int_{v_2}^{u_2} t_2 d[\bar{p}_2(t_2)] + a_2(\bar{p}_2(v_2) - \bar{p}_2(u_2)) & \text{for } v_2 \leq v_2 \leq u_2 \\
\int_{v_2}^{u_2} t_2 d[\bar{p}_2(t_2)] + a_2(\bar{p}_2(v_2) - \bar{p}_2(a_2)) + \bar{v}_2(\bar{p}_2(b_2) - \bar{p}_2(v_2)) & \text{for } v_2 = b_2 
\end{cases}
\]

and

\[
\chi_1(v_1) = \begin{cases} 
0 & \text{for } v_1 = a_1 \\
\int_{v_1}^{u_1} t_1 d[-\bar{p}_1(t_1)] - a_1(\bar{p}_1(v_1) - \bar{p}_1(a_1)) & \text{for } v_1 \leq v_1 \leq u_1 \\
\int_{v_1}^{u_1} t_1 d[-\bar{p}_1(t_1)] - a_1(\bar{p}_1(v_1) - \bar{p}_1(a_1)) - \bar{v}_1(\bar{p}_1(b_1) - \bar{p}_1(v_1)) & \text{for } v_1 = b_1 
\end{cases}
\]

and the constant is chosen such that \( U_2(a_2) = 0 \). To check the incentive compatibility of this payment function, observe that the difference \( U_1(v_1, v_1') - U_1(v_1', v_1) \) when both \( v_1' \) and \( v_1 \) are on the continuous part of the buyer’s support is given by:

\[
-v_1(\bar{p}_1(v_1) - \bar{p}_1(v_1')) - \chi_1(v_1) + \chi_1(v_1') = v_1 \int_{t_1 = v_1'}^{v_1} d[-\bar{p}_1(t_1)] - \int_{t_1 = v_1'}^{v_1} t_1 d[-\bar{p}_1(t_1)] \\
= \int_{t_1 = v_1'}^{v_1} (v_1 - t_1) d[-\bar{p}_1(t_1)] \geq 0 ,
\]

since \( \bar{p}_1 \) is a weakly decreasing function. First, suppose the \( v_1 \)-type reports \( b_1 \), in
which case the difference \( U_1(v_1, v_1) - U_1(b_1, v_1) \) becomes:

\[
- v_1(\overline{p}_1(v_1) - \overline{p}_1(b_1)) - \chi_1(v_1) + \chi_1(b_1) = \\
v_1 \int_{t_1=v_1}^{\overline{v}_1} d[-\overline{p}_1(t_1)] - v_1(\overline{p}_1(\overline{v}_1) - \overline{p}_1(b_1)) - \int_{v_1}^{\overline{v}_1} t_1 d[-\overline{p}_1(t_1)] - v_1(\overline{p}_1(b_1) - \overline{p}_1(\overline{v}_1)) = \\
\int_{[v_1]}^{{\overline{v}_1}} (v_1 - t_1) d[-\overline{p}_1(t_1)] \geq 0. \quad (28)
\]

Alternatively, suppose the \( v_1 \)-type seller reports \( a_1 \). We have for \( U_1(v_1, v_1) - U_1(a_1, v_1) \):

\[
- v_1(\overline{p}_1(v_1) - \overline{p}_1(a_1)) - \chi_1(v_1) + \chi_1(a_1) = \\
v_1 \int_{t_1=v_1}^{\overline{v}_1} d[-\overline{p}_1(t_1)] - v_1(\overline{p}_1(v_1) - \overline{p}_1(a_1)) - \int_{t_1=v_1}^{\overline{v}_1} t_1 d[-\overline{p}_1(t_1)] + a_1(\overline{p}_1(v_1) - \overline{p}_1(a_1)) = \\
\int_{t_1=v_1}^{v_1} (v_1 - t_1) d[-\overline{p}_1(t_1)] + (a_1 - v_1)(\overline{p}_1(v_1) - \overline{p}_1(a_1)) \geq 0 \quad (29)
\]

