Exact Soft-Covering Exponent

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Abstract

This work establishes the exact exponents for the soft-covering phenomenon of a memoryless channel under the total variation metric when random (i.i.d. and constant-composition) channel codes are used. The exponents, established herein, are strict improvements in both directions on bounds found in the literature. This complements the recent literature establishing the exact exponents under the relative entropy metric; however, the proof techniques have significant differences, and thus, neither result trivially implies the other.

The found exponents imply new and improved bounds for various problems that use soft-covering as their achievability argument, including new lower bounds for the resolvability exponent and the secrecy exponent in the wiretap channel.

Keywords: Soft-covering lemma, total variation distance, channel resolvability, random coding exponent, random i.i.d. coding ensemble, random constant-composition coding ensemble.

I. Introduction

The soft-covering lemma is a strong and useful tool commonly used for proving achievability results for information theoretic security, resolvability, channel synthesis and lossy source coding. The roots of the soft-covering concept originate back to Wyner [1, Theorem 6.3] where he developed this tool with the aim of proving achievability in his work on the common information of two random variables. Coincidentally, the most widespread current application of soft-covering is security proofs in wiretap channels, e.g., [2], which Wyner also introduced in that same year in [3] but apparently did not see how soft-covering applied.

The soft-covering lemma states that given a stationary memoryless channel $P_{Y^n|X^n}$ with stationary memoryless input distribution $P_{X^n}$ yielding an output distribution $P_{Y^n}$, the distribution $P_{Y^n|C^n}$ induced by instead selecting a sequence $X^n$ at random from a codebook $C^n_M$ and passing it through the channel, see
Definition 13, will be a good approximation of the output distribution $P_{Y^n}$ in the limit as $n$ goes to infinity so long as the codebook is of size $M$ greater than $\exp(nR)$ where $R$ is greater than the single-shot mutual information between the input and output, i.e., $R > I(P_X, P_{Y|X})$. In fact, the aforementioned codebook $\mathcal{C}_M^n$ can be chosen quite carelessly, e.g., by drawing each codeword independently from $P_{X^n}$ or by drawing each codeword uniformly at random from the type class $\mathcal{T}_{P_X}^n$.

The concept of soft-covering is fundamentally related to that of channel resolvability [4], in that the former is a property of random codebooks while the latter is the fundamental limit of optimal codebooks. As a matter of fact, soft-covering establishes the direct proof (also known as “achievability”) for resolvability. Furthermore, given the chronology of the literature, the resolvability problem can be viewed as a question about soft-covering—how much better can an optimized codebook match an output distribution than a random codebook? The first order, the answer is that it does no better.

In the literature, various versions of the soft-covering lemma use various distinctness measures on distributions (commonly relative entropy or total variation distance, see Definitions 6 and 8) and claim that the distance between the induced distribution $P_{Y^n|\mathcal{C}_M^n}$ and the desired distribution $P_{Y^n}$ vanishes in expectation over the random selection of the codebook $\mathcal{C}_M^n$. Regarding the most notable contributions, [4] studies the fundamental limits of soft-covering under the name of “resolvability”, [5] develops the lemma calling it a “cloud mixing” lemma, [6] provides achievable rates of exponential convergence, [7] improves the exponent and extends the framework, [8] and [9, Chapter 16] refer to soft-covering simply as “covering” in the quantum context, [10] refers to it as a “sampling lemma” and points out that it holds for the stronger metric of relative entropy, [11] gives a direct proof of the relative entropy result, and [12] and [13] move away from expected value analysis and show that a random codebook achieves soft-covering phenomenon with a doubly exponentially high probability under the relative entropy measure and total variation distance, respectively.

The motivation of this work is to complement the results of Parizi et al. [14, Theorem 4], and Yu and Tan [15, Theorem 5], where they pin down the exact soft-covering exponents in the expected value analysis of the relative entropy, and of the Rényi divergence of order $\alpha \in (0, 1) \cup (1, 2)$, respectively. In this paper, we first highlight that the total variation distance between the i.i.d. codebook induced distribution $P_{Y^n|\mathcal{C}_M^n}$ and the desired output distribution $P_{Y^n}$ concentrates to its expected value with doubly exponential certainty [16, Theorem 31], a property which is neither satisfied by relative entropy nor by the Rényi divergence. The first main result of this paper, stated in Theorem 1, provides the exact soft-covering exponent for the expected value of the total variation distance between $P_{Y^n|\mathcal{C}_M^n}$ and $P_{Y^n}$. Next, we consider the setting when the random codebook is restricted to contain codewords of the same empirical distributions. Calling this the random constant-composition codebook and denoting it by $\mathcal{D}_M^n$, in Lemma 2, we show the counterpart of [16, Theorem 31]. In other words, we prove the fact that the total variation distance between the constant-composition induced distribution $P_{Y^n|\mathcal{D}_M^n}$ and the desired output distribution $R_{Y^n}$ concentrates to its expected value in a doubly exponential fashion as well. Finally, we present our second main result in Theorem 2, providing the exact soft-covering exponent for the expected value of the total variation distance.
between $P_{Y^n|X^n}$ and $R_{Y^n}$. The exponents for soft-covering, established in this work, provide improved lower bounds on the exponents for resolvability. It may be the case that use of an optimized codebook provides a better exponents, even though this work provides the exact exponents (both upper and lower bounds) for the random codebooks.

In the remainder of this paper, Section II establishes the basic notation and definitions adopted throughout, and Section III highlights [16, Theorem 31], shows its counterpart in the constant-composition setting, and states the main results of this paper, namely, the exact soft-covering exponents for the cases of random i.i.d. codebooks and random constant-composition codebooks, along with a number of remarks. Sections IV and V prove the lower and upper bound directions of the main result in Theorem 1 together with the remarks of how one would recover the proof of Theorem 2 based on the proof provided. As Section VI proves alternative expressions for the exact soft-covering exponents, Section VII compares the exact exponents to their previously discovered lower bounds, and finally, Appendices I–V and VIII provide the lemmas and corollaries that are invoked in the main proofs while Appendices VI and VII provide the full proof of Theorem 2 and the finite block-length results that appear as the byproduct of our proof, respectively.

II. Notation and Definitions

This section introduces the basic notation and fundamental concepts as well as several definitions and properties to be used in the sequel.

Given a finite alphabet $\mathcal{X}$, let $\mathcal{P}(\mathcal{X})$ denote the set of all distributions defined on it. For a random variable $X$ on $\mathcal{X}$, a central measure in information theory, namely the amount of information provided by $X = x \in \mathcal{X}$, is defined as follows.

**Definition 1. Information.** Suppose $X \sim P_X \in \mathcal{P}(\mathcal{X})$, the information in $X = x \in \mathcal{X}$ is

$$\mathcal{I}_{P_X}(x) = \log \frac{1}{P_X(x)}. \quad (1)$$

When we investigate the interplay between two random variables $(X, Y) \in \mathcal{X} \times \mathcal{Y}$, the amount of information provided by $Y = y$ after observing $X = x$ is measured by conditional information.

**Definition 2. Conditional Information.** Suppose that given $X = x$, $Y \sim P_{Y|X=x}$. The conditional information provided by $Y = y$, given $X = x$, is

$$\mathcal{I}_{P_{Y|X}}(y|x) = \log \frac{1}{P_{Y|X}(y|x)}. \quad (2)$$

Notice that information $\mathcal{I}_{P_X}(x)$ is a deterministic function depending on the random variable $X \sim P_X$ only through its probability mass function. If one considers the average of $\mathcal{I}_{P_X}(X)$, the random information provided by $X$, this gives rise to the definition of the most famous information theoretic quantity, entropy, which is defined next.

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1Unless otherwise stated, logarithms and exponentials are of arbitrary (but matching) bases throughout this paper.
Definition 3. **Entropy.** The entropy of a discrete random variable $X \sim P_X \in \mathcal{P}(\mathcal{X})$ is the average information provided by $X$, that is

$$H(P_X) = \mathbb{E}[\hat{t}_{P_X}(X)].$$

(3)

When the distribution of the discrete random variable $X$ is clear from the context, it is customary to denote its entropy by $H(X)$. Given $(X,Y) \sim P_{X|Y}P_Y$ the average entropy remaining in $X$ when given $Y$ is measured by **conditional entropy** which is defined as follows.

**Definition 4. Conditional Entropy.** Suppose that $(X,Y) \sim P_{X|Y}P_Y \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$. The conditional entropy of a discrete random variable $X$ given $Y$ is

$$H(X|Y) = \mathbb{E}[\hat{t}_{P_{X|Y}}(X|Y)]$$

(4)

$$= \sum_{b \in \mathcal{Y}} H(P_{X|Y=b}P_Y(b)).$$

(5)

Given two random variables $X$ and $\tilde{X}$ on the same alphabet $\mathcal{X}$, the information provided by the event $X = x$ relative to the information provided by $\tilde{X} = x$ is captured by **relative information**, whose definition is given below.

**Definition 5. Relative Information.** Let $P_X$ and $Q_X$ be two distributions in $\mathcal{P}(\mathcal{X})$, the relative information in $x \in \mathcal{X}$ according to $(P_X, Q_X)$ is

$$\hat{t}_{P_X\|Q_X}(x) = \log \frac{P_X(x)}{Q_X(x)}.$$ 

(6)

Although it neither satisfies symmetry nor the triangular inequality, widely used in probability theory, statistical inference, and physics, the expectation of the random variable $\hat{t}_{P_X\|Q_X}(X)$ when $X \sim P_X$ is a non-negative measure of distinctness between $P_X$ and $Q_X$. This expectation is **relative entropy**, defined as follows.

**Definition 6. Relative Entropy.** Suppose $P_X$ and $Q_X$ are two distributions in $\mathcal{P}(\mathcal{X})$ such that $P_X$ is absolutely continuous with respect to $Q_X$, i.e., $P_X \ll Q_X$. The relative entropy between $P_X$ and $Q_X$ is

$$D(P_X\|Q_X) = \mathbb{E}[\hat{t}_{P_X\|Q_X}(X)],$$

(7)

where $X \sim P_X$. If $P_X \not\ll Q_X$, then $D(P_X\|Q_X) = +\infty$.

Several key properties of the relative entropy, including but not limited to its non-negativity and convexity, can be found in standard information theory books such as [17, 18].

We define a conditional version of the relative entropy as below.

**Definition 7. Conditional Relative Entropy.** Let $P_Y \in \mathcal{P}(\mathcal{Y})$ and suppose that $P_{X|Y} : \mathcal{Y} \to \mathcal{X}$ and $Q_{X|Y} : \mathcal{Y} \to \mathcal{X}$ are two conditional distributions on the finite alphabet $\mathcal{X}$. The conditional relative entropy between $P_{X|Y}$ and $Q_{X|Y}$ given $Y \sim P_Y$ is defined as

$$D(P_{X|Y}\|Q_{X|Y}|P_Y) = D(P_{X|Y}P_Y\|Q_{X|Y}P_Y)$$

(8)
\[
\sum_{b \in \mathcal{Y}} P_Y(b) D(P_X|Y=b)\|Q_X|Y=b). \tag{9}
\]

As mentioned above, since \(D(P_X||Q_X)\) does not satisfy all of the metric axioms, it is not a proper measure of distance between \(P_X\) and \(Q_X\) in the topological sense. One such metric that measures topological distance between two distributions \(P_X\) and \(Q_X\) is total variation distance which is defined next.

**Definition 8. Total Variation Distance.** Suppose \(P_X\) and \(Q_X\) are two distributions in \(\mathcal{P}(\mathcal{X})\), the total variation distance\(^2\) (or \(\ell_1\)-distance) between \(P_X\) and \(Q_X\) is

\[
\|P_X - Q_X\|_1 = \sum_{x \in \mathcal{X}} |P_X(x) - Q_X(x)| \tag{10}
\]

\[
= 2 \sup_{A \subseteq \mathcal{X}} |P_X(A) - Q_X(A)|. \tag{11}
\]

Letting \(\mathcal{X}\) and \(\mathcal{Y}\) denote finite input and output alphabets, respectively, and using the standard notation \(a^n = (a_1, \ldots, a_n)\) to denote an \(n\)-dimensional array, a stationary discrete memoryless channel is defined through the sequence of random transformations as follows.

**Definition 9. Discrete Memoryless Channel.** Suppose that \(P_{Y|X}: \mathcal{X} \to \mathcal{Y}\) is a random transformation between the finite alphabets \(\mathcal{X}\) and \(\mathcal{Y}\). A stationary discrete memoryless channel with input and output alphabets, \(\mathcal{X}\) and \(\mathcal{Y}\), respectively, is a sequence of random transformations \(\{P_{Y^n|X^n}: \mathcal{X}^n \to \mathcal{Y}^n\}_{n=1}^{\infty}\) such that

\[
P_{Y^n|X^n}(y^n|x^n) = \prod_{i=1}^{n} P_{Y_i|X_i}(y_i|x_i), \tag{12}
\]

where for each \(i, P_{Y_i|X_i} = P_{Y|X}\).

If the input and the output of the stationary discrete memoryless channel are independent from each other, i.e., \(P_{Y^n|X^n} = P_{Y^n}\), then we call this channel a degenerate channel as it is impossible to communicate a meaningful message through it.

Assume that \(P_X \in \mathcal{P}(\mathcal{X})\), unless otherwise stated, the product distribution \(P_X^n \in \mathcal{P}(\mathcal{X}^n)\) denotes its independent identically distributed (i.i.d.) extension, i.e.,

\[
P_X^n(x^n) = \prod_{i=1}^{n} P_X(x_i), \tag{13}
\]

where \(X_i\) are i.i.d. according to \(P_X\). If we input an \(n\)-shot stationary discrete memoryless channel \(P_{Y^n|X^n}\) with \(X^n \sim P_X^n\), then at the output we get \(Y^n \sim P_{Y^n}\) where

\[
P_{Y^n}(y^n) = \sum_{x^n \in \mathcal{X}^n} P_X^n(x^n) P_{Y^n|X^n}(y^n|x^n). \tag{14}
\]

**Remark 1.** Throughout this paper, \(P_{X^n}\) and \(P_{Y^n}\) always denote the product distributions of \(P_X \in \mathcal{P}(\mathcal{X})\) and \(P_Y \in \mathcal{P}(\mathcal{Y})\), respectively.

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\(^2\)Also known as variational distance. Notice that our definition in (10) does not have the normalization factor of 1/2, and for this reason, given \(P_X, Q_X \in \mathcal{P}(\mathcal{X})\), we have \(0 \leq \|P_X - Q_X\|_1 \leq 2\). The main results of this work do not change if the normalization factor is included.
In what follows, we occasionally make use of the notation

\[ P^X_n \rightarrow P_{Y^n|X^n} \rightarrow P^Y_n \]

to indicate that the \( n \)-shot channel \( P_{Y^n|X^n} : \mathcal{X}^n \rightarrow \mathcal{Y}^n \) is inputted with a random variable \( X^n \) whose distribution is \( P^X_n \), and the resulting random variable \( Y^n \) at the output of the channel has distribution \( P^Y_n = \sum_{x^n \in \mathcal{X}^n} P^X_n(x^n)P_{Y^n|X^n}(\cdot|x^n) \). Indeed, \( P^X_n \rightarrow P_{Y^n|X^n} \rightarrow P^Y_n \) also defines a joint distribution \( P^{X^nY^n} = P^X_nP_{Y^n|X^n} \), and furthermore, it allows us to define a key quantity in information theory, namely the information density.

**Definition 10. Information Density.** Given \( P_X \rightarrow P_{Y|X} \rightarrow P_Y \), the information density of \((x,y) \in \mathcal{X} \times \mathcal{Y} \) is

\[ i_{X,Y}(x,y) = i_{P_{XY}}(x,y) \]

\[ = \log \frac{P_{Y|X}(y|x)}{P_Y(y)}. \]

Granted that the correlation between \( X \sim P_X \) and \( Y \sim P_Y \) is through \( P_X \rightarrow P_{Y|X} \rightarrow P_Y \), the expected value of the random variable \( i_{X,Y}(X,Y) \) is a measure of dependency between \( X \) and \( Y \), which gives rise to the definition of mutual information.

**Definition 11. Mutual Information.** Given \( P_X \rightarrow P_{Y|X} \rightarrow P_Y \), the mutual information of \((X,Y) \sim P_XP_{Y|X} \) is

\[ I(P_X,P_{Y|X}) = \mathbb{E}[i_{X,Y}(X,Y)] \]

\[ = D(P_{XY}||P_XP_Y) \]

\[ = D(P_{Y|X}||P_Y|P_X). \]

The heart of the proof in channel coding theorem, random i.i.d. coding ensemble can be defined as follows.

**Definition 12. Random (i.i.d.) Codebook.** Given \( P_X \in \mathcal{P}(\mathcal{X}) \), let \( P_{X^n} \in \mathcal{P}(\mathcal{X}^n) \) be its i.i.d. extension. A random (i.i.d.) codebook \( \mathcal{C}_M^n \) of size \( M \) and block-length \( n \) satisfies

\[ \mathcal{C}_M^n = \{X_1^n, \ldots, X_M^n\}, \]

where \( X_j^n \) are independently drawn from \( P_X^n \) for each \( j \in \{1, \ldots, M\} \).

Given a random codebook \( \mathcal{C}_M^n \), the distribution at the output of the channel induced by \( \mathcal{C}_M^n \) is defined next.