Conversely, consider an \( a_1 \)-type seller reporting \( v'_1 > a_1 \). The difference \( U_1(a_1, a_1) - U_1(v'_1, a_1) \) is given by:

\[
- a_1(\overline{p}_1(a_1) - \overline{p}_1(v'_1)) + \chi_1(v'_1) = -a_1 \left( \int_{t_1=v_1}^{v'_1} d[-\overline{p}_1(t_1)] - \overline{p}_1(v'_1) + \overline{p}_1(a_1) \right) \\
- \int_{t_1=v_1}^{v'_1} t_1 d[-\overline{p}_1(t_1)] - a_1(\overline{p}_1(v'_1) - \overline{p}_1(a_1)) = \\
- a_1 \int_{t_1=v_1}^{v'_1} d[-\overline{p}_1(t_1)] + \int_{t_1=v_1}^{v'_1} t_1 d[-\overline{p}_1(t_1)] = \int_{t_1=v_1}^{v'_1} (t_1 - a_1) d[-\overline{p}_1(t_1)] \geq 0. \quad (30)
\]
Finally, consider a $b_1$-type seller reporting $v'_1$; the difference $U_1(b_1, b_1) - U_1(v'_1, b_1)$ is:

$$-b_1(\bar{p}_1(b_1) - \bar{p}_1(v'_1)) - \chi_1(b_1) = \chi_1(v'_1) =$$

$$b_1 \left( \int_{t_1=v'_1}^{\bar{t}_1} d[-\bar{p}_1(t_1)] - (\bar{p}_1(b_1) - \bar{p}_1(\bar{v}_1)) \right) + \bar{v}_1(\bar{p}_1(b_1) - \bar{p}_1(\bar{v}_1)) - \int_{t_1=v'_1}^{\bar{t}_1} t_1 d[-\bar{p}_1(t_1)]$$

$$= \int_{t_1=v'_1}^{\bar{t}_1} (b_1 - t_1)d[-\bar{p}_1(t_1)] + (\bar{v}_1 - b_1)(\bar{p}_1(b_1) - \bar{p}_1(\bar{v}_1)) \geq 0. \quad (31)$$

The proof for the buyers proceeds analogously. □

### 5.1 Efficient mechanisms

With Theorem 3, we have the following Lemma:

**Lemma 1.** Efficient trading mechanisms that satisfy the IC and IP conditions exist for some distributions of buyers’ and sellers’ valuations.

To prove this lemma, we only need to find an example distribution that satisfies the constraint (10). To that end, suppose the distributions satisfy the following inequalities: $b_2 > b_1 > \bar{v}_2 > \bar{v}_1 > v_2 > v_1$. Assuming an efficient mechanism (i.e., when $p(v_1, v_2) = 1$ iff $v_2 > v_1$, and zero otherwise), we can evaluate inequality (10) and find that it equals:

$$-\int_{\underline{v}_2}^{\bar{v}_2} \mathcal{F}_1(t) \left( 1 - q_2 - \mathcal{F}_2(t) \right) dt - q_1 \int_{\underline{v}_2}^{\bar{v}_2} \mathcal{F}_2(t) dt - q_1 (1 - q_2) (\bar{v}_2 - v_2)$$

$$+ \bar{q}_2 (\bar{v}_2 - b_1 - b_2(1 - \bar{q}_1)) + \bar{q}_1 \bar{q}_2 b_2 + \bar{q}_1 \bar{q}_2 (b_2 - b_1). \quad (32)$$

Now, consider the following numerical example where the continuous parts of the distributions are uniform, and the parameters are given by: $b_2 = 1$, $b_1 = 0.7$, $\bar{v}_2 =$
0.65, \( v_1 = 0.6, v_2 = 0.4, v_1 = 0.2, q_1 = 0.1, \overline{q}_1 = 0.8, q_2 = 0.1, \) and \( \overline{q}_2 = 0.8 \). With these assumptions, expression (32) can be numerically evaluated, and is found to be \( \approx 0.035 \).

This example illustrates why the failure of the Myerson-Satterthwaite Theorem when atoms are allowed is not surprising: efficiency tends to be possible when most of the probability weight is assigned to the atoms, leaving little probability density in the overlapping parts of the distributions, and when the gaps between the continuous range and the atoms are large. Both make it easier to incentivize truth-telling.

5.2 Second-best mechanisms

When efficient mechanisms are not possible to implement with an incentive compatible, individually rational mechanism, we have the following results about the second-best mechanisms:

**Lemma 2.** When an efficient mechanism does not satisfy both IC and IP, the optimal mechanism maximizing aggregate gains from trade has:

\[
U_2(b_2) = U_2(\overline{v}_2) + (b_2 - \overline{v}_2)p_2(\overline{v}_2)
\]
\[
U_2(v_2) = U_2(a_2) + (v_2 - a_2)p_2(a_2)
\]
\[
U_1(\overline{v}_1) = U_1(b_1) + (b_1 - \overline{v}_1)p_1(b_1)
\]
\[
U_1(a_1) = U_1(\overline{v}_1) + (\overline{v}_1 - a_1)p_1(\overline{v}_1)
\]
Proof. Consider a mechanism that has:

\[ U_2(b_2) = U_2(v_2) + (b_2 - v_2)p_2(v_2) + \gamma_2 \]
\[ U_2(a_2) = U_2(v_2) + (v_2 - a_2)p_2(a_2) + \gamma_2 \]
\[ U_1(v_1) = U_1(b_1) + (b_1 - v_1)p_1(b_1) + \gamma_1 \]
\[ U_1(a_1) = U_1(v_1) + (v_1 - a_1)p_1(v_1) + \gamma_1 \]

where \( \gamma_i, \overline{\gamma}_i \geq 0 \). Repeating the steps leading up to (23), we can write the condition for a mechanism to satisfy IC and IP as:

\[
0 \leq \int_{\Xi_2} \left( v_2 - \frac{1 - q_2 - F_2(v_2)}{f_2(v_2)} \right) p(v_2)f_2(v_2)dv_2 - \int_{\Xi_1} \left( v_1 + \frac{F_1(v_1) + q_1}{f_1(v_1)} \right) p(v_1)f_1(v_1)dv_1
\]
\[ - (1 - \overline{q}_1)(b_1 - v_1)p_1(b_1) - q_1(v_1 - a_1)p_1(v_1) - (1 - q_2)(v_2 - a_2)p_2(a_2) - \overline{q}_2(b_2 - v_2)p_2(v_2) + q_1q_2(a_2 - a_1)p(a_1, a_2) + q_1\overline{q}_2(b_2 - a_1)p(a_1, b_2) + \overline{q}_1q_2(a_2 - b_1)p(b_1, a_2) + \overline{q}_1\overline{q}_2(b_2 - b_1)p(b_1, b_2)
\]
\[ - \overline{q}_1\overline{q}_2 - (1 - q_1)\gamma_2 - q_1\gamma_1 \equiv G(\gamma_1, \overline{\gamma}_1, \gamma_2, \overline{\gamma}_2). \] (33)

The second-best mechanism is given by maximizing the aggregate welfare subject to (33), i.e., maximizing the Lagrangian:

\[
\int_{\Xi_2} \int_{\Xi_1} (v_2 - v_1)p(v_1, v_2)f_1(v_1)f_2(v_2)dv_1dv_2 + \int_{\Xi_1} \overline{q}_2(b_2 - v_1)p(v_1, b_2)f_1(v_1)dv_1
\]
\[ + q_2 \int_{\Xi_1} (a_2 - v_1)p(v_1, a_2)f_1(v_1)dv_1 + \int_{\Xi_2} \overline{q}_1(v_2 - a_1)p(a_1, v_2)f_2(v_2)dv_2
\]
\[ + \int_{\Xi_2} \overline{q}_1(v_2 - b_1)p(b_1, v_2)f_2(v_2)dv_2 + q_1\overline{q}_2(b_2 - a_1)p(a_1, b_2) + \overline{q}_1\overline{q}_2(b_2 - b_1)p(b_1, b_2) + \overline{q}_1q_2(a_2 - b_1)p(b_1, a_2)
\]
\[ + \lambda G(\gamma_1, \overline{\gamma}_1, \gamma_2, \overline{\gamma}_2). \] (34)
where $\lambda \geq 0$ is a Lagrange multiplier. But since $G(\gamma_1, \tau_1, \gamma_2, \tau_2)$ is decreasing in all of $\gamma$'s, the maximum of the Lagrangian requires them to be zero, which finishes the proof. 