**Definition 13. Induced Output Distribution.** Given an \( n \)-shot stationary discrete memoryless channel \( P_{Y^n|X^n} : \mathcal{X}^n \rightarrow \mathcal{Y}^n \), let \( \mathcal{C}_M^n \) be the random codebook defined as in (20). Then, \( P_{Y^n|\mathcal{C}_M^n} \) denotes the induced output distribution when a uniformly chosen codeword from \( \mathcal{C}_M^n \) is transmitted through \( P_{Y^n|X^n} \). In
other words, for any \( y^n \in \mathcal{Y}^n \),
\[
P_{Y^n|X^n_M}(y^n) = \frac{1}{M} \sum_{j=1}^{M} P_{Y^n|X^n_j}(y^n|X^n_j),
\]
where \( X^n_j \sim P_{X^n} \) for each \( j \in \{1, \ldots, M\} \).

Remark 2. Due to its dependence on the random codebook \( \mathcal{C}^n_M \), \( P_{Y^n|X^n} \) is, in fact, a random variable.

Often times, it is combinatorially convenient to treat the sequences with identical empirical distributions on an equal footing. Given a sequence \( x^n \in \mathcal{X}^n \), its empirical distribution is called an \( n \)-type which we define as follows.

**Definition 14.** \( n \)-Type. For any positive integer \( n \), a probability distribution \( Q_{\bar{X}} \in \mathcal{P}(\mathcal{X}) \) is called an \( n \)-type if for any \( x \in \mathcal{X} \)
\[
Q_{\bar{X}}(x) \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, 1 \right\},
\]
and the set of all \( n \)-types is denoted by \( \mathcal{P}_n(\mathcal{X}) \subset \mathcal{P}(\mathcal{X}) \).

Remark 3. For \( m, k \in \mathbb{N} \), if \( Q_{\bar{X}} \) is an \( m \)-type, it is also an \( km \)-type.

Note that, see, e.g., [18, Problem 2.1], the exact number of \( n \)-types in \( \mathcal{X}^n \) is \( |\mathcal{P}_n(\mathcal{X})| = \binom{n+|\mathcal{X}|-1}{|\mathcal{X}|-1} \) which grows polynomially with \( n \). Since \( n \)-types play a significant role in our proofs, from this point onward, we reserve the overbar random variable notation for \( n \)-types. That is, for example, \( \bar{X} \sim Q_{\bar{X}} \) denotes a random variable whose distribution is an \( n \)-type \( Q_{\bar{X}} \in \mathcal{P}_n(\mathcal{X}) \). Similarly, \( (\bar{X}, \bar{Y}) \sim Q_{\bar{X}\bar{Y}} \) denotes a random variable whose distribution is a joint \( n \)-type \( Q_{\bar{X}\bar{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y}) \).

It is easy to see that given a sequence \( x^n = (x_1, \ldots, x_n) \in \mathcal{X}^n \) of block-length \( n \), its empirical distribution defines an \( n \)-type \( Q_{\bar{X}} \in \mathcal{P}_n(\mathcal{X}) \) as
\[
Q_{\bar{X}}(a) = \frac{1}{n} \sum_{i=1}^{n} 1\{a = x_i\}.
\]
Conversely, given an \( n \)-type \( Q_{\bar{X}} \in \mathcal{P}_n(\mathcal{X}) \), one can find a sequence \( x^n \in \mathcal{X} \) whose empirical distribution is \( Q_{\bar{X}} \). This gives rise to the following definition.

**Definition 15.** Type Class. Given an \( n \)-type \( Q_{\bar{X}} \in \mathcal{P}_n(\mathcal{X}) \), the subset \( \mathcal{T}_{Q_{\bar{X}}}^n \subset \mathcal{X}^n \) is called the the type class of \( Q_{\bar{X}} \), and it denotes the set of all \( x^n \in \mathcal{X}^n \) whose empirical distribution is \( Q_{\bar{X}} \).

To better understand the interplay of the joint sequences, the concept of conditional \( n \)-type will be required. Let
\[
\mathcal{P}(\mathcal{X}|\mathcal{Y}) = \{ P_{X|Y} : \mathcal{Y} \rightarrow \mathcal{X} \}
\]
denote the set of all random transformations\(^3\) from \( \mathcal{Y} \) to \( \mathcal{X} \).

\(^3\)Since both \( \mathcal{X} \) and \( \mathcal{Y} \) are finite alphabets, under the convention that probability distributions are column vectors, \( \mathcal{P}(\mathcal{X}|\mathcal{Y}) \) denotes the set of size \(|\mathcal{X}| \times |\mathcal{Y}|\) stochastic matrices.
Definition 16. Conditional Type. Given an n-type $Q_Y$, fix $y^n \in T^n_{Q_Y}$. A random transformation$^4$ $Q_{X|Y}: \mathcal{X} \rightarrow \mathcal{Y} \in \mathcal{P}(\mathcal{X}|\mathcal{Y})$ is called the conditional type of $x^n \in \mathcal{X}^n$ given $y^n$ if for any $(a,b) \in \mathcal{X} \times \mathcal{Y}$

$$Q_{X|Y}(a,b) = Q_{X|Y}(a|b)Q_Y(b),$$

where $Q_{X|Y}$ denotes the joint n-type of $(x^n,y^n)$.

Remark 4. Note that if $Q_Y(b) = 0$ for some $b \in \mathcal{Y}$, then $Q_{X|Y}(a,b) = 0$ for any $a \in \mathcal{X}$ and $Q_{X|Y}(-|b)$ is not defined. If $Q_Y(b) > 0$, then $Q_{X|Y}(-|b)$ is a t-type where $t = nQ_Y(b)$ is the number of times $b$ appears in $y^n$.

Given a fixed $y^n \in T^n_{Q_Y}$, the joint type $Q_{X|Y}$ of the sequence $(x^n,y^n)$ can be determined by the conditional type $Q_{X|Y}$ of $x^n$ given $y^n$, in which case $Q_{X|Y} = Q_{X|Y}Q_Y$. As this concept is utilized throughout this paper, a notation for the set of all conditional types is in order.

Definition 17. Set of Conditional Types. Given an n-type $Q_Y \in \mathcal{P}_n(\mathcal{Y})$, $\mathcal{P}_n(\mathcal{X}|Q_Y)$ denotes$^5$ the set of all conditional types given $y^n \in T^n_{Q_Y}$.

Remark 5. As suggested by our careful choice of notation, it is easy to see that $\mathcal{P}_n(\mathcal{X}|Q_Y)$ depends on $y^n \in T^n_{Q_Y}$ only through its type $Q_Y$.

Remark 6. With Definition 17 at hand, the set of the joint n-types on $\mathcal{X} \times \mathcal{Y}$ can be written as the disjoint union over n-types $\mathcal{P}_n(\mathcal{Y})$ of the right $Q_Y$ coset$^6$ of the set of conditional types $\mathcal{P}_n(\mathcal{X}|Q_Y)$. That is, borrowing the coset notation from algebra,

$$\mathcal{P}_n(\mathcal{X} \times \mathcal{Y}) = \bigsqcup_{Q_Y \in \mathcal{P}_n(\mathcal{Y})} \mathcal{P}_n(\mathcal{X}|Q_Y) \times Q_Y,$$

where the notation $\bigsqcup$ emphasizes that the unionization is disjoint.

It is straightforward that given $y^n \in T^n_{Q_Y}$, the empirical distribution of $x^n \in \mathcal{X}^n$ in comparison with $y^n$ defines a conditional type $Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_Y)$ as

$$Q_{X|Y}(a|b) = \frac{1}{nQ_Y(b)} \sum_{i=1}^{nQ_Y(b)} 1\{(x_i, y_i) = (a, b)\}.$$  

Conversely, suppose we have a conditional type $Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_Y)$ given $y^n \in T^n_{Q_Y}$, we can construct a sequence $x^n \in \mathcal{X}^n$ whose empirical distribution in comparison with $y^n$ is $Q_{X|Y}$. This gives rise to the definition of conditional type class.

Definition 18. Conditional Type Class. Let $Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_Y)$ be a conditional type given $y^n \in T^n_{Q_Y}$, the subset $T^n_{Q_{X|Y}}(y^n)$ is called the conditional type class of $Q_{X|Y}$ given $y^n$, and it denotes the set of all $x^n \in \mathcal{X}^n$ whose empirical distribution in comparison with $y^n$ is $Q_{X|Y}$.

$^4$Under the convention of Footnote 3, a stochastic matrix of dimension $|\mathcal{X}| \times |\mathcal{Y}|$.

$^5$The subscript $n$ in the notation $\mathcal{P}_n(\mathcal{X}|Q_Y)$ is to denote that $Q_Y$ is an n-type. Elements of $\mathcal{P}_n(\mathcal{X}|Q_Y)$ are conditional types, which are not necessarily n-types, see Remark 4.

$^6$Abuse of terminology. $\mathcal{P}_n(\mathcal{X}|Q_Y)$ does not have a group structure.
Remark 7. The size of the conditional type class, namely $|T_{Q_{X,Y}^{n}}(y^n)|$, depends on $y^n$ only through its type. This is because shuffling the order of terms in which they appear in $y^n$, one can always shuffle $x^n$ in the same manner preserving the conditional type of $x^n$ given $y^n$.

Using the established familiarity with types, a random constant-composition codebook can be defined as follows.

**Definition 19. Random Constant-Composition Codebook.** For a fixed integer $m$, suppose we are given an $m$-type $P_X \in \mathcal{P}_m(\mathcal{X})$. Let $n$ be a multiple of $m$ (i.e., $n \in m\mathbb{N}$) and define a constant-composition distribution on $\mathcal{X}^n$ based on $P_X$ as

$$R_{\mathcal{X}^n}(x^n) = \frac{1}{|T_{P_X}^n|} \mathbb{1}\left\{ x^n \in T_{P_X}^n \right\}. \tag{28}$$

Then, a random constant-composition codebook of size $M$, and block-length $n$, that is based on $P_X$ is defined as

$$\mathcal{D}_M^n = \left\{ \hat{X}_1^n, \ldots, \hat{X}_M^n \right\}, \tag{29}$$

where $\hat{X}_j^n$ are pairwise independent and identically distributed with $R_{\mathcal{X}^n}$, for each $j \in \{1, \ldots, M\}$.

Remark 8. Each codeword in $\mathcal{D}_M^n$ has the same $m$-type $P_X$ as they are taken uniformly at random from the type class $T_{P_X}^n$, hence the name constant-composition.

Remark 9. In the constant-composition case, $m$ is always fixed and $n$ is always a multiple of $m$. This ensures that the type class $T_{P_X}^n$ is a well-defined non-empty set as the $m$-type $P_X$ is also an $n$-type, see Remark 3.

Remark 10. Throughout this paper, the distributions with breve accent “˘” either denote constant-composition distributions or denote output distributions that are induced by constant-composition distributions. That is, unlike $P_{\mathcal{X}^n} \in \mathcal{P}(\mathcal{X}^n)$, or $P_{\mathcal{Y}^n} \in \mathcal{P}(\mathcal{Y}^n)$; $R_{\mathcal{X}^n} \in \mathcal{P}(\mathcal{X}^n)$, nor $R_{\mathcal{Y}^n} \in \mathcal{P}(\mathcal{Y}^n)$, is not a product distribution.

Given a random constant-composition codebook $\mathcal{D}_M^n$, the constant-composition induced output distribution $P_{\mathcal{Y}^n|\mathcal{D}_M^n}$, in other words, the distribution induced by $\mathcal{D}_M^n$ at the channel output, is defined as follows.

**Definition 20. Constant-Composition Induced Output Distribution.** Given an $n$-shot stationary discrete memoryless channel $P_{\mathcal{Y}^n|\mathcal{X}^n}: \mathcal{X}^n \rightarrow \mathcal{Y}^n$, let $\mathcal{D}_M^n$ be a random constant-composition codebook defined as in (29). Then, $P_{\mathcal{Y}^n|\mathcal{D}_M^n}$ denotes the constant-composition induced output distribution when a uniformly chosen codeword from $\mathcal{D}_M^n$ is transmitted through $P_{\mathcal{Y}^n|\mathcal{X}^n}$. In other words, for any $y^n \in \mathcal{Y}^n$,

$$P_{\mathcal{Y}^n|\mathcal{D}_M^n}(y^n) = \frac{1}{M} \sum_{j=1}^{M} P_{\mathcal{Y}^n|\mathcal{X}^n}(y^n|\hat{X}_j^n), \tag{30}$$

where for each $j \in \{1, \ldots, M\}$ the random variable $\hat{X}_j^n$ is distributed according to a constant-composition distribution $R_{\mathcal{X}^n}$ that is based on an $m$-type $P_X \in \mathcal{P}_m(\mathcal{X})$, namely $\hat{X}_j^n \sim R_{\mathcal{X}^n}$ as in (28).

Remark 11. Similar to $P_{\mathcal{Y}^n|\mathcal{D}_M^n}$, due to its dependence on the random (constant-composition) codebook $\mathcal{D}_M^n$, $P_{\mathcal{Y}^n|\mathcal{D}_M^n}$ is, indeed, a random variable.
III. Exact Soft-Covering Exponent

We begin by citing [16, Theorem 31] which establishes that the total variation distance between the induced output distribution $P_{Y^n|X^n}$ and the desired output distribution $P_Y^n$, in contrast to the relative entropy or the Rényi divergence between the two, has a concentration property. As the block-length $n$ increases, the total variation distance between these two distributions (a random quantity, due to the randomness of the codebook) concentrates tightly to its exponentially vanishing expected value with double-exponential certainty.

**Lemma 1.** Suppose $P_{X^n} \rightarrow P_{Y^n|X^n} \rightarrow P_{Y^n}$ and denote by $P_{Y^n|X^n}(y^n)$ the induced output distribution when a uniformly chosen codeword from the random (i.i.d.) codebook $C^n_M$ is transmitted through the channel $P_{Y^n|X^n}$, see Definitions 12 and 13. Then, for any $t > 0$,

$$
\mathbb{P} \left[ \left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 - \mathbb{E} \left[ \left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 \right] \geq t \right] \leq 2 \exp \left( -\frac{Mt^2}{2} \right). \tag{31}
$$

Predictably, replacing the random (i.i.d.) codebook $C^n_M$ with a random constant-composition codebook $\mathcal{R}^n_M$ in Lemma 1 and looking at the total variation distance between the constant-composition induced output distribution $P_{Y^n|\mathcal{R}^n_M}$ and the desired output distribution $R_Y^n$, we see that the same concentration property holds:

**Lemma 2.** Suppose $R_{X^n} \rightarrow P_{Y^n|X^n} \rightarrow R_Y^n$ and denote by $P_{Y^n|\mathcal{R}^n_M}(y^n)$ the induced output distribution when a uniformly chosen codeword from the random (constant-composition) codebook $\mathcal{R}^n_M$ is transmitted through the channel $P_{Y^n|X^n}$, see Definitions 19 and 20. Then, for any $t > 0$,

$$
\mathbb{P} \left[ \left\| P_{Y^n|\mathcal{R}^n_M} - R_Y^n \right\|_1 - \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{R}^n_M} - R_Y^n \right\|_1 \right] \geq t \right] \leq 2 \exp \left( -\frac{Mt^2}{2} \right). \tag{32}
$$

The main results of this paper, stated in Theorems 1 and 2, give the exact asymptotic exponential decay rate of the expected total variation distance between the induced distribution $P_{Y^n|X^n}$ (respectively, $P_{Y^n|\mathcal{R}^n_M}$) and the desired output distribution $P_Y^n$ (respectively, $R_Y^n$).

**Theorem 1.** Exact Soft-Covering Exponent (i.i.d.). Suppose $P_{X^n} \rightarrow P_{Y^n|X^n} \rightarrow P_{Y^n}$, where the $n$-shot stationary memoryless channel $P_{Y^n|X^n}$ is non-degenerate, i.e., $P_{Y^n|X^n} \neq P_Y^n$. For any $R > I(P_X, P_Y|X)$, let $M = \lceil \exp(nR) \rceil$, and denote by $P_{Y^n|\mathcal{C}^n_M}(y^n)$ the induced output distribution when a uniformly chosen codeword from the random codebook $\mathcal{C}^n_M$ is transmitted through the channel, see Definitions 12 and 13. Then,

$$
\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{C}^n_M} - P_Y^n \right\|_1 \right] = \min_{Q_{XY}} \left\{ D(Q_{XY}\|P_{XY}) + \frac{1}{2} [R - D(Q_{XY}\|P_XQ_Y)]_+ \right\}, \tag{33}
$$

$$
\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{R}^n_M} - P_Y^n \right\|_1 \right] = \max_{\lambda \in [1,2]} \left\{ \frac{\lambda - 1}{\lambda} \left( R - I_\lambda(P_X, P_Y|X) \right) \right\}. \tag{34}
$$
where in (33) \( [f]_+ = \max\{0, f\} \) and the optimization is carried over all distributions \( Q_{XY} \) that are absolutely continuous with respect to \( P_{XY} = P_X P_{Y|X} \), i.e., \( Q_{XY} \ll P_{XY} \); and in (34) \( I_\lambda(P_X, P_{Y|X}) \) denotes the \( \alpha \)-mutual information\(^7\) of order \( \lambda \).