Using Lemma 2, we can now prove that a second-best mechanism when efficiency is not possible rules out a mixing investment equilibrium.

**Lemma 3. (Mixed investment unravels)** Suppose that given $F_1, F_2$ with atoms and gaps there is no IC and IP mechanism that is efficient, and the designer chooses a second-best mechanism $(p, x)$ to maximize the aggregate gains from trade given these lotteries, then it is not possible to support $F_1, F_2$ as equilibrium mixed investment strategies with $(p, x)$ for any strictly convex and continuous cost functions.

To prove Lemma 3, we will focus on the lower gap of the buyer at a candidate mixed investment equilibrium; similar arguments apply for the upper gap of the buyer and both gaps of the seller. First, note that the mixing condition for the buyer is given by:

$$U_2(v_2) - c_2(v_2) = U_2(v'_2) - c_2(v'_2),$$  \hspace{1cm} (35)

for any $v_2, v'_2$ in the support of the buyer’s mixed strategy. Hence, we have $U_2(v_2) - U_2(a_2) = c_2(v_2) - c_2(a_2)$. Due to the convexity of the cost function, we have:

$$c'(a_2) < \frac{c_2(v_2) - c_2(a_2)}{v_2 - a_2} = p_2(a_2),$$  \hspace{1cm} (36)

where the last equality follows from Lemma 2. Now, consider a deviation from a candidate mixed-strategy equilibrium where a buyer invests at $a_2 + \epsilon$, but reports $a_2$. 

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For small $\epsilon$, the gain from this deviation is given by:

$$\left( \overline{p}_2(a_2) - c'_2(a_2) \right) \epsilon > 0,$$

meaning that such a deviation will be profitable. Hence, the candidate equilibrium is not an equilibrium strategy. This implies there cannot be a gap on the lower end of the buyer’s investment strategy. The following theorem follows directly from Lemmas (1) and (3).

**Theorem 4.** With pre-trade investment there is no equilibrium to any trading game with form of inefficiency in Myerson-Satterthwaite.

### 6 Allocation inefficiency

In the previous section we showed that with pre-trade investment, there is no equilibrium of a trading game that has the inefficiency identified by Myerson and Satterthwaite, and inefficient mixed investment strategies will unravel. However, even if the agent with the item invests optimally with probability 1, a third form of inefficiency might persist, in that the agent who ends up with the item might not be the one capable of producing the highest surplus. An important question is then, in bilateral trade with investments, are there failures of Coasian efficiency? That is, do there exist initial allocations of the good such that the individual who extracts less utility from ownership maintains or obtains the property right to the good in equilibrium?

We maintain the assumption that the good is initially assigned to one player (the seller). In principle, a larger class of schemes involving a market maker or auctioneer can be considered. We will say a player is the efficient owner if he/she is the player
who is able to generate the highest utility from the use of the good. In practice, such a user would have a cost function that allows for lower production costs. First, suppose that the seller, who initially owns the good, is the efficient user. In this situation, allocation efficiency always occurs in equilibrium. If such a seller keeps the item with probability 1, then in equilibrium, he will choose an investment level to maximize his utility of ownership. In any equilibrium in which the buyer gets the good with probability 0, her investment must be equal to 0.

To see that nothing else is an equilibrium, suppose the seller trades the good. At most the buyer pays an amount

\[ x^* = \max_{v_2} (v_2 - c_2(v_2)) . \]

However, we have assumed for this case that

\[ \max_{v_2} (v_2 - c_2(v_2)) < \max_{v_1} (v_1 - c_1(v_1)) \]

and the seller is better off not participating in the trade, keeping the good, and investing optimally. Importantly, the buyer is never willing to pay a price that makes the seller willing to sell and there is no equilibrium where the seller trades the good when he is the efficient owner.