**Theorem 2.** Exact Soft-Covering Exponent (constant-composition). Let \( m \) be a fixed integer and \( P_X \in \mathcal{P}_m(\mathcal{X}) \) be a fixed \( m \)-type. For \( n \in m\mathbb{N} \), suppose that \( R_{X^n} \) is a constant-composition distribution based on \( P_X \) defined as in (28), and let \( R_{X^n} \rightarrow P_{Y^n|X^n} \rightarrow R_{Y^n} \), where the \( n \)-shot stationary discrete memoryless channel \( P_{Y^n|X^n} \) is non-degenerate, i.e., \( P_{Y^n|X^n} \neq R_{Y^n} \). For any \( R > I(P_X, P_{Y|X}) \), let \( M = [\exp(nR)] \), and denote by \( P_{Y^n|\mathcal{Q}_M^n} \) the constant-composition induced output distribution when a uniformly chosen codeword from the random constant-composition codebook \( \mathcal{Q}_M^n \) is transmitted through the channel, see Definitions 19 and 20. Then,

\[
\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{E}\left[ \|P_{Y^n|\mathcal{Q}_M^n} - R_{Y^n}\|_1 \right] = \min_{Q_{Y|X}} \left\{ D(P_X Q_{Y|X} \| P_{XY}) + \frac{1}{2} [R - D(P_X Q_{Y|X} \| P_{XY})]_+ \right\} \tag{35}
\]

\[
= \max_{\lambda \in [1, 2]} \max_{S_Y} \left\{ \frac{\lambda - 1}{\lambda} R - \mathbb{E} \left[ \log \mathbb{E}^{\frac{1}{2}} \left[ \exp \left( (\lambda - 1) I_{P_{XY}} \parallel P_X S_Y (X, Y) | X \right) \right] X \right] \right\}, \tag{36}
\]

where in (35) \([f]_+ = \max\{0, f\}\), \( P_X \rightarrow Q_{Y|X} \rightarrow Q_Y \) and the optimization is carried over all random transformations \( Q_{Y|X} \) such that \( P_X Q_{Y|X} \ll P_X P_{Y|X} = P_{XY} \); and in (36) \((X, Y) \sim P_{XY} \).

Some remarks are in order.

**Remark 12.** To make it easier to refer, assuming \( R > I(P_X, P_{Y|X}) > 0 \), define

\[
\alpha(R, P_X, P_{Y|X}) = \min_{Q_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY} \parallel P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \parallel P_X Q_Y)]_+ \right\} \tag{37}
\]

\[
= \max_{\lambda \in [1, 2]} \frac{\lambda - 1}{\lambda} \left( R - I_\lambda(P_X, P_{Y|X}) \right), \tag{38}
\]

where the minimization in (37) is over all joint distributions on \( \mathcal{X} \times \mathcal{Y} \). Similarly, assuming \( R > I(P_X, P_{Y|X}) > 0 \), define

\[
\kappa(R, P_X, P_{Y|X}) = \min_{Q_{Y|X} \in \mathcal{P}(\mathcal{Y}|X)} \left\{ D(P_X Q_{Y|X} \parallel P_{XY}) + \frac{1}{2} [R - D(P_X Q_{Y|X} \parallel P_X Q_Y)]_+ \right\} \tag{39}
\]

\[
= \max_{\lambda \in [1, 2]} \max_{S_Y \in \mathcal{P}(\mathcal{Y})} \left\{ \frac{\lambda - 1}{\lambda} R - \mathbb{E} \left[ \log \mathbb{E}^{\frac{1}{2}} \left[ \exp \left( (\lambda - 1) I_{P_{XY}} \parallel P_X S_Y (X, Y) | X \right) \right] X \right] \right\}, \tag{40}
\]

where the minimization in (39) is over all random transformations from \( \mathcal{X} \) to \( \mathcal{Y} \), and the inner maximization in (40) is over all distributions on \( \mathcal{Y} \).

**Remark 13.** Perhaps surprisingly, the proof of Lemma 1, which can be found in [16, Theorem 31], easily follows from McDiarmid’s inequality [19, Theorem 2.2.3]. As Lemma 1 is an integral part of the spirit of this paper, Appendix I-A repeats its simple proof. Contained in Appendix I-B, the proof of Lemma 2 follows the footprints of that of Lemma 1.

\(^7\)See Remark 23.
Remark 14. By further assuming that the codebooks $\mathcal{C}_M^n$ and $\mathcal{D}_M^n$ contain a random number of codewords $M$, thanks to the total probability law, it is possible to get corollaries to the results of Lemmas 1 and 2. Indeed, two such examples, which assume that $M$ is Poisson distributed and are useful in the proofs of the upper bounds in Theorems 1 and 2, are provided in Lemmas 11 and 17 in Appendices II-A and II-B, respectively.

Remark 15. In order to provide a better presentation, the proof of Theorem 1 is divided into three parts, which can be found in Sections IV, V and VI-A. In proving the lower bound direction in (33), see Section IV, the key steps are the use of the type method and an upper bound on the absolute mean deviation of a binomial distribution in terms of its mean and standard deviation. To prove the upper bound direction, on the other hand, the biggest problem turns out to be dealing with the weakly dependent binomial random variables, see Section V. To solve this weak dependence puzzle, first, the codebook size $M$ is treated as if it were a Poisson distributed random variable with mean $\mu_n = \exp(nR)$. This surplus assumption on the codebook size grants the desired independence property and provides the gateway to prove the pseudo-upper bound in the case when $M$ is Poisson distributed. Then, to prove the upper bound to the original problem where $M$ is deterministically equals to $\lceil \exp(nR) \rceil$, the extra Poisson assumption is removed by conditioning on $M = \lceil \exp(nR) \rceil$ and the result provided by Lemma 1 is enjoyed. As for the proof of the dual representation of the exact soft-covering exponent in (34), see Section VI-A, the main tools are provided by Lemma 30 and several corollaries that follow, all of which are contained in Appendix VIII.

Remark 16. In presenting the proof of Theorem 2, much effort has been made to make the proof look identical to that of Theorem 1. Still, there are key differences between these two proofs, which is why neither theorem is a corollary of the other. One example to these key differences is that, in the case of Theorem 2, in applying the type method, one needs to keep in mind that $\mathcal{X}$-marginal of the joint types is fixed to be $P_{\bar{X}}$, whereas this is not the case in the proof of Theorem 1. Another key difference is that, in the case of Theorem 2, the codewords of the random constant-composition codebook $\mathcal{D}_M^n$ are distributed according to the non-product distribution $R_{\bar{X}^n}$, while the codewords of the random (i.i.d.) codebook $\mathcal{C}_M^n$ are distributed according to the product distribution $P_{X^n}$. Luckily, using a minimalist approach, it is possible to emphasize the similarities in the techniques used. To do so, while proving Theorem 1 in Sections IV and V, several remarks have been made to convince the reader in regard to Theorem 2 without having them refer to its entire proof. Though, for the sake of completeness, the full proof of Theorem 2 can be found in Appendix VI. The proof for the equivalence of the primal and dual forms of the exact constant-composition soft-covering exponent, namely (36), can be found in Section VI-B.

Remark 17. The result of Theorem 1 can alternatively be interpreted as the exact random coding exponent for resolvability. Note, however, that we are not claiming to have found “the” exact resolvability exponent.\footnote{Also see [20], which discusses the lower bound in (34) without showing the equivalence in (33). Note that our proof technique in Section IV is far simpler to follow.}
Finding the exact resolvability exponent is a hard problem as it requires the search over all sequences of codes. Here, we restrict ourselves to random codebooks, as are typically used in achievability proofs (e.g. wiretap channels) where soft covering may be only one of several objectives. This choice of focus has a side benefit of finding the exact exponent.

Remark 18. As it is evident from the upper bound in (124) in Section V, \( \alpha(R, P_X, P_{Y|X}) \) is the best possible soft-covering exponent. Sections VII-A and VII-B confirm that \( \alpha(R, P_X, P_{Y|X}) \) provides an upper bound to the previously known lower bounds on the soft-covering exponent contained in [6, Theorem 6], and [7, Lemma VII.9].

Remark 19. Parizi et al. [14, Theorem 4] provide the soft-covering exponent (both i.i.d. and constant-composition cases) when the relative entropy rather than total variation is used as the measure of distinctness. While Pinsker’s and Jensen’s inequalities immediately imply that the half of the exponents in [14] are lower bounds on the soft-covering exponents in Theorems 1 and 2, as shown in Sections VII-C and VII-D, \( \alpha(R, P_X, P_{Y|X}) \) and \( R(R, P_X, P_{Y|X}) \) are greater than the half of their respective counterparts in [14, Theorem 4].

Remark 20. From the proofs provided, it is possible to deduce the following finite block-length results, see Theorems 3 and 4 in Appendix VII:

\[
\alpha_n(R, P_X, P_{Y|X}) - \kappa_n \leq -\frac{1}{n} \log E \left[ \| P_{Y^n|X^n} - P_{Y^n} \|_1 \right] \leq \alpha_n(R, P_X, P_{Y|X}) + \nu_n, \tag{41}
\]

\[
\underline{\alpha}_n(R, P_X, P_{Y|X}) - \overline{\eta}_n \leq -\frac{1}{n} \log E \left[ \| P_{Y^n|X^n} - R_{Y^n} \|_1 \right] \leq \underline{\alpha}_n(R, P_X, P_{Y|X}) + \overline{\nu}_n, \tag{42}
\]

where

\[
\alpha_n(R, P_X, P_{Y|X}) = \min_{Q_{XY} \in P_n(X \times Y)} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \| P_X Q_Y)]_+ \right\}, \tag{43}
\]

\[
\underline{\alpha}_n(R, P_X, P_{Y|X}) = \min_{Q_{Y|X} \in P_n(Y|X)} \left\{ D(P_{XY} Q_{Y|X} \| P_{XY}) + \frac{1}{2} [R - D(P_{XY} Q_{Y|X} \| P_X Q_Y)]_+ \right\}, \tag{44}
\]

and among the vanishing constants \( \kappa_n, \nu_n, \overline{\eta}_n, \overline{\nu}_n \), the ones in the lower bounds in (41) and (42), i.e., \( \kappa_n \) and \( \overline{\eta}_n \), depend only on the block-length \( n \) and the alphabet sizes \(|X|\) and \(|Y|\), while the ones in the upper bounds, i.e., \( \nu_n \) and \( \overline{\nu}_n \), additionally depend mildly\(^9\) on \( P_X, P_X \) and \( P_{X|Y} \). The definitions of these vanishing constants, along with the proofs of (41) and (42), are contained in Appendix VII.

Remark 21. In the degenerate channel case, i.e., when channel input and output are independent from each other, we have \( P_{Y^n|X^n} = P_{Y^n} \) (\( P_{Y^n|X^n} = R_{Y^n} \) in the constant-composition codes setting) and

\[
E \left[ \| P_{Y^n|X^n} - P_{Y^n} \|_1 \right] = 0, \tag{45}
\]

\[
E \left[ \| P_{Y^n|X^n} - R_{Y^n} \|_1 \right] = 0. \tag{46}
\]

\(^9\)Also see Remark 47.
In an allegorical spirit, one can say that the exact soft-covering exponents are \(\infty\) in this case. Although, it should be noted that (33), (34), (35), and (36) become inconclusive. A similar discontinuity occurs in the case when the distinctness measure is relative entropy instead of total variation distance, see [14, Theorem 4]. In our case, the reason for these discontinuities is captured in (129) and (450) in the upper bound proofs.

**Remark 22.** The optimization in (37) can also be written as

\[
\alpha(R, P_X, P_{Y|X}) = \min_{Q_{XY}} \left\{ D(Q_Y \| P_Y) + \min_{Q_{X|Y}} \left\{ D(Q_{X|Y} \| P_{X|Y}|Q_Y) + \frac{1}{2} [R - D(Q_{X|Y} \| P_X|Q_Y)]_+ \right\} \right\}.
\]

As shown in Appendix V, without loss of optimality, the inner minimization can be constrained to be over the random transformations \(Q_{X|Y}\) satisfying \(D(Q_{X|Y} \| P_{X|Y}) \geq D(P_{X|Y} \| P_X|Q_Y)\).

**Remark 23.** In the optimization in the right side of (38)

\[
I_\alpha(P_X, P_{Y|X}) = \frac{\lambda}{\lambda - 1} \log E \left[ \exp(\lambda E_{X,Y}(X; Y)) \right] = \frac{\lambda}{\lambda - 1} \log \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P_X(x) P_{Y|X}^\lambda(y|x) \right)^{\frac{1}{\lambda}}
\]

is the \(\alpha\)-mutual information of order \(\lambda\) as defined by Sibson [21]. Its more general definition, basic properties, relation to the other variations of \(\alpha\)-mutual information, and connection to Gallager error exponent function [22, Eq. (5.6.14)] are explored in [23].

**Remark 24.** Given an arbitrary non-degenerate channel \(P_{Y|X}: \mathcal{Y} \rightarrow \mathcal{X}\), and an \(m\)-type \(P_X \in \mathcal{T}_m(\mathcal{X})\) as the input distribution, proving \(\alpha \leq \aleph\) is simple:

\[
\alpha(R, P_X, P_{Y|X}) = \min_{Q_{XY}} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \| P_X|Q_Y)]_+ \right\}
\]

\[
= \min_{Q_{XY} \in \mathcal{X}} \left\{ \min_{Q_X \in \mathcal{X}} \left\{ D(Q_X \| P_X) + \frac{1}{2} [R - D(Q_X \| P_X)]_+ \right\} \right\}
\]

\[
\leq \min_{Q_{XY} \in \mathcal{X}} \left\{ D(P_X \| Q_{XY}) + \frac{1}{2} [R - D(P_X \| Q_{XY})]_+ \right\}
\]

\[
= \aleph(R, P_X, P_{Y|X}),
\]

where (52) follows from the suboptimal choice of \(Q_X = P_X\). Though, as the following example illustrates, there are settings for which \(\alpha < \aleph\).

**Example 1.** *Binary Symmetric Channel.* Suppose \(\mathcal{X} = \mathcal{Y} = \{0, 1\}\), and let \(P_{Y|X}: \mathcal{X} \rightarrow \mathcal{Y}\) be a binary symmetric channel with crossover probability \(p = 0.05\ [17, \text{Section 7.1.4}]\). If \(P_X(0) = 2/5\, and \(R = 0.85 > I(P_X, P_{Y|X}) \approx 0.69\) bits,

\[
\alpha(0.85, P_X, P_{Y|X}) \approx 2.0429 \times 10^{-2},
\]

\[
\aleph(0.85, P_X, P_{Y|X}) \approx 2.2216 \times 10^{-2},
\]
implying \( \alpha \neq \infty \), in general.

Figure 1 below depicts the gap between \( \alpha(R, P_X, P_{Y|X}) \) and \( \aleph(R, P_X, P_{Y|X}) \). While Figure 1a illustrates Example 1 for different values of rate \( R \), Figure 1b illustrates the case for Binary Z-Channel with the same input distribution \( P_X \) and the same error probability \( p = 0.05 \).

(a) BSC: When \( R - I(P_X, P_{Y|X}) \in [0.001, 0.26] \).

(b) BZC: When \( R - I(P_X, P_{Y|X}) \in [0.001, 0.26] \).

Figure 1: Comparison between \( \alpha(R, P_X, P_{Y|X}) \) and \( \aleph(R, P_X, P_{Y|X}) \).

Remark 25. Suppose \( P_X \rightarrow P_{Y|X} \rightarrow P_Y \). Let \((\tilde{X}, Y) \sim P_X P_{Y|X} \) and \((\tilde{X}, Y) \sim P_{\tilde{X}} P_Y \)

\[
\aleph(R, P_X, P_{Y|X}) = \max_{\lambda \in [1,2]} \max_{S_Y} \left\{ \frac{\lambda - 1}{\lambda} R - \mathbb{E} \left[ \log \mathbb{E}^{\frac{1}{\lambda}} \left[ \exp \left( (\lambda - 1) t_{P_{X|Y}} P_{S_Y} (\tilde{X}, Y) \right) | \tilde{X} \right] \right] \right\} \tag{56}
\]

\[
\geq \max_{\lambda \in [1,2]} \max_{S_Y} \left\{ \frac{\lambda - 1}{\lambda} R - \mathbb{E} \left[ \mathbb{E}^{\frac{1}{\lambda}} \left[ \exp \left( (\lambda - 1) t_{P_{X|Y}} P_{S_Y} (\tilde{X}, Y) \right) | \tilde{X} \right] \right] \right\} \tag{57}
\]

\[
\geq \max_{\lambda \in [1,2]} \left\{ \frac{\lambda - 1}{\lambda} R - \log \mathbb{E} \left[ \mathbb{E}^{\frac{1}{\lambda}} \left[ \exp \left( (\lambda - 1) t_{P_{X|Y}} P_X (\tilde{X}, Y) \right) | \tilde{X} \right] \right] \right\} \tag{58}
\]

\[
= \max_{\lambda \in [1,2]} \left\{ \frac{\lambda - 1}{\lambda} R - \log \mathbb{E} \left[ \mathbb{E}^{\frac{1}{\lambda}} \left[ \exp \left( (\lambda - 1) t_{P_{X|Y}} P_X (\tilde{X}, Y) \right) | \tilde{X} \right] \right] \right\} \tag{59}
\]

\[
= \max_{\lambda \in [1,2]} \left\{ \frac{\lambda - 1}{\lambda} \left( R - I_{\lambda}(P_Y, P_X) \right) \right\} \tag{60}
\]

\[= \alpha(R, P_Y, P_{X|Y}), \tag{61}\]

where (57) follows from Jensen’s inequality; (58) follows from the suboptimal choice of \( S_Y = P_Y \); (59) follows from the change of measure property; and finally, in (60) the reverse channel \( P_{X|Y} \) is such that \( P_Y \rightarrow P_{X|Y} \rightarrow P_X \) and the equality follows from the definition\(^{10}\) of \( \alpha \)-mutual information in (48).

Together with (53), (61) implies that

\[
\aleph(R, P_X, P_{Y|X}) \geq \max \left\{ \alpha(R, P_X, P_{Y|X}), \alpha(R, P_Y, P_{X|Y}) \right\}. \tag{62}
\]

\(^{10}\)Warning: In general, \( \alpha \)-mutual information is not a symmetric information measure [23, Example 4]. Hence, \( I_{\lambda}(P_Y, P_{X|Y}) \neq I_{\lambda}(P_X, P_{Y|X}) \).
Remark 26. If \( \bar{P}_X \in \mathcal{P}_m(\mathcal{X}) \) is such that

\[
\lim_{m \to \infty} P_{\bar{X}} = P_X,
\]

for some \( P_X \in \mathcal{P}(\mathcal{X}) \), then, assuming \( R > I(P_{\bar{X}}, P_{Y|X}) > 0 \) for all \( m \in \mathbb{N} \), it is straightforward to see that \( \kappa(R, P_{\bar{X}}, P_{Y|X}) \) is sequentially continuous in \( P_{\bar{X}} \). That is,

\[
\lim_{m \to \infty} \kappa(R, P_{\bar{X}}, P_{Y|X}) = \kappa(R, P_X, P_{Y|X}).
\]

Remark 27. Regarding the computation of the exact soft-covering exponents \( \alpha \) and \( \kappa \), the dual forms in (38) and (40) are far easier to calculate than their primal counterparts in (37) and (39). This is because, in calculating the former pair, the optimizations are carried over spaces of dimensions 1, and \(|\mathcal{Y}| \), respectively, whereas in calculating the latter pair the optimizations are carried over spaces of dimensions \(|\mathcal{X}||\mathcal{Y}| - 1 \) and \(|\mathcal{X}|(|\mathcal{Y}| - 1) \), respectively.

Remark 28. Taylor expansion of \( I_\lambda(P_X, P_{Y|X}) \) around \( \lambda = 1 \) yields

\[
I_\lambda(P_X, P_{Y|X}) = I(P_X, P_{Y|X}) + \frac{1}{2} \text{Var}[i_{X,Y}(X;Y)] \left( (\lambda - 1) + O((\lambda - 1)^2) \right),
\]

where \( (X,Y) \sim P_X P_{Y|X} \), and \( \text{Var}[i_{X,Y}(X;Y)] \) denotes the variance of \( i_{X,Y}(X;Y) \). Hence, when \( R = I(P_X, P_{Y|X}) + \epsilon \) for some small\(^{12} \) \( \epsilon \),

\[
\alpha(R, P_X, P_{Y|X}) = \max_{\lambda \in [1,2]} \left\{ \frac{\lambda - 1}{\lambda} \left( R - I_\lambda(P_X, P_{Y|X}) \right) \right\},
\]

where the maximum in the right side of (67) is achieved when \( \lambda = 1 + 2 \epsilon \text{Var}^{-1}[i_{X,Y}(X;Y)] \)^\( \frac{1}{2} \). For the sake of simplicity, suppose \( \lambda = 1 \) in the denominator of the right hand side of (67), the approximate maximizer becomes \( \lambda \approx 1 + \epsilon \text{Var}^{-1}[i_{X,Y}(X;Y)] \) and (68) follows.

Remark 29. In a similar spirit to Remark 28, Taylor expansion of \( E[\log E[\exp((\lambda - 1) i_{P_{XY} \| P_X S_Y} (\bar{X}, Y))|\bar{X}]]\) around \( \lambda = 1 \) yields

\[
E[\log E[\exp((\lambda - 1) i_{P_{XY} \| P_X S_Y} (\bar{X}, Y))|\bar{X}]] = (\lambda - 1) D(P_{XY} \| P_X S_Y) + \frac{1}{2} (\lambda - 1)^2 \text{Var}[i_{P_{XY} \| P_X S_Y} (\bar{X}, Y)] + O((\lambda - 1)^3),
\]

\(^{11}\) If \( P_X \) is a capacity-achieving distribution, then \( \text{Var}[i_{X,Y}(X;Y)] \) is a property of the channel known as the channel dispersion \cite{24}. \(^{12}\) When \( R = I(P_X, P_{Y|X}) + \epsilon \), since \( I_\lambda(P_X, P_{Y|X}) \) is non-decreasing in \( \lambda \), the maximum in (66) is achieved at a \( \lambda \) value that is close to 1.
where \((\bar{X}, Y) \sim P_X P_{Y|X}\). Hence, when \(R = I(P_X, P_{Y|X}) + \epsilon\) for some small \(\epsilon\),

\[
\mathbb{N}(R, P_X, P_{Y|X}) \approx \max_{\lambda \in [1, 2]} \max_{S_Y} \left\{ \frac{\lambda-1}{\lambda} \left( R - D(P_{XY} \parallel P_X S_Y) - \frac{\lambda-1}{2} \operatorname{Var} \left[ \log \left( \frac{1}{P_{XY}^{P_X S_Y} (\bar{X}, Y)} \right) \right] \right) \right\}
\]

\[
\approx \max_{\lambda \in [1, 2]} \left\{ \frac{\lambda-1}{\lambda} \left( \epsilon - \frac{\lambda-1}{2} \operatorname{Var} \left[ t_{X;Y} (X;Y) \right] \right) \right\}
\]

\[
\approx \frac{\epsilon}{2} \operatorname{Var}^{-1} \left[ t_{X;Y} (X;Y) \right]
\]

\[
\approx \alpha(R, P_X, P_{Y|X}),
\]

which can also be observed in Figure 1.

**Remark 30.** Observing that \(f_\lambda(x) = x^{\frac{2-\lambda}{2}}\) is concave in \(x\) for any \(\lambda \in [0, 1]\), Jensen’s inequality implies

\[
D_{1 + \frac{\lambda}{2}}(P_{XY} \parallel P_X P_Y) \leq I_{\frac{2}{\lambda}}(P_X, P_{Y|X})
\]

\[
= \frac{\lambda}{2} \log \mathbb{E} \left[ \left( \mathbb{E} \left[ \exp \left( \frac{\lambda}{2-\lambda} t_{X;Y} (X;Y) \right) \right] Y \right)^{2-\lambda} \right]
\]

\[
\leq D_{1 + \frac{\lambda}{2}}(P_{XY} \parallel P_X P_Y),
\]

where \(D_\lambda(P \parallel Q)\) denotes the Rényi divergence (see, e.g., [26]) of order \(\lambda\) between \(P\) and \(Q\). As a consequence of (76) and (77),

\[
\max_{\lambda \in [0, 1]} \frac{\lambda}{2} \left\{ R - D_{1 + \frac{\lambda}{2}}(P_{XY} \parallel P_X P_Y) \right\} \leq \alpha(R, P_X, P_{Y|X})
\]

\[
\leq \max_{\lambda \in [0, 1]} \frac{\lambda}{2} \left\{ R - D_{1 + \frac{\lambda}{2}}(P_{XY} \parallel P_X P_Y) \right\}.
\]

**Remark 31.** In light of (78) and (79), the monotonicity of Rényi divergence in its order [26, Theorem 3] implies

\[
\frac{1}{2} (R - D_2(P_{XY} \parallel P_X P_Y)) \leq \alpha(R, P_X, P_{Y|X})
\]

\[
\leq \frac{1}{2} (R - D(P_{XY} \parallel P_X P_Y)),
\]

which means that, for a high enough rate \(R\),

\[
\alpha(R, P_X, P_{Y|X}) \approx \frac{1}{2} \left( R - \frac{1}{2} (I(P_X, P_{Y|X}) + D_2(P_{XY} \parallel P_X P_Y)) \right).
\]

**Remark 32.** Regarding the dual form of \(\mathbb{N}\), using the concavity of \(\log(x)\), one gets the following bound:

\[
\mathbb{N}(R, P_X, P_{Y|X}) \leq \max_{\lambda \in [1, 2]} \max_{S_Y} \left\{ \frac{\lambda-1}{\lambda} \left( R - D(P_{XY} \parallel P_X S_Y) \right) \right\}
\]

\[
= \frac{1}{2} \left[ R - I(P_X, P_{Y|X}) \right].
\]

On the other hand, the concavity of \(f_\lambda(x) = x^{\frac{\lambda}{2}}\) yields a tighter bound

\[
\mathbb{N}(R, P_X, P_{Y|X}) \leq \max_{\lambda \in [1, 2]} \max_{S_Y} \left\{ \frac{\lambda-1}{\lambda} \left( R - \mathbb{E} \left[ \log \mathbb{E} \left[ \exp \left( \frac{\lambda-1}{\lambda} t_{X;Y} (X, Y), Y \right) \right] \left( \bar{X} \right) \right] \right) \right\}.
\]

Capitalizing on (62), (80), and (84), for high enough rate \(R\), \(\mathbb{N}\) also satisfies

\[
\mathbb{N}(R, P_X, P_{Y|X}) \approx \frac{1}{2} \left( R - \frac{1}{2} (I(P_X, P_{Y|X}) + D_2(P_{XY} \parallel P_X P_Y)) \right).
\]
Remark 33. Since $Q_{XY} = P_{XY}$ and $Q_{Y|X} = P_{Y|X}$ are suboptimal choices, it is easy to see that

$$
\alpha(R, P_X, P_{Y|X}) = \min_{Q_{XY}} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q_{XY} \| P_{XY}) \right]_+ \right\} \tag{87}
$$

$$\leq \frac{1}{2} \left[ R - I(P_X, P_{Y|X}) \right]_+ \tag{88}$$

$$< \frac{R}{2}, \tag{89}$$

and

$$\mathcal{N}(R, P_X, P_{Y|X}) = \min_{Q_{Y|X}} \left\{ D(P_X Q_{Y|X} \| P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|X} \| P_{XY}) \right]_+ \right\} \tag{90}
$$

$$\leq \frac{1}{2} \left[ R - I(P_X, P_{Y|X}) \right]_+ \tag{91}$$

$$< \frac{R}{2}, \tag{92}$$

where (89) and (92) follow because the channel $P_{Y|X}$ is assumed to be non-degenerate. In light of (81) and (84), the same observation can be made from the dual forms of $\alpha(R, P_X, P_{Y|X})$ and $\mathcal{N}(R, P_X, P_{Y|X})$ in (38) and (40), respectively.

In what follows, Sections IV and V prove the lower and upper bound directions in (33), respectively. Section VI proves the equivalence of the primal and dual forms of the exact soft-covering exponents, namely (34) and (36), finally Section VII is devoted to the comparison of the previously known lower bounds on the soft-covering exponents $\alpha$ and $\mathcal{N}$.

IV. Proof of the Lower Bound in Theorem 1

This section establishes

$$\liminf_{n \to \infty} - \frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 \right] \geq \alpha(R, P_X, P_{Y|X}). \tag{93}$$

Indeed, using the finite block-length analysis, we shall prove the following stronger claim (see Theorem 3 in Appendix VII):

$$\frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 \right] \geq \min_{Q_{XY} \in P_n(X \times Y)} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q_{XY} \| P_{XY}) \right]_+ \right\} - \kappa_n, \tag{94}$$

where the vanishing constant $\kappa_n$ depends only on the block-length $n$ and the alphabet sizes $|X|$ and $|Y|$.

Suppose that $P_X^n$ is the i.i.d. input distribution to the memoryless channel $P_{Y^n|X^n}$ generating the i.i.d. output distribution $P_{Y^n}$, i.e., suppose $P_X^n \to P_{Y^n|X^n} \to P_{Y^n}$. Inspired by [14], given $y^n \in Y^n$, let

$$L_{\Theta^n_M}(y^n) = \begin{cases} \frac{P_{Y^n|X^n}(y^n)}{P_{Y^n}(y^n)} & \text{if } P_{Y^n}(y^n) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{95}$$
\[
E[L_{\mathcal{C}_M^n}(y^n)] = 1. \tag{97}
\]
Suppose \(y^n \in \mathcal{Y}^n\), and let \(Q_{X|Y}\) denote the conditional type of \(x^n \in \mathcal{X}^n\) given \(y^n\) so that the joint type \(Q_{X|Y}\) of the sequence \((x^n, y^n)\) satisfies
\[
Q_{X|Y}(a, b) = Q_{X|Y}(a|b)Q_Y(b), \tag{98}
\]
where \(Q_Y\) denotes the type of \(y^n\). Note that \(y^n \in \mathcal{T}_Q^n\) and \(Q_{X|Y} \in \mathcal{P}_n(X|Q_Y)\) together induce a joint type \(Q_{X|Y}\) via the relation in (98).

Assume \(P_{Y^n}(y^n) > 0\), since \(P_{Y^n|X^n}(y^n|x^n)\) and \(P_{Y^n}(y^n)\) depend on \((x^n, y^n)\) only through its joint type, using the type enumeration method [27, 28], one can write
\[
L_{\mathcal{C}_M^n}(y^n) = \frac{1}{M} \sum_{Q_{X|Y} \in \mathcal{P}_n(X|Q_Y)} N_{Q_{X|Y}}(y^n)l_{Q_{X|Y}}(y^n), \tag{99}
\]
where
\[
l_{Q_{X|Y}}(y^n) = \frac{P_{Y^n|X^n}(y^n|x^n)}{P_{Y^n}(y^n)} \tag{100}
\]
for some \(x^n_{Q_{X|Y}} \in \mathcal{T}_{Q_{X|Y}}(y^n)\), and the random variable
\[
N_{Q_{X|Y}}(y^n) = \left| \left\{ X^n \in \mathcal{E}_M^n : X^n \in \mathcal{T}_{Q_{X|Y}}(y^n) \right\} \right| \tag{101}
\]
\[
= \sum_{x^n \in \mathcal{E}_M^n} 1 \left\{ X^n \in \mathcal{T}_{Q_{X|Y}}(y^n) \right\} \tag{102}
\]
denotes the number of random codewords in \(\mathcal{E}_M^n\) which have conditional type \(Q_{X|Y}\) given \(y^n\). Since \(\mathcal{E}_M^n\) contains \(M\) independent codewords, it follows that \(N_{Q_{X|Y}}(y^n)\) is a binomial random variable with cluster size \(M\) and success probability
\[
p_{Q_{X|Y}}(y^n) = \Pr\left[ X^n \in \mathcal{T}_{Q_{X|Y}}(y^n) \right]. \tag{103}
\]
For the remainder of this paper, it is crucial to note that both \(l_{Q_{X|Y}}(y^n)\) and \(p_{Q_{X|Y}}(y^n)\) depend on \(y^n\) only through its type.

Given \(y^n \in \mathcal{T}_Q^n\) and \(Q_{X|Y} \in \mathcal{P}_n(X|Q_Y)\), define
\[
Z_{Q_{X|Y}}(y^n) = \frac{1}{M} N_{Q_{X|Y}}(y^n)l_{Q_{X|Y}}(y^n), \tag{104}
\]
\[
\Psi(M, Q_{X|Y}) = \min \left\{ 2p_{Q_{X|Y}}(y^n), M^{-\frac{1}{2}} p_{Q_{X|Y}}^{\frac{1}{2}}(y^n) \right\}, \tag{105}
\]

Note that \(L_{\mathcal{C}_M^n}(y^n)\) is a random variable as it depends on the random codebook \(\mathcal{C}_M^n\), and it is easy to see that
\[
E[L_{\mathcal{C}_M^n}(y^n)] = 1. \tag{97}
\]
and observe that

\[ \mathbb{E} \left[ \left\| P_{Y^n | \mathcal{G}^n_M} - P_{Y^n} \right\|_1 \right] \]

\[ = \sum_{y^n \in \mathcal{Y}^n} P_{Y^n}(y^n) \mathbb{E} \left[ \left\| L_{\mathcal{G}^n_M}(y^n) - 1 \right\|_1 \right] \tag{106} \]

\[ = \sum_{y^n \in \mathcal{Y}^n} P_{Y^n}(y^n) \mathbb{E} \left[ \left\| \sum_{Q_{XY} \in \mathbb{P}_n(X|Q_Y)} Z_{Q_{XY}}(y^n) - \mathbb{E}[Z_{Q_{XY}}(y^n)] \right\|_1 \right] \tag{107} \]

\[ \leq \sum_{y^n \in \mathcal{Y}^n} P_{Y^n}(y^n) \sum_{Q_{XY} \in \mathbb{P}_n(X|Q_Y)} \mathbb{E} \left[ \left\| Z_{Q_{XY}}(y^n) - \mathbb{E}[Z_{Q_{XY}}(y^n)] \right\|_1 \right] \tag{108} \]

\[ \leq \sum_{Q_Y \in \mathbb{P}_n(Y)} \sum_{y^n \in \mathcal{T}_{Q_Y}^n} \sum_{Q_{XY} \in \mathbb{P}_n(X|Q_Y)} P_{Y^n\mid X^n}(y^n | x^n_{Q_{XY}}) \mathbb{P}(M, Q_{\bar{X}Y}) \tag{109} \]

\[ = \sum_{Q_{XY} \in \mathbb{P}_n(X \times Y)} |\mathcal{T}_{Q_Y}^n| \exp(-n \mathbb{E}[t_{P_{Y|X}}(\bar{Y}, \bar{X})]) \mathbb{P}(M, Q_{\bar{X}Y}) \tag{110} \]

\[ \leq |\mathbb{P}_n(X \times Y)| \max_{Q_{XY} \in \mathbb{P}_n(X \times Y)} \left\{ |\mathcal{T}_{Q_Y}^n| \exp(-n \mathbb{E}[t_{P_{Y|X}}(\bar{Y}, \bar{X})]) \mathbb{P}(M, Q_{\bar{X}Y}) \right\}, \tag{111} \]

where (106) follows from the definition of \( L_{\mathcal{G}^n_M}(y^n) \) in (95); (107) follows from (99) and the definition of \( Z_{Q_{XY}}(y^n) \) in (104); (108) follows from the triangle inequality; (109) is due to Lemma 3 in Appendix II-A and the fact that

\[ \mathcal{Y}^n = \bigcup_{Q_Y \in \mathbb{P}_n(Y)} \mathcal{T}_{Q_Y}^n; \tag{112} \]

in (110) \((\bar{X}, \bar{Y}) \sim Q_{\bar{X}Y}\) and the equality follows from Lemma 19 in Appendix III. Denoting

\[ \mathbb{P}_\infty(X \times Y) = \bigcup_{n \in \mathbb{N}} \mathbb{P}_n(X \times Y), \tag{113} \]

it follows from (111) that

\[ \liminf_{n \to \infty} - \frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n | \mathcal{G}^n_M} - P_{Y^n} \right\|_1 \right] \geq \inf_{Q_{XY} \in \mathbb{P}_\infty(X \times Y)} \left\{ D(Q_{\bar{X}Y} \| P_{XY}) + \frac{1}{2} [R - D(Q_{\bar{X}Y} \| P_{XQ_{\bar{Y}}})]_+ \right\} \tag{114} \]

\[ = \min_{Q_{XY} \in \mathbb{P}(X \times Y)} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \| P_{XQ_{Y}})]_+ \right\} \tag{115} \]

\[ = \alpha(R, P_X, P_{Y|X}), \tag{116} \]

where in (114) we use Lemmas 20 and 25 in Appendix III; and finally (115) follows from Lemma 27 in Appendix IV-A.

\[ \quad \blacksquare \]

Remark 34. In the constant-composition case,\(^{14}\)

\[ L_{\mathcal{G}^n_M}(y^n) = \begin{cases} \frac{P_{Y^n | \mathcal{G}^n_M}(y^n)}{R_{Y^n}(y^n)} & \text{if } R_{Y^n}(y^n) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{117} \]

\(^{13}\)Also see Remarks 5 and 6.