When the buyer is the efficient owner of the good, i.e., when

\[ \max_{v_2} (v_2 - c_2(v_2)) > \max_{v_1} (v_1 - c_1(v_1)) , \]

the picture is more complicated and there are cases where there are equilibria with
only the seller (less efficient type) investing and not trading. This happens under two conditions. The first one is when the optimal investment by the seller is greater than the surplus generated by the buyer’s optimal investment, i.e., when:

$$\arg \max_{v_1} (v_1 - c_1(v_1)) > \max_{v_2} (v_2 - c_2(v_2)) .$$

When this condition holds, there is no price that the buyer can pay to an optimally investing seller that would induce trade and result in a positive total utility to the buyer. Hence, the seller investing optimally, and the buyer not investing occurs in an equilibrium. However, the seller not investing, the buyer investing optimally, and trade taking place, is also an equilibrium provided that the price $p$ satisfies

$$\max_{v_2} (v_2 - c_2(v_2)) > p > \max_{v_1} (v_1 - c_1(v_1)) .$$

The second case that admits an inefficient equilibrium stems from perhaps rather peculiar cost functions. In particular, suppose again the buyer is the efficient owner, but that

$$\arg \max_{v_1} (v_1 - c_1(v_1)) > \arg \max_{v_2} (v_2 - c_2(v_2)) .$$

In other words, even though the buyer can generate more surplus, she does this with a lower absolute investment level than the seller’s maximum surplus. In this case again, trade between efficiently invested sellers and buyers is impossible; hence, the buyer investing efficiently and the seller not investing occurs in an equilibrium. Furthermore, as in the previous case, the efficient outcome (seller not investing, buyer investing efficiently, and trading) is also supportable in an equilibrium, as long as the
price $p$ satisfies the same inequalities above.

Note that in both cases, the inefficient equilibrium could be dispensed with if, prior to the investment stage, the buyer can sell an option to the seller that commits the buyer to buy at a price within the range identified with these inequalities. In this case, the seller can always ensure a higher payoff by not investing and selling at that price, rather than investing optimally and not trading, regardless of the buyer’s actions. Accordingly, in equilibrium, the seller will invest 0 and sell the item. The buyer’s best response, then, is to invest optimally. Hence, only the efficient equilibrium remains.

In conclusion of this section, if the initial allocation of the good is to the inefficient player (one who can generate less surplus), then allowing trade after unobservable investments does not necessarily guarantee that the best allocation will be achieved. However, this result does not stem from private information during the trading phase; in fact, all equilibria discussed above imply that players will follow pure investment strategies, and will have beliefs accordingly. Rather, the inefficiencies result from the inability of the parties to trade at a high enough price. If either a mechanism designer or the buyer can commit to trade at a high enough price, the inefficient equilibria can be made to go away. Even though our results do not guarantee first-best outcomes in the sense that the player who can make the most out of owning the good possesses it in all equilibria, we find that informational asymmetries are not the cause of the inefficiencies that remain at equilibrium.
7 Conclusion

Sometimes the value of a trade between two economic agents is determined by choices that the traders make prior to the transaction. In these circumstances a rational expectation about how the trading game might be played can be seen to have important effects on the incentives to invest and, as a consequence, generate bilateral trade games where the information environment looks very different from those well-studied in the economics literature. When valuations are the product of hidden pre-trade investments, the standard connected set of types cannot emerge as the result of equilibrium mixing. Furthermore in every equilibrium of the trading game, given investments, trade occurs in every instance where the net gains are positive. In short, this means that the Myerson-Satterthwaite inefficiencies do not arise in these environments.

But this is not all there is to say when it comes to social welfare and trade. In thinking about the set of equilibria to all trading rules, it is useful to distinguish between three potential forms of “inefficiencies.” Given equilibrium investment decisions, the trading rule might misallocate the item. This is the form of inefficiency that Myerson and Satterthwaite find is inescapable in the context of IC and ex-interim mechanisms with exogenous distributions over valuations that exhibit overlap. A second form of inefficiency stems from the possibility that players may influence their bargaining power by investing prior to bargaining. This feature is central to the hold-up problem. A third inefficiency can surface even if the item is allocated to the player with the highest valuation and the players optimally invest given the ex-post allocation. This form involves optimal investment and ownership of the good by the player with the less efficient technology for turning investment into valuation. Simply allocating transferable ownership to one of the players can avoid the first two-forms
of inefficiency but would potentially result in the third, depending on the allocation and the cost functions.
References


Figure 1: Diagram of the buyer and seller distributions. Solid lines signify continuous supports of the players’ investment strategies.