\(^{14}\)See Definition 22 for the definition of the set of conditional types with fixed marginals, i.e., \( \mathbb{P}_n(X|Q_Y; P_X) \).
\[
\frac{1}{M} \sum_{Q_{X|Y} \in \mathcal{P}_n(X|Q_{Y};P_X)} \tilde{N}_{Q_{X|Y}}(y^n) \tilde{l}_{Q_{X|Y}}(y^n),
\] (118)

with
\[
\tilde{l}_{Q_{X|Y}}(y^n) = \frac{P_{Y^n|X^n}(y^n|x^n_{Q_{X|Y}})}{R_{Y^n}(y^n)},
\] (119)
\[
\tilde{N}_{Q_{X|Y}}(y^n) = \left| \left\{ \tilde{X}^n \in \mathcal{D}_{M^n} : \tilde{X}^n \in T_{Q_{X|Y}}^{n_1}(y^n) \right\} \right|,
\] (120)

and
\[
\bar{p}_{Q_{X|Y}}(y^n) = \mathbb{P} \left[ \tilde{X}^n \in T_{Q_{X|Y}}^{n_1}(y^n) \right],
\] (121)
\[
\bar{Z}_{Q_{X|Y}}(y^n) = \frac{1}{M} \tilde{N}_{Q_{X|Y}}(y^n) \tilde{l}_{Q_{X|Y}}(y^n),
\] (122)
\[
\bar{Q}(M, Q_{X|Y}) = \min \left\{ 2\bar{p}_{Q_{X|Y}}(y^n), M^{-\frac{1}{2}} \bar{Z}_{Q_{X|Y}}(y^n) \right\}.
\] (123)

The steps (106)–(115) remain almost identical except one needs to keep in mind that X-marginal of the joint types \( Q_{X,Y} \) is fixed to be \( P_X \) and replace
\[
P_{Y^n|\mathcal{D}_M} \leftarrow P_{Y^n|\mathcal{D}_M},
\]
\[
P_Y \leftarrow R_Y,
\]
\[
L_{\mathcal{D}_M} \leftarrow L_{\mathcal{D}_M},
\]

Lemmas 25 and 27 \( \leftarrow \) Lemmas 26 and 28,

together with proper replacement of the terms defined in (119)–(123). For the complete proof of lower bound in the constant-composition case, refer to Appendix VI-A.

Remark 35. It should be noted that the key step of the lower bound proof is the bound in (109). In that step, the mean and the standard deviation of each of the random variables \( Z_{Q_{X|Y}} \) are directly used as the upper bound for each conditional type \( Q_{X|Y} \in \mathcal{P}_n(X|Q_Y) \). In previous soft-covering exponent analysis [6, 7], the set of the conditional types \( \mathcal{P}_n(X|Q_Y) \) is first partitioned into two sets containing the so-called typical and atypical conditional types according to a threshold on \( l_{Q_{X|Y}}(y^n) \). Then, the standard deviation bound is applied on the typical set whereas the mean bound is applied on the atypical one. Although this “partition by joint probability first, bound later” technique is also espoused in the exact exponent analysis of the relative entropy variant of the soft-covering lemma [14], it turns out to be a suboptimal method for the total variation distance.

Remark 36. Thanks to the analysis on the absolute mean deviation of binomial distribution provided in [29, Theorem 1], the mean and standard deviation bound applied in Lemma 3 can be shown to be tight.
V. Proof of the Upper Bound in Theorem 1

This section establishes

\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{C}_n} - P_{Y^n} \right\|_1 \right] \leq \alpha(R, P_X, P_{Y|X}).
\] (124)

Indeed, using the finite block-length analysis, we shall prove the following stronger claim (see Theorem 3 in Appendix VII):

\[
-\frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{C}_n} - P_{Y^n} \right\|_1 \right] \leq \min_{Q_{X,Y} \in \mathcal{P}_n(X \times Y)} \left\{ D(Q_{X|Y} \parallel P_X) + \frac{1}{2} [R - D(Q_{X|Y} \parallel P_X Q_{Y|X})]_+ \right\} + v_n,
\] (125)

where the vanishing constant \( v_n \) depends on the block-length \( n \), the alphabet sizes \( |X| \) and \( |Y| \), and the joint distribution \( P_{X,Y|X} \).

The biggest obstacle in showing (124) is the mutual dependences of the random variables \( N_{Q_{X|Y}}(y^n) \), as defined in (101). Note that, given two distinct conditional types (given \( y^n \in \mathcal{Y}^n \)), say \( Q_{X|Y} \) and \( R_{X|Y} \), the random variables \( N_{Q_{X|Y}}(y^n) \) and \( N_{R_{X|Y}}(y^n) \) are not independent from each other. Fortunately, their dependence can be shown to be negligible. Indeed, instead of assuming that the number of codewords \( M \) in the codebook \( \mathcal{C}_n^M \) is a deterministic number \( \lceil \exp(nR) \rceil \), if one assumes that it is Poisson distributed with mean \( \mu_n = \exp(nR) \), then \( N_{Q_{X|Y}}(y^n) \) becomes a Poisson splitting of the codewords in \( \mathcal{C}_n^M \). In that case, given two distinct conditional types \( Q_{X|Y} \) and \( R_{X|Y} \), the random variables \( N_{Q_{X|Y}}(y^n) \) and \( N_{R_{X|Y}}(y^n) \) correspond to two distinct Poisson splits and they become independent from one another. This turns out to be the gateway in proving the pseudo-upper bound in the case when \( M \) is Poisson distributed. However, to prove the upper bound for the actual statement in Theorem 1, the auxiliary assumption that the codebook \( \mathcal{C}_n^M \) contains a random number of codewords needs to be eliminated, which can be done with the help of Lemma 1. As already mentioned in Remark 14, it is possible to prove a result similar to Lemma 1 with the assumption that \( M \) is Poisson distributed, see Lemma 11 in Appendix II-A. This result can be utilized to show that it is immaterial whether \( M \) is Poisson distributed or \( M = \lceil \exp(nR) \rceil \) that (124) holds.

To provide a more transparent presentation, the upper bound proof is divided into three subsections: Section V-A introduces the auxiliary assumption that the codebook size \( M \) is Poisson distributed with mean \( \mu_n = \exp(nR) \), Section V-B provides the pseudo-upper bound proof under the assumption that \( M \) is Poisson distributed, and finally, Section V-C shows that, removing the auxiliary assumption by conditioning on \( M = \lceil \mu_n \rceil \), one still cannot do better than \( \alpha(R, P_X, P_{Y|X}) \).

\[15\] One quick way to see these mutual dependences is that the sum of \( N_{Q_{X|Y}}(y^n) \) over all conditional types \( Q_{X|Y} \in \mathcal{P}_n(X|Q_{Y}) \) is equal to \( M \).
V-A. Poissonization

Suppose, for the moment, that $M$ is Poisson distributed with mean $\mu_n = \exp(nR)$. In that case, using the established notation so far, for each $y^n \in T^n_{Q_Y}$ and each $Q_{X|Y} \in \mathcal{P}_n(X'|Q_Y)$, the random variable

$$N_{Q_{X|Y}}(y^n) = \sum_{X^n \in \mathcal{P}^n_M} 1\{|X^n| \in T^n_{Q_{X|Y}}(y^n)\}$$

is a Poisson splitting of $M$ with mean

$$\mu_n p_{Q_{X|Y}}(y^n) = \exp(nR) \mathbb{P}\left(|X^n| \in T^n_{Q_{X|Y}}(y^n)\right).$$

Moreover, as the random variables $N_{Q_{X|Y}}(y^n)$ and $N_{R_{X|Y}}(y^n)$ correspond to different bins defined by different conditional types $Q_{X|Y}$ and $R_{X|Y} \in \mathcal{P}_n(X'|Q_Y)$, they are independent from each other.

Choose $\delta \in (0, 1)$, and note that for any $y^n \in \mathcal{Y}^n$ an application of Lemma 4 in Appendix II-A with

$$W \leftarrow M \left| P_{Y^n|\mathcal{P}^n_M}(y^n) - P_{Y^n}(y^n) \right|,$$

$$X \leftarrow M,$$

$$c \leftarrow (1 + \delta) \mu_n,$$

yields

$$(1 + \delta) \mu_n \mathbb{E}\left[\left| P_{Y^n|\mathcal{P}^n_M}(y^n) - P_{Y^n}(y^n) \right|\right] \geq \mathbb{E}\left[\left| M \left| P_{Y^n|\mathcal{P}^n_M}(y^n) - P_{Y^n}(y^n) \right|\right| - \mathbb{E}[M]\{M > (1 + \delta) \mu_n\}\right].$$

(128)

On one hand, regarding the first term in the right side of (128), the triangle inequality implies

$$\mathbb{E}\left[\left| M \left| P_{Y^n|\mathcal{P}^n_M}(y^n) - P_{Y^n}(y^n) \right|\right| \geq \mathbb{E}\left[\left| MP_{Y^n|\mathcal{P}^n_M}(y^n) - \mu_n P_{Y^n}(y^n) \right|\right| - \mathbb{E}[|M - \mu_n| P_{Y^n}(y^n)]\right] \geq \mathbb{E}\left[\left| MP_{Y^n|\mathcal{P}^n_M}(y^n) - \mu_n P_{Y^n}(y^n) \right|\right] - \frac{\mu_n P_{Y^n}(y^n)}{\sqrt{\mu_n - 1}},$$

(129)

(130)

where (130) follows from Lemma 5 in Appendix II-A. On the other hand, regarding the second term in the right side of (128),

$$\mathbb{E}[M\{M > (1 + \delta) \mu_n\}] \leq \mu_n a_\epsilon \frac{\mu_n}{\sqrt{\mu_n - 1}},$$

(131)

which is a consequence of Lemma 6 in Appendix II-A. Note that, in the right side of (131), $a_\epsilon$ is a constant that satisfies $a_\epsilon < 1$ for all $\epsilon \in (0, 1)$, which is explicitly defined in (276).

Assembling (128), (130) and (131),

$$(1 + \delta) \mathbb{E}\left[\left| P_{Y^n|\mathcal{P}^n_M}(y^n) - P_{Y^n}(y^n) \right|\right]$$

16The bound in (131) is valid only when $\delta > \frac{1}{\mu_n}$. Even though the choice of $\delta \in (0, 1)$ does not depend on $\mu_n = \exp(nR)$, the applicability of Lemma 6 is guaranteed for large enough $n$. 23
greater than

As will be seen, the remaining terms in the right side of (134) are residual terms whose exponents are

This section focuses on the summation in the right side of (134) and shows that its exponent is

Remark 37. In the constant-composition case, to get the counter-part of (132), all one needs to do is replace

To this end, invoking the lemmas provided in Appendix II-A,

In the sense that they vanish with a faster rate with

This section focuses on the summation in the right side of (134) and shows that its exponent is \( \alpha(R, P_X, P_{Y|X}) \). As will be seen, the remaining terms in the right side of (134) are residual terms whose exponents are greater than\(^\text{17}\) \( \alpha(R, P_X, P_{Y|X}) \), and therefore, they do not contribute to the overall exponential decay rate of \( E[\|P_{Y^n|\mathcal{E}_M^n} - P_{Y^n}\|_1] \).

To this end, invoking the lemmas provided in Appendix II-A,

\(^{17}\text{In the sense that they vanish with a faster rate with } n.\)}
\[ \geq \frac{1}{4} \sum_{Q_Y \in \mathcal{P}_n(Y)} \sum_{y^n \in T_{Q_Y}} \max_{Q_X | Y \in \mathcal{P}_n(X|Q_Y)} \left\{ P^X_{y^n} | X^n(y^n | x^n_Q_{X|Y}) \mathcal{Y}(\mu_n, Q_{X\bar{Y}}) \right\} \]  

(138)

\[ = \frac{1}{4} \sum_{Q_Y \in \mathcal{P}_n(Y)} \max_{Q_X | Y \in \mathcal{P}_n(X|Q_Y)} \left\{ |T_{Q_Y} | \exp(-n\mathbb{E}[i_{P_Y|X}(\bar{Y}|\bar{X})])\mathcal{Y}(\mu_n, Q_{X\bar{Y}}) \right\} \]  

(139)

\[ \geq \frac{1}{4} \max_{Q_X | Y \in \mathcal{P}_n(X \times Y)} \left\{ |T_{Q_Y} | \exp(-n\mathbb{E}[i_{P_Y|X}(\bar{Y}|\bar{X})])\mathcal{Y}(\mu_n, Q_{X\bar{Y}}) \right\} , \]  

(140)

where (135) follows from the definition of \( L_{\mathcal{M}_n}^n(y^n) \) in (95); (136) follows from the type enumeration method, see (99), and Lemma 7; the key step in (137) follows from Lemma 8 and the definition of \( l_{Q_{X|Y}}(y^n) \) in (100); in (138) the function \( \mathcal{Y}(\mu_n, Q_{X\bar{Y}}) \) is as defined in (105) and the bound follows from Lemma 9; in (139) \( \bar{X}, \bar{Y} \sim Q_{X\bar{Y}} Q_{\bar{Y}} \) and we invoke\(^{18} \) Lemma 19 in Appendix III; and finally, (140) follows because the right side of (139) is a sum of non-negative numbers.\(^{19} \)

Note that

\[ \lim_{n \to \infty} \frac{1}{n} \log \max_{Q_X | Y \in \mathcal{P}(X \times Y)} \left\{ |T_{Q_Y} | \exp(-n\mathbb{E}[i_{P_Y|X}(\bar{Y}|\bar{X})])\mathcal{Y}(\mu_n, Q_{X\bar{Y}}) \right\} \]

(141)

\[ = \inf_{Q_X | Y \in \mathcal{P}(X \times Y)} \left\{ D(Q_X Y \| P_X Y) + \frac{1}{2} [R - D(Q_X Y \| P_X Q_Y)]_+ \right\} \]

(142)

\[ = \min_{Q_X | Y \in \mathcal{P}(X \times Y)} \left\{ D(Q_X Y \| P_X Y) + \frac{1}{2} [R - D(Q_X Y \| P_X Q_Y)]_+ \right\} \]

(143)

where (141) is thanks to Lemma 25 in Appendix III while (142) follows from Lemma 27 in Appendix IV-A.

On the other hand, going back to (134), the fact that \( \mu_n = \exp(nR) \) and \( a_\epsilon < 1 \) for all \( \epsilon \in (0, 1) \) implies

\[ \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\sqrt{\mu_n - 1}} = \frac{R}{2} , \]

(144)

\[ \lim_{n \to \infty} \frac{1}{n} \log \left( |Y|^n a_\epsilon^{\mu_n \frac{1}{1 - a_\epsilon}} \right) = \infty. \]

(145)

Since the right side of (142) is strictly less than \( R/2 \), see Remark 33, it follows from (134), and (140)–(145) that, when \( M \) is a Poisson distributed random variable with mean \( \mu_n = \exp(nR) \),

\[ \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 \right] \leq \alpha(R, P_X, P_{Y|X}) . \]

(146)

Remark 38. In the constant-composition case, in addition to the replacements mentioned in Remark 37, replace

\[ L_{\mathcal{M}_n}^n \leftarrow \tilde{L}_{\mathcal{M}_n}^n , \]

\[ l_{Q_{X|Y}}(y^n) \leftarrow \tilde{l}_{Q_{X|Y}}(y^n) , \]

\[ \mathcal{Y}(\mu_n, Q_{X\bar{Y}}) \leftarrow \tilde{\mathcal{Y}}(\mu_n, Q_{X\bar{Y}}) , \]

Lemmas 25 and 27 \( \leftrightarrow \) Lemmas 26 and 28,

\(^{18} \) Also see Remark 5.

\(^{19} \) Also see Remark 6.
and keep in mind that the $\mathcal{X}$-marginal of the joint types $Q_{X,Y}$ is fixed to be $P_{X}$. See Appendix VI-B.2 for further details.

Remark 39. In order for the key step in (137) to be valid, independence among $N_{Q_{X,Y}}(y^n)$ is a must. This is the reason why poissonization was applied.

V-C. Depoissonization

To prove the upper bound in Theorem 1, it remains to show that the result established in (146) still holds when $M = \lceil \exp(nR) \rceil$. To this end, once again utilizing the fact that $\alpha(R, P_{X}, P_{Y|X}) < R/2$, choose $r \in (\alpha, R/2)$, let $\epsilon_n = \exp(-nr)$, define the random variable

$$T_n(m) = \|P_{Y^n|\mathcal{C}_n} - P_{Y^n}\|_1,$$

and consider the following three events:

$$\mathcal{A}_n = \{|E[T_n([\mu_n])] - E[T_n(M)]| < 2\epsilon_n\},$$

$$\mathcal{B}_n = \{|T_n([\mu_n]) - E[T_n([\mu_n])]| < \epsilon_n\},$$

$$\mathcal{C}_n = \{|T_n([\mu_n]) - E[T_n(M)]| < \epsilon_n\},$$

where $T_n([\mu_n])$ denotes the case when the codebook is assumed to have a deterministic number of codewords and $T_n(M)$ denotes the case when the codebook is assumed to have a random (Poisson) number of codewords.

Observe that

$$\mathbb{P}[\mathcal{A}_n] \geq \mathbb{P}[\mathcal{B}_n \cap \mathcal{C}_n] \quad (151)$$

$$\geq 1 - \mathbb{P}[\mathcal{B}_n^c] - \mathbb{P}[\mathcal{C}_n^c] \quad (152)$$

$$\geq 1 - \left(2 + 16[\mu_n]^{1/2}\right) \exp\left(-\frac{\mu_n\epsilon_n^2}{2}\right), \quad (153)$$

where (151) is because $\mathcal{A}_n \supset \mathcal{B}_n \cap \mathcal{C}_n$; (152) is the union bound; and (153) follows from Lemmas 1 and 12 in Section III and Appendix II-A, respectively. Thanks to the choice of $\epsilon_n$, for large enough $n$, the right side of (153) is strictly greater than 0. Moreover, since $\mathcal{A}_n$ is a deterministic event, $\mathbb{P}[\mathcal{A}_n] > 0$ implies that $\mathbb{P}[\mathcal{A}_n] = 1$. That is, for large enough $n$, and $r \in (\alpha, R/2)$,

$$E[T_n([\mu_n])] > E[T_n(M)] - 2\exp(-nr). \quad (154)$$

Hence, it follows that

$$\limsup_{n \to \infty} -\frac{1}{n} \log E[T_n([\mu_n])] \leq \limsup_{n \to \infty} -\frac{1}{n} \log(E[T_n(M)] - 2\exp(-nr)) \leq \alpha(R, P_{X}, P_{Y|X}), \quad (155)$$

where (156) is due to (146).
Remark 40. In addition to the replacements mentioned in Remarks 37 and 38, replacing Lemmas 1 and 12 ← Lemmas 2 and 18 recovers the proof in the constant-composition case. Refer to Appendix VI-B.3 for details.

VI. Proof of the Dual Representations

This section provides proofs for (34) and (36), which are alternative representations of the exact soft-covering exponents in the random i.i.d. codebook and random constant-composition codebook cases, respectively.

VI-A. Proof of the Dual Representation of $\alpha$

Proposition 1. Given $P_X \to P_{Y|X} \to P_Y$, and $R > I(P_X, P_{Y|X})$

$$\min_{Q_{XY}} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q_{XY} \| P_X Q_Y) \right] \right\}$$

$$= \max_{\lambda \in [1,2]} \left\{ \frac{\lambda - 1}{\lambda} \left( R - I_\lambda (P_X, P_{Y|X}) \right) \right\}. \quad (157)$$

Proof. Note that

$$\min_{Q_{XY}} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q_{XY} \| P_X Q_Y) \right] \right\}$$

$$= \min_{Q_Y} \min_{Q_{X|Y}} \max_{\lambda \in [0,1]} \left\{ D(Q_Y \| P_Y) + D(Q_{X|Y} \| P_{X|Y} Q_Y) + \frac{\lambda}{2} \left( R - D(Q_{X|Y} \| P_X Q_Y) \right) \right\} \quad (158)$$

$$= \min_{Q_Y} \min_{Q_{X|Y}} \max_{\lambda \in [0,1]} \left\{ D(Q_Y \| P_Y) + \frac{2 - \lambda}{2} D(Q_{X|Y} \| P_{X|Y} Q_Y) + \frac{\lambda}{2} \left( R - \mathbb{E}[i_{X,Y}(\hat{X}; \hat{Y})] \right) \right\} \quad (159)$$

$$= \min_{Q_Y} \max_{\lambda \in [0,1]} \min_{Q_{X|Y}} \left\{ D(Q_Y \| P_Y) + \frac{2 - \lambda}{2} D(Q_{X|Y} \| P_{X|Y} Q_Y) + \frac{\lambda}{2} \left( R - \mathbb{E}[i_{X,Y}(\hat{X}; \hat{Y})] \right) \right\} \quad (160)$$

$$= \min_{Q_Y} \max_{\lambda \in [0,1]} \left\{ D(Q_Y \| P_Y) + \frac{\lambda}{2} R + \min_{Q_{X|Y}} \left\{ 2 - \lambda D(Q_{X|Y} \| P_{X|Y} Q_Y) - \frac{\lambda}{2} \mathbb{E}[i_{X,Y}(\hat{X}; \hat{Y})] \right\} \right\} \quad (161)$$

$$= \min_{Q_Y} \max_{\lambda \in [0,1]} \left\{ D(Q_Y \| P_Y) + \frac{\lambda}{2} R - \frac{2 - \lambda}{2} \mathbb{E} \left[ \log \mathbb{E} \left[ \exp \left( \frac{\lambda}{2 - \lambda} i_{X,Y}(\hat{X}; \hat{Y}) \right) \hat{Y} \right] \right] \right\} \quad (162)$$

$$= \max_{\lambda \in [0,1]} \min_{Q_Y} \left\{ D(Q_Y \| P_Y) + \frac{\lambda}{2} R - \frac{2 - \lambda}{2} \mathbb{E} \left[ \log \mathbb{E} \left[ \exp \left( \frac{\lambda}{2 - \lambda} i_{X,Y}(\hat{X}; \hat{Y}) \right) \hat{Y} \right] \right] \right\} \quad (163)$$

$$= \max_{\lambda \in [0,1]} \left\{ \frac{\lambda}{2} R + \min_{Q_Y} \left\{ D(Q_Y \| P_Y) - \frac{2 - \lambda}{2} \mathbb{E} \left[ \log \mathbb{E} \left[ \exp \left( \frac{\lambda}{2 - \lambda} i_{X,Y}(X; Y) \right) Y \right] \right] \right\} \right\} \quad (164)$$

$$= \max_{\lambda \in [0,1]} \left\{ \frac{\lambda}{2} R - \log \mathbb{E} \left[ \mathbb{E}^{\frac{2 - \lambda}{2}} \left[ \exp \left( \frac{\lambda}{2 - \lambda} i_{X,Y}(X; Y) \right) Y \right] \right] \right\} \quad (165)$$

$$= \max_{\lambda \in [0,1]} \left\{ \frac{\lambda}{2} \left( R - I_{\frac{2 - \lambda}{2}} (P_X, P_{Y|X}) \right) \right\}, \quad (166)$$

where in (159) $(\hat{X}, \hat{Y}) \sim Q_{X|Y} Q_Y$ and the fact that

$$D(Q_{X|Y} \| P_X Q_Y) = D(Q_{X|Y} \| P_{X|Y} Q_Y) + \mathbb{E}[i_{X,Y}(\hat{X}; \hat{Y})] \quad (167)$$
is used; in (160) there is no duality gap in changing the minimax to maximin because the optimized quantity is convex in \( Q_{X|Y} \) and linear in \( \lambda \); in (162) \((\tilde{X}, \tilde{Y}) \sim P_{X|Y} Q_Y\) and Corollary 1 in Appendix VIII is used; in (163), once again, there is no duality gap in changing minimax to maximin because the optimized quantity is convex in \( Q_Y \) while this time it is concave in \( \lambda \) because the minimum of a collection of linear functions is concave; (165) is an application of Lemma 30 in Appendix VIII such that

\[
f(y) = \frac{2 - \lambda}{2} \log \mathbb{E} \left[ \exp \left( \frac{\lambda}{2 - \lambda} t_{X,Y}(X;Y) \right) \right] = y,
\]

with the random transformation from \( \mathcal{Y} \) to \( \mathcal{X} \) in (168) is fixed to be \( P_{X|Y} \); and finally (166) follows by first noting that for \((X, \tilde{Y}) \sim P_X P_Y\)

\[
\mathbb{E}[g(X,Y)|\tilde{Y}] = \mathbb{E} \left[ g(X, \tilde{Y}) \exp(t_{X,Y}(X;\tilde{Y})) | \tilde{Y} \right],
\]

and then recalling the definition of \( \alpha \)-mutual information, see, e.g., [23, Eq. (46)].

### VI-B. Proof of the Dual Representation of \( \mathcal{N} \)

**Proposition 2.** Given \( P_X \rightarrow P_{Y|X} \rightarrow P_Y \) and \( R > I(P_X, P_{Y|X}) \)

\[
\min_{Q_{Y|X}} \left\{ D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) + \frac{1}{2} [R - D(P_X Q_{Y|X} \| P_{X|X} Q_{Y})]_+ \right\} = \max_{\lambda \in [0,1]} \max_{S_Y \in P(\mathcal{Y})} \left\{ \frac{\lambda}{2} R + \left( 1 - \frac{\lambda}{2} \right) D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) - \frac{\lambda}{2} H(Q_Y) + \frac{\lambda}{2} \mathbb{E} \left[ t_{Y|X}(Y|\tilde{X}) \right] \right\}
\]

where \((\tilde{X}, Y) \sim P_{X|Y} \), and \( Q_Y \) is such that \( P_X \rightarrow Q_{Y|X} \rightarrow Q_Y \).

**Proof.** Observe that

\[
\min_{Q_{Y|X}} \left\{ D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) + \frac{1}{2} [R - D(P_X Q_{Y|X} \| P_{X|X} Q_{Y})]_+ \right\} = \min_{Q_{Y|X}} \left\{ D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) + \frac{\lambda}{2} (R - D(P_X Q_{Y|X} \| P_{X|X} Q_{Y})) \right\}
\]

(171)

\[
= \min_{Q_{Y|X}} \max_{\lambda \in [0,1]} \left\{ D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) + \frac{\lambda}{2} (R - D(P_X Q_{Y|X} \| P_{X|X} Q_{Y})) \right\}
\]

(172)

\[
= \max_{\lambda \in [0,1]} \min_{Q_{Y|X}} \left\{ \frac{\lambda}{2} R + \left( 1 - \frac{\lambda}{2} \right) D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) - \frac{\lambda}{2} H(Q_Y) + \frac{\lambda}{2} \mathbb{E} \left[ t_{Y|X}(Y|\tilde{X}) \right] \right\}
\]

(173)

\[
= \max_{\lambda \in [0,1]} \min_{Q_{Y|X}} \left\{ \frac{\lambda}{2} R + \left( 1 - \frac{\lambda}{2} \right) D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) - \frac{\lambda}{2} H(Q_Y) + \frac{\lambda}{2} \mathbb{E} \left[ t_{Y|X}(Y|\tilde{X}) \right] \right\}
\]

(174)

\[
= \max_{\lambda \in [0,1]} \min_{Q_{Y|X}} \left\{ \frac{\lambda}{2} R + \min_{S_Y} \left\{ \left( 1 - \frac{\lambda}{2} \right) D(P_X Q_{Y|X} \| P_{X|X} Q_{Y}) - \frac{\lambda}{2} \mathbb{E} \left[ t_{Y|X}(Y|\tilde{X}) \right] \right\} \right\}
\]

(175)

\[
= \max_{\lambda \in [0,1]} \min_{S_Y} \left\{ \frac{\lambda}{2} R - \mathbb{E} \left[ \log \mathbb{E} \left[ \exp \left( \frac{\lambda}{2 - \lambda} t_{X,Y}(X;Y) \right) \right] \right] \right\}
\]

(176)

where in (172) \((\tilde{Y}, \tilde{X}) \sim Q_{Y|X} P_X\); in (173) there is no duality gap as the optimized quantity is linear in \( \lambda \) and convex in \( Q_{Y|X} \); (174) follows from the variational representation of entropy:

\[
H(Q_Y) = \min_{S_Y} \mathbb{E} \left[ t_{Y}(Y) \right];
\]

(177)
in (175) there is no duality gap as the optimized quantity is convex in $Q_{Y|X}$ and concave in $S_Y$; finally, in (176) we use Corollary 5.

\section{Comparison with the Known Lower Bounds on the Soft-Covering Exponent}

This section compares the exact soft-covering exponents in Theorems 1 and 2 to their previously known lower bounds. In particular, Sections VII-A and VII-B provide comparisons of $\alpha(R, P_X, P_{Y|X})$ with the exponents that can be found in [7, Lemma VII.9] and [6, Theorem 6], respectively. Additionally, Sections VII-C and VII-D compare $\alpha(R, P_X, P_{Y|X})$ (in the i.i.d. case) and $\lambda(R, P_X, P_{Y|X})$ (in the constant-composition case) with the half of the relative entropy variant of the soft-covering exponents that can be found in [14, Theorem 4], respectively.

\subsection{Comparison with Cuff’s Lower Bound}

Prior to our result in Theorem 1, the best known-to-date lower bound on the soft-covering exponent was provided in [7, Lemma VII.9] which was shown to be

$$
\beta(R, P_X, P_{Y|X}) = \max_{\lambda \geq 0} \max_{\lambda' \leq 1} \left\{ \frac{\lambda}{2\lambda + 1 - \lambda'} \left( R - (1 - \lambda')D_{1+\lambda}(P_{XY}||P_XP_Y) - \lambda'\tilde{D}_{1+\lambda'}(P_{XY}||P_XP_Y) \right) \right\},
$$

(178)

where, supposing $(X,Y) \sim P_XP_{Y|X}$,

$$
D_{1+\lambda}(P_{XY}||P_XP_Y) = \frac{1}{\lambda} \log \mathbb{E} \left[ \exp(\lambda t_{X,Y}(X,Y)) \right]
$$

(179)

is the Rényi divergence (see, e.g., [26]) of order $1 + \lambda$ between the joint and product distributions, and

$$
\tilde{D}_{1+\lambda'}(P_{XY}||P_XP_Y) = \frac{2}{\lambda'} \log \mathbb{E} \left[ \mathbb{E}^{\frac{1}{\lambda'}} \left[ \exp \left( \lambda' t_{X,Y}(X,Y) \right) \right] \right].
$$

(180)

Using the results provided in Appendix VIII, Proposition 3 proves the fact that $\alpha(R, P_X, P_{Y|X})$ captures the exponential decay rate in soft-covering lemma better than $\beta(R, P_X, P_{Y|X})$.

\textbf{Proposition 3.} Suppose $P_X \rightarrow P_{Y|X} \rightarrow P_Y$, and $R > I(P_X, P_{Y|X}) > 0$. Then

$$
\alpha(R, P_X, P_{Y|X}) \geq \beta(R, P_X, P_{Y|X}),
$$

(181)

where $\alpha(R, P_X, P_{Y|X})$ and $\beta(R, P_X, P_{Y|X})$ are as defined in (37) and (178), respectively.

\textbf{Proof.} Suppose $(X,Y) \sim P_XP_{Y|X}$, $(\tilde{X}, \tilde{Y}) \sim Q_X|YQ_Y$, and $(\hat{X}, \hat{Y}) \sim S_{X|Y}S_Y$. It follows that

$$
\beta(R, P_X, P_{Y|X}) = \max_{\lambda \geq 0} \max_{\lambda' \leq 1} \left\{ \frac{\lambda}{2\lambda + 1 - \lambda'} \left( R - (1 - \lambda')D_{1+\lambda}(P_{XY}||P_XP_Y) - \lambda'\tilde{D}_{1+\lambda'}(P_{XY}||P_XP_Y) \right) \right\}
$$

(182)

$$
= \max_{\lambda \geq 0} \max_{\lambda' \leq 1} \left\{ \frac{\lambda}{2\lambda + 1 - \lambda'} \left( R + \frac{1 - \lambda'}{\lambda} \min_{Q_{XY}} \left\{ D(Q_{XY}||P_{XY}) - \lambda\mathbb{E}[t_{X,Y}(\hat{X};\hat{Y})] \right\} \right) \right\}
$$

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\[
+ \min_{S_{XY}} \left\{ 2D(S_{XY} \parallel P_{XY}) - D(S_{XY} \parallel P_{X|Y} S_Y) - \lambda' \mathbb{E}[t_{X,Y}(\hat{X};\hat{Y})] \right\} \right) \quad (183)
\]
\[
\leq \max_{Q_{XY}} \min_{\lambda \geq 0} \left\{ D(Q_{XY} \parallel P_{XY}) + \frac{\lambda}{2\lambda + 1} (R - D(Q_{XY} \parallel P_X Q_Y)) \right\} \quad (184)
\]
\[
\leq \min_{Q_{XY}} \max_{\lambda \geq 0} \left\{ D(Q_{XY} \parallel P_{XY}) + \frac{\lambda}{2\lambda + 1} (R - D(Q_{XY} \parallel P_X Q_Y)) \right\} \quad (185)
\]
\[
= \min_{Q_{XY}} \left\{ D(Q_{XY} \parallel P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \parallel P_X Q_Y)]_+ \right\} \quad (186)
\]
\[
= \alpha(R, P_X, P_{Y|X}), \quad (187)
\]
where (183) uses Corollaries 3 and 4 in Appendix VIII; (184) constrains the two minimizations by assuming that their minimizers are equivalent and uses the fact that
\[
D(Q_{XY} \parallel P_{X|Y} Q_Y) + \mathbb{E}[t_{X,Y}(\hat{X};\hat{Y})] = D(Q_{XY} \parallel P_X Q_Y); \quad (188)
\]
(185) is due to the duality gap; and finally (186) follows because \( \frac{\alpha}{2\lambda + 1 - \lambda'} \) is monotone decreasing or increasing in \( \lambda' \) depending on whether \( a < 0 \) or \( a > 0 \).

\[\framebox{\[
\text{VII-B. Comparison with Hayashi’s Lower Bound}
\]
}

In [6, Theorem 6] Hayashi proves that
\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n} - P_{Y^n} \right\|_1 \right] \geq \gamma(R, P_X, P_{Y|X}), \quad (189)
\]
where
\[
\gamma(R, P_X, P_{Y|X}) = \max_{\lambda \in [0,1]} \left\{ \frac{\lambda}{1 + \lambda} (R - D_{1+\lambda}(P_{XY} \parallel P_X P_Y)) \right\}. \quad (190)
\]
As shown in [7], thanks to Jensen’s inequality, noting that
\[
\tilde{D}_{1+\lambda}(P_{XY} \parallel P_X P_Y) \leq D_{1+\lambda}(P_{XY} \parallel P_X P_Y), \quad (191)
\]
and altering the maximization domain in the right side of (178) by restricting \( \lambda' = \lambda \) yields
\[
\beta(R, P_X, P_{Y|X}) \geq \gamma(R, P_X, P_{Y|X}). \quad (192)
\]
Together with Proposition 3, (192) implies
\[
\alpha(R, P_X, P_{Y|X}) \geq \gamma(R, P_X, P_{Y|X}). \quad (193)
\]
For the sake of demonstration, a non-trivial alternative way of proving (193) using the duality gap instead of (192) is presented in Proposition 4.

**Proposition 4.** Suppose \( P_X \to P_{Y|X} \to P_Y \), and \( R > I(P_X, P_{Y|X}) > 0 \). Then
\[
\alpha(R, P_X, P_{Y|X}) \geq \gamma(R, P_X, P_{Y|X}), \quad (194)
\]
deing that \( \alpha(R, P_X, P_{Y|X}) \) and \( \gamma(R, P_X, P_{Y|X}) \) are as defined in (37) and (190), respectively.
Proof. Suppose \((X, Y) \sim P_X P_{Y|X}\) and \((\tilde{X}, \tilde{Y}) \sim Q_{X|Y} Q_Y\)

\[
\gamma(R, P_X, P_{Y|X}) = \max_{\lambda \in [0,1]} \left\{ \frac{\lambda}{1 + \lambda} \left( R - D_{1+\lambda}(P_{XY} \| P_X P_Y) \right) \right\}
\]

(195)

\[
\begin{align*}
\gamma(R, P_X, P_{Y|X}) &= \max_{\lambda \in [0,1]} \min_{Q_{XY}} \left\{ D(Q_{XY} \| P_{XY}) + \frac{\lambda}{1 + \lambda} (R - D(Q_{XY} \| P_{X} P_{Y})) \right\} \\
&\leq \min_{Q_{XY}} \max_{\lambda \in [0,1]} \left\{ D(Q_{XY} \| P_{XY}) + \frac{\lambda}{1 + \lambda} (R - D(Q_{XY} \| P_{X} P_{Y})) \right\}
\end{align*}
\]

(196)

(197)

where (196) is due to Corollary 3 in Appendix VIII and the fact that

\[
E[i_{X,Y}(\tilde{X}; \tilde{Y})] + D(Q_{XY} \| P_{XY}) = D(Q_{XY} \| P_X P_Y);
\]

(201)

(197) is because of the duality gap; (198) follows since the function \(\frac{\alpha \lambda}{1 + \lambda}\) is decreasing or increasing in \(\lambda\) depending on whether \(\alpha < 0\) or \(\alpha > 0\); and finally (199) follows because

\[
D(Q_{XY} \| P_X P_Y) - D(Q_{XY} \| P_X Q_Y) = D(Q_Y \| P_Y)
\]

(202)

\[
\geq 0.
\]

(203)

\[
\blacksquare
\]

VII-C. Comparison with the Half of the Exact Exponent of Parizi et al. (i.i.d.)

Parizi et al. [14, Theorem 4(i)] show that\(^{20}\)

\[
\lim_{n \to \infty} - \frac{1}{n} \log \mathbb{E} \left[ D \left( P_{Y^n|\xi^n_M} \| P_{Y^n} \right) \right] = \zeta(R, P_X, P_{Y|X}),
\]

(204)

where

\[
\zeta(R, P_X, P_{Y|X}) = \min_{Q_{XY}} \left\{ D(Q_{XY} \| P_{XY}) + \left[ R - E[i_{X,Y}(\tilde{X}; \tilde{Y})] \right] \right\}
\]

(205)

\[
= \max_{\lambda \in [0,1]} \lambda \left( R - D_{1+\lambda}(P_{XY} \| P_X P_Y) \right)
\]

(206)

with \((\tilde{X}, \tilde{Y}) \sim Q_{X|Y} Q_Y\). Using Pinsker’s [18, Problem 3.18] and Jensen’s inequalities

\[
\begin{align*}
\mathbb{E} \left[ D \left( P_{Y^n|\xi^n_M} \| P_{Y^n} \right) \right] &\geq \frac{\log e}{2} \mathbb{E} \left[ \left\| P_{Y^n|\xi^n_M} - P_{Y^n} \right\|_1^2 \right] \\
&\geq \frac{\log e}{2} \mathbb{E} \left[ \left\| P_{Y^n|\xi^n_M} - P_{Y^n} \right\|_1^2 \right]
\end{align*}
\]

(207)

(208)

and one can easily see the following lower bound on the soft-covering exponent

\[
\lim_{n \to \infty} - \frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|\xi^n_M} - P_{Y^n} \right\|_1 \right] \geq \frac{1}{2} \zeta(R, P_X, P_{Y|X}).
\]

(209)

\(^{20}\)Previously, Hayashi argues the lower bound in (204) without showing the primal form of \(\zeta\) in (205), see [30].
It is easy to see that
\[ \gamma(R, P_X, P_{Y|X}) \geq \frac{1}{2} \zeta(R, P_X, P_{Y|X}), \]  
(210)
and together with Proposition 4, (210) already implies
\[ \alpha(R, P_X, P_{Y|X}) \geq \frac{1}{2} \zeta(R, P_X, P_{Y|X}). \]  
(211)
Still, non-trivial alternative ways of proving this fact are demonstrated in the proof of Proposition 5.

**Proposition 5.** Suppose \( P_X \rightarrow P_{Y|X} \rightarrow P_Y \), and \( R > I(P_X, P_{Y|X}) > 0 \). Then
\[ \alpha(R, P_X, P_{Y|X}) \geq \frac{1}{2} \zeta(R, P_X, P_{Y|X}). \]  
(212)
where \( \alpha(R, P_X, P_{Y|X}) \) and \( \zeta(R, P_X, P_{Y|X}) \) are as defined in (37) and (205), respectively.

**Proof.** Let \((X, Y) \sim P_X P_{Y|X}\) and \((\tilde{X}, \tilde{Y}) \sim Q_X Q_Y\). After noticing (cf. (188))
\[ \mathbb{E}[i_{X,Y}(\tilde{X}; \tilde{Y})] \leq D(Q_{XY} \parallel P_X P_Y), \]  
(213)
and considering the three cases where\(^{21}\)

- \( R \leq \mathbb{E}[i_{X,Y}(\tilde{X}; \tilde{Y})] \),
- \( \mathbb{E}[i_{X,Y}(\tilde{X}; \tilde{Y})] \leq R \leq D(Q_{XY} \parallel P_X P_Y) \), and
- \( R \geq D(Q_{XY} \parallel P_X P_Y) \)
yields the desired result. However, the proof is even simpler when the dual forms of \( \alpha \) and \( \frac{1}{2} \zeta \) are compared, see (166) and (206), respectively. This simple proof stems from the following inequalities: for any \( \lambda \in [0, 1] \),
\[ I_{\frac{1}{2} \lambda}(P_X, P_{Y|X}) = \frac{2}{\lambda} \log \mathbb{E} \left[ \exp \left( \frac{\lambda}{2 - \lambda} i_{X,Y}(X; Y) \right) \right] \]  
(214)
\[ \leq D_{1 + \frac{1}{2} \lambda}(P_{XY} \parallel P_X P_Y) \]  
(215)
\[ \leq D_{1 + \lambda}(P_{XY} \parallel P_X P_Y), \]  
(216)
where (215) follows from Jensen’s inequality and the concavity of \( f(x) = x^{\frac{1}{2} \lambda} \); and (216) follows from the monotonicity of Rényi divergence in its order \([26, \text{Theorem 3}]\), and the fact that \( \frac{1}{2 - \lambda} \leq \lambda \) when \( \lambda \in [0, 1] \). \( \blacksquare \)

**Example 2.** Binary Symmetric Channel. Consider the setting in Example 1, where \( P_{Y|X} : \mathcal{X} \rightarrow \mathcal{Y} \) is a binary symmetric channel with crossover probability \( p = 0.05 \), and \( P_X(0) = 2/5 \). If \( R = 0.85 > I(P_X, P_{Y|X}) \approx 0.69 \) bits,
\[ \alpha(0.85, P_X, P_{Y|X}) \approx 2.0429 \times 10^{-2}, \]  
(217)
\[ \beta(0.85, P_X, P_{Y|X}) \approx 2.0331 \times 10^{-2}, \]  
(218)
\(^{21}\)Useful identities are presented in (201) and (202).
\[ \gamma(0.85, P_X, P_{Y|X}) \approx 2.0116 \times 10^{-2}, \] (219)
\[ 0.5 \times \zeta(0.85, P_X, P_{Y|X}) \approx 1.3767 \times 10^{-2}, \] (220)

implying \( \alpha > \beta > \gamma > \frac{1}{2} \zeta \), in general.

Below, Figure 2 shows the computed \( \alpha, \beta, \gamma \) and \( \frac{1}{2} \zeta \) values for various rates \( R \). Note that, although Figure 2a shows that \( \alpha, \beta, \gamma \), are almost equal to one another for a range of \( R \) values, there exists a small but strictly positive gap between them, see, e.g., Figure 2b.

![Figure 2: Comparison of the Soft-Covering Exponents (i.i.d.)](image)

(a) When \( R - I(P_X, P_{Y|X}) \in [0.001, 0.26] \).
(b) When \( R \in [0.8, 0.805] \).

VII-D. Comparison with the Half of the Exact Exponent of Parizi et al. (constant-composition)

When the constant-composition coding ensemble \( D_M^n \) is used instead of the i.i.d. coding ensemble \( C_M^n \), Parizi et al. [14, Theorem 4(ii)] show that,

\[ \lim_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ D \left( P_{Y^n|\mathcal{D}_M^n} \| R_{Y^n} \right) \right] = \Xi(R, P_X, P_{Y|X}), \] (221)

such that

\[ \Xi(R, P_X, P_{Y|X}) = \min_{Q_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \left\{ D(P_X Q_{Y|X} \| P_{XY}) + [R - G(Q_{Y|X} \| P_{Y|X} P_X)]_+ \right\}, \] (222)

where \( P_{XY} = P_X P_{Y|X} \), and for \( P_X \to Q_{Y|X} \to Q_Y \) and \( (X, \tilde{Y}) \sim P_X Q_{Y|X} \)

\[ G(Q_{Y|X} \| P_{Y|X} P_X) = H(Q_Y) - E[t_{P_Y}(\tilde{Y}|X)] + \min_{P_X \to Q_{Y|X} \to Q_Y} D(P_X R_{Y|X} \| P_{XY}). \] (223)

Once again, using Pinsker’s [18, Problem 3.18] and Jensen’s inequalities

\[ \mathbb{E} \left[ D \left( P_{Y^n|\mathcal{D}_M^n} \| R_{Y^n} \right) \right] \geq \frac{\log e}{2} \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{D}_M^n} - R_{Y^n} \right\|_1 \right] \] (224)
Then, Hayashi and Matsumoto [31, Theorem 10] discuss that

\[ \mathcal{H}(R, P_X, P_{Y|X}) \leq \frac{1}{2} \log(1 + H(Y)) = \frac{1}{2} \mathcal{H}(R, P_X, P_{Y|X}). \]

(225)

one can easily see the following lower bound on the soft-covering exponent in the constant-composition case:

\[ \liminf_{n \to \infty} - \frac{1}{n} \log E \left[ \left\| P_{Y^n|\mathcal{G}^n} - R_{Y^n} \right\|_1 \right] \geq \frac{1}{2} \mathcal{H}(R, P_X, P_{Y|X}). \]

(226)

Since \( \mathcal{H}(R, P_X, P_{Y|X}) \) is the exact soft-covering exponent in the constant composition case, it is expected that \( \mathcal{H} \geq \frac{1}{2} \mathcal{H} \). This result is formally established by Proposition 6.

**Proposition 6.** Given an \( m \)-type \( P_X \in \mathcal{P}_m(X) \) suppose \( P_X \to P_Y \to P_Y \), and \( R > I(P_X, P_{Y|X}) > 0 \). Then,

\[ \mathcal{H}(R, P_X, P_{Y|X}) \geq \frac{1}{2} \mathcal{H}(R, P_X, P_{Y|X}), \]

(227)

where \( \mathcal{H}(R, P_X, P_{Y|X}) \) and \( \mathcal{H}(R, P_X, P_{Y|X}) \) are as defined in (39) and (222), respectively.

**Proof.** Assume \( (\tilde{X}, \tilde{Y}) \sim P_X Q_{Y|X} \). Realizing (cf. (223))

\[ G(Q_{Y|X}||P_{Y|X}|P_X) \geq H(Q_Y) - E[I_{P_X}(\tilde{Y}|\tilde{X})], \]

(228)

note that

\[ \mathcal{H}(R, P_X, P_{Y|X}) = \min_{Q_{Y|X} \in \mathcal{P}(\mathcal{Y}|X)} \left\{ D(P_X Q_{Y|X}||P_X Y) + \frac{1}{2} \left[ R - D(P_X Q_{Y|X}||P_X Q_Y) \right] \right\} \]

(229)

\[ \geq \min_{Q_{Y|X} \in \mathcal{P}(\mathcal{Y}|X)} \left\{ \frac{1}{2} D(P_X Q_{Y|X}||P_X Y) + \frac{1}{2} \left[ R + E[I_{P_X}(\tilde{Y}|\tilde{X})] - H(Q_Y) \right] \right\} \]

(230)

\[ \geq \min_{Q_{Y|X} \in \mathcal{P}(\mathcal{Y}|X)} \left\{ \frac{1}{2} D(P_X Q_{Y|X}||P_X Y) + \frac{1}{2} \left[ R - G(Q_Y||P_{Y|X}|P_X) \right] \right\} \]

(231)

\[ = \frac{1}{2} \mathcal{H}(R, P_X, P_{Y|X}) \]

(232)

where (230) follows from the facts that

\[ D(P_X Q_{Y|X}||P_X Y) = D(P_X Q_{Y|X}||P_{X Y}) + H(Q_Y) - E[I_{P_X}(\tilde{Y}|\tilde{X})] \]

(233)

and \( [f - h]_+ \geq [f]_+ - h \) for any non-negative \( h \).

**VII-E. Comparison with the Half of the Exponent of Hayashi and Matsumoto**

Hayashi and Matsumoto [31, Theorem 10] discuss that

\[ \liminf_{n \to \infty} - \frac{1}{n} \log E \left[ D \left( P_{Y^n|\mathcal{G}^n} \right| R_{Y^n} \right] \geq \mathcal{H}(R, P_X, P_{Y|X}), \]

(234)

where

\[ \mathcal{H}(R, P_X, P_{Y|X}) = \max_{\lambda \in [0,1]} \left\{ \lambda \left( R - I \frac{1}{\lambda} (P_X, P_{Y|X}) \right) \right\}. \]

(235)
Using (158)-(166) and the fact that (cf. (223) and (233))
\[ G(Q_{Y|X}||P_{Y|X}|P_X) \leq D(P_X Q_{Y|X}||P_X Q_Y), \] (236)
it is easy to establish\(^{22}\)
\[ \gamma(R, P_X, P_{Y|X}) = \max_{\lambda \in [0,1]} \left\{ \lambda \left( R - I_{\frac{1}{R}}(P_X, P_{Y|X}) \right) \right\} \]
\[ = \min_{Q_{XY}} \left\{ D(Q_{XY}||P_{XY}) + [R - D(Q_{XY}||P_X Q_Y)]_+ \right\} \]
\[ \leq \min_{Q_{Y|X}} \left\{ D(P_X Q_{Y|X}||P_{XY}) + [R - D(P_X Q_{Y|X}||P_X Q_Y)]_+ \right\} \]
\[ \leq \mathfrak{H}(R, P_X, P_{Y|X}), \] (240)
where (239) follows from the suboptimal choice \( Q_X = P_X \). Together with Proposition 6, (240) already implies that
\[ \mathfrak{H}(R, P_X, P_{Y|X}) \geq \frac{1}{2} \gamma(R, P_X, P_{Y|X}). \] (241)
An alternative proof of (241) that does not rely on (240) is presented below.

**Proposition 7.** Given an m-type \( P_X \in \mathcal{P}_m(X) \) suppose \( P_X \to P_{Y|X} \to P_Y \), and \( R > I(P_X, P_{Y|X}) > 0 \). Then,
\[ \mathfrak{H}(R, P_X, P_{Y|X}) \geq \frac{1}{2} \gamma(R, P_X, P_{Y|X}), \] (242)
where \( \mathfrak{H}(R, P_X, P_{Y|X}) \) and \( \gamma(R, P_X, P_{Y|X}) \) are as defined in (39) and (235), respectively.

**Proof.** For an arbitrary discrete distribution \( P_X \), since \( I_{\lambda}(P_X, P_{Y|X}) \) is non-decreasing in \( \lambda \), [25, Theorem 4]
\[ \alpha(R, P_X, P_{Y|X}) = \max_{\lambda \in [0,1]} \left\{ \lambda \left( R - I_{\frac{1}{R}}(P_X, P_{Y|X}) \right) \right\} \]
\[ \geq \max_{\lambda \in [0,1]} \left\{ \frac{\lambda}{2} \left( R - I_{\frac{1}{R}}(P_X, P_{Y|X}) \right) \right\}. \] (243)
Particularizing (244) by replacing \( P_X \leftarrow P_X \), the desired result is a consequence of the fact that \( \mathfrak{H} \geq \alpha \), see Remark 24. \( \blacksquare \)

**Remark 41.** In a different paper, Hayashi and Matsumoto [32, Eq. (177)] also argue that
\[ \liminf_{n \to \infty} \frac{-1}{n} \log E \left[ D \left( \hat{P}_{Y^n|X^n} \big|\big| P_{Y^n} \right) \right] \geq \mathfrak{H}(R, P_X, P_{Y|X}), \] (245)
where
\[ \mathfrak{H}(R, P_X, P_{Y|X}) = \min_{Q_{Y|X}} \left\{ D(P_X Q_{Y|X}||P_{XY}) + [R - D(P_X Q_{Y|X}||P_X Q_Y)]_+ \right\}. \] (246)
Though, since\(^{23}\) \( D(P_X Q_{Y|X}||P_{XY}) \geq 0 \), it is trivial to see that \( \mathfrak{H} \geq \frac{1}{2} \mathfrak{H} \) in this case.

\(^{22}\) Also see [14, Appendix C] for a different proof of (240).
\(^{23}\) Also see (240) together with Proposition 6.
Example 3. Binary Symmetric Channel. Consider the setting in Examples 1 and 2, where $P_{Y|X} : \mathcal{X} \rightarrow \mathcal{Y}$ is a binary symmetric channel with crossover probability $p = 0.05$, and $P_X(0) = 2/5$. If $R = 0.85 > I(P_X, P_{Y|X}) \approx 0.69 \text{ bits}$

\begin{align}
\aleph(0.85, P_X, P_{Y|X}) &\approx 2.21595 \times 10^{-2}, \\
\frac{1}{2} \aleph(0.85, P_X, P_{Y|X}) &\approx 1.60663 \times 10^{-2}, \\
\frac{1}{2} \aleph^*(0.85, P_X, P_{Y|X}) &\approx 1.10797 \times 10^{-2}, \\
\frac{1}{2} \aleph^*(0.85, P_X, P_{Y|X}) &\approx 1.02143 \times 10^{-2},
\end{align}

implying $\aleph > \frac{1}{2} \aleph > \frac{1}{2} \aleph^* > \frac{1}{2} \aleph^*$, in general.

Figure 3 illustrates the computed $\aleph$, $\frac{1}{2} \aleph$, $\frac{1}{2} \aleph^*$, and $\frac{1}{2} \aleph^*$ values for various rates $R$.

![Figure 3: Comparison of the Soft-Covering Exponents (constant-composition).](image)

Appendices

I. Proofs of Lemmas 1 and 2

This section provides the proofs of Lemmas 1 and 2 that are presented in Section III. The simple proof of Lemma 1, which can be found in [16, Theorem 31] and [33, Lemma 2], is repeated in Appendix I-A. The proof of Lemma 2, which follows the footprints of that of Lemma 1, is contained in Appendix I-B.
I-A. Proof of Lemma 1

Denoting $x_i^j = (x_i, x_{i+1}, \ldots, x_j)$, define the variation of a function $f: \mathcal{X}^m \to \mathbb{R}$ at coordinate $i$ as

$$d_i(f(x^m)) = \sup_{z, z'} |f(x_i^{i-1}, z, x_{i+1}^m) - f(x_i^{i-1}, z', x_{i+1}^m)|,$$

and observe that

$$\left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 = f(X^n_1, \ldots, X^n_M),$$

where for the given discrete memoryless channel, $P_{Y^n|X^n}$, the function $f: (\mathcal{X}^n)^M \to \mathbb{R}$ is defined as

$$f(X^n_1, \ldots, X^n_M) = \sum_{y^n \in \mathcal{Y}^n} \left| \frac{1}{M} \sum_{j=1}^M P_{Y^n|X^n}(y^n|X^n_j) - P_{Y^n}(y^n) \right|.$$

Since for any $i \in \{1, \ldots, M\}$

$$\sum_{y^n \in \mathcal{Y}^n} \left| \frac{1}{M} \sum_{j \neq i} P_{Y^n|X^n}(y^n|X^n_j) - P_{Y^n}(y^n) \right| - \frac{1}{M} \leq \sum_{y^n \in \mathcal{Y}^n} \left| \frac{1}{M} \sum_{j=1}^M P_{Y^n|X^n}(y^n|X^n_j) - P_{Y^n}(y^n) \right| \leq \sum_{y^n \in \mathcal{Y}^n} \left| \frac{1}{M} \sum_{j \neq i} P_{Y^n|X^n}(y^n|X^n_j) - P_{Y^n}(y^n) \right| + \frac{1}{M},$$

it follows that, for any $i \in \{1, \ldots, M\}$,

$$d_i \left( \left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 \right) \leq \frac{2}{M}.$$

Finally, the desired result follows from the McDiarmid’s inequality, see, e.g., [19, Theorem 2.2.3].

I-B. Proof of Lemma 2

Following the footprints of the proof of Lemma 1 in Appendix I-A, observe that

$$d_i \left( \left\| P_{Y^n|X^n} - R_{Y^n} \right\|_1 \right) \leq \frac{2}{M}.$$

Hence, once again, McDiarmid’s inequality [19, Theorem 2.2.3] yields the desired result.

II. Preliminary Lemmas for the Proofs of Theorems 1 and 2

II-A. Lemmas for the Proof of Theorem 1

This section provides several non-asymptotic results that are used in the proof of Theorem 1.

Lemma 3. Given $y^n \in \mathcal{T}_Q^n$, and $Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_{Y})$, let $Z_{Q_{X|Y}}(y^n)$ be the random variable as defined in (104), and let $P_{Y^n}$ be the i.i.d. output distribution. Then,

$$P_{Y^n}(y^n) E \left[ Z_{Q_{X|Y}}(y^n) - E[Z_{Q_{X|Y}}(y^n)] \right] \leq P_{Y^n|X^n}(y^n|a^n_{Q_{X|Y}}) \mathcal{Y}(M, Q_{X,Y}),$$

where $a^n_{Q_{X|Y}}$ represents an element from the conditional type class $\mathcal{T}_Q^n$, and $\mathcal{Y}(M, Q_{X,Y})$ is as defined in (105).
Proof. Thanks to the triangle inequality and the fact that $Z_{Q_{X^N}}(y^n) \geq 0$ almost surely,

$$E \left[ \left| Z_{Q_{X^N}}(y^n) - E[Z_{Q_{X^N}}(y^n)] \right| \right] \leq 2E[Z_{Q_{X^N}}(y^n)]$$

$$= 2l_{Q_{X^N}}(y^n)p_{Q_{X^N}}(y^n). \quad (259)$$

On the other hand, by Jensen’s inequality,

$$E \left[ \left| Z_{Q_{X^N}}(y^n) - E[Z_{Q_{X^N}}(y^n)] \right| \right] \leq E^{\frac{1}{2}} \left[ \left| Z_{Q_{X^N}}(y^n) - E[Z_{Q_{X^N}}(y^n)] \right|^2 \right]$$

$$= l_{Q_{X^N}}(y^n)M^{-\frac{1}{2}}p_{Q_{X^N}}(y^n)(1 - p_{Q_{X^N}}(y^n))^\frac{1}{2}$$

$$\leq l_{Q_{X^N}}(y^n)M^{-\frac{1}{2}}p_{Q_{X^N}}(y^n). \quad (260)$$

Combining (260) and (263) together with the fact that $l_{Q_{X^N}}(y^n) = P_{Y^n|X^n}(y^n|x^n)P_{Y^n}(y^n)$ yields (258).

Lemma 4. Let $W$ and $X$ be non-negative random variables such that $W \leq X$ almost surely. Then, for any $c \in (0, \infty)$,

$$E \left[ \frac{W}{X} \right] \geq \frac{1}{c}E[W] - \frac{1}{c}E[X1\{X > c\}]. \quad (265)$$

Proof. Since both $W$ and $X$ are non-negative,

$$E \left[ \frac{W}{X} \right] \geq E \left[ \frac{W}{X}1\{X \leq c\} \right]$$

$$\geq \frac{1}{c}E[W1\{X \leq c\}]$$

$$= \frac{1}{c}E[W] - \frac{1}{c}E[W1\{X > c\}]$$

$$\geq \frac{1}{c}E[W] - \frac{1}{c}E[X1\{X > c\}], \quad (269)$$

where (269) is due to the fact that $W \leq X$ almost surely.

Lemma 5. Let $M$ be a Poisson distributed random variable with mean $\mu > 1$. Then

$$E[|M - \mu|] \leq \frac{\mu}{\sqrt{\mu - 1}}. \quad (270)$$

Proof. Thanks to Robbins’ sharpening of Stirling’s approximation [34],

$$|\mu|! \geq |\mu|e^{-|\mu| + \frac{1}{12|\mu| + 1}}\sqrt{2\pi|\mu|}. \quad (271)$$

Let $\tau = \mu - |\mu|$, using the lower bound in (271),

$$E[|M - \mu|] = \frac{\mu|\mu|+1}{|\mu|!}2e^{-\mu} \quad (272)$$
\[ \leq \frac{2\mu e^{-\frac{\tau - 1}{\mu} \tau}}{\sqrt{2\pi |\mu|}} \left( 1 + \frac{\tau}{|\mu|} \right)^{|\mu|} \] (273)

\[ \leq \frac{\mu}{\sqrt{\mu - 1}}, \] (274)

where a proof for (272) can be found in [35], and (274) follows because \( \mu - 1 < |\mu| \) and \( (1 + \frac{\tau}{|\mu|})^{|\mu|} \leq e^\tau \). □

**Lemma 6.** Suppose that \( M \) is a Poisson distributed random variable with mean \( \mu > 1 \). Assuming \( \delta \in (\frac{1}{\mu}, 1) \)

\[ \mathbb{E}[M \{ M > (1 + \delta)\mu \}] \leq \mu a_{\delta - \frac{1}{\mu}}, \] (275)

where

\[ a_\epsilon = \frac{e^\epsilon}{(1 + \epsilon)^{1 + \epsilon}} \] (276)

is a constant which is strictly less than 1 for all \( \epsilon \in (0, 1) \).

**Proof.** Note that

\[ \mathbb{E}[M \{ M > (1 + \delta)\mu \}] \leq \mu \mathbb{P}[M > (1 + \delta)\mu - 1] \] (277)

\[ \leq \mu a_{\delta - \frac{1}{\mu}}, \] (278)

where (277) holds because \( M \) is Poisson distributed; and (278) follows from [36, Theorem 5.4].

To see \( a_\epsilon < 1 \) for any \( \epsilon \in (0, 1) \), observe that \( a_0 = 1 \) and \( a_\epsilon \) is strictly monotone decreasing in \( \epsilon \in (0, 1) \) as

\[ \frac{d \log \, a_\epsilon}{d \epsilon} = \log_e \frac{1}{1 + \epsilon} \] (279)

\[ < 0. \] (280)

□

**Lemma 7.** Suppose that \( M \) is a Poisson distributed random variable with mean \( \mu \). Given \( y^n \in \mathcal{Y}^n \),

\[ \mathbb{E}[ML_{\phi^n_M}(y^n)] = \mu. \] (281)

In particular, if \( P_{Y^n}(y^n) > 0 \),

\[ \mathbb{E}[ML_{\phi^n_M}(y^n)] = \mathbb{E} \left[ \sum_{j=1}^{M} \frac{P_{Y^n \mid X^n}(y^n \mid X^n_j)}{P_{Y^n}(y^n)} \right] \] (282)

\[ = \sum_{Q_{X^n \mid Y} \in \mathcal{P}_n(X \mid Q_Y)} \mathbb{I}_{Q_{X^n \mid Y}}(y^n) \mathbb{E}[N_{Q_{X^n \mid Y}}(y^n)] \] (283)

\[ = \mu. \] (284)

**Proof.** If \( P_{Y^n}(y^n) = 0 \), then \( L_{\phi^n_M}(y^n) = 1 \), and

\[ \mathbb{E}[ML_{\phi^n_M}(y^n)] = \mathbb{E}[M] \] (285)
Suppose $P_{Y^n}(y^n) > 0$, then by definition of $L_{Q_n}^n(y^n)$,

$$
\mathbb{E}[ML_{Q_n}^n(y^n)] = \mathbb{E} \left[ \sum_{j=1}^{M} \frac{P_{Y^n|X^n}(y^n|X^n_j)}{P_{Y^n}(y^n)} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{j=1}^{M} \frac{P_{Y^n|X^n}(y^n|X^n_j)}{P_{Y^n}(y^n)} | M \right] \right] = \mathbb{E}[M] = \mu, \quad (287)
$$

where (288) follows from the tower property of expectation.

Note that (283) follows from the linearity of expectation and the fact that both $P_{Y^n|X^n}(y^n|x^n)$ and $P_{Y^n}(y^n)$ depend on $(x^n, y^n)$ through its joint type, see (99) and the discussion therein. ■

**Lemma 8.** Suppose that $X_1, \ldots, X_m$ are mutually independent zero-mean random variables, then

$$
\mathbb{E} \left[ \sum_{i=1}^{m} X_i \right] \geq \max_{i \in \{1, \ldots, m\}} \mathbb{E} |X_i|. \quad (291)
$$

**Proof.** Without loss of generality assume

$$
\mathbb{E} |X_i| = \max_{i \in \{1, \ldots, m\}} \mathbb{E} |X_i|, \quad (292)
$$

and note that

$$
\mathbb{E} \left[ \sum_{i=1}^{m} X_i \right] = \mathbb{E} \left[ X_1 + \sum_{i=2}^{m} X_i \right] \geq \mathbb{E} \left[ X_1 + \mathbb{E} \left[ \sum_{i=2}^{m} X_i \right] \right] = \mathbb{E} |X_1|, \quad (295)
$$

where (293) follows from the tower property of expectation; (294) follows from modulus inequality and the independence of $X_1$ from $X_i$ for $i \neq 1$; lastly (295) follows as the random variables are all zero-mean. ■

**Lemma 9.** Let $N$ be a Poisson distributed random variable with mean $\xi > 0$, then

$$
\mathbb{E}[|N - \xi|] \geq \frac{1}{4} \min \left\{ 2\xi, \xi^{\frac{1}{2}} \right\}. \quad (296)
$$

**Proof.** As can be seen in [35], one can show that

$$
\mathbb{E}[|N - \xi|] = \frac{\xi^{\lfloor \xi \rfloor + 1}}{\lfloor \xi \rfloor !} 2e^{-\xi}. \quad (297)
$$

The inequality in (296) is the lower bound counter part of ‘upper bounding the absolute mean deviation of binomial random variable by either twice its mean or its standard deviation’ that can be seen in the proof of Lemma 3.
To see (296), observe that \( \xi \in (0, 1] \) implies
\[
\frac{\xi^{\lfloor \xi \rfloor + 1}}{\lfloor \xi \rfloor !} 2e^{-\xi} = 2\xi e^{-\xi} \geq \frac{1}{2}\xi.
\] (298)

On the other hand, when \( \xi \in (1, \infty) \), by Robbins’ sharpening of Stirling’s approximation [34],
\[
|\xi|! \leq |\xi|^{\lfloor \xi \rfloor} e^{-|\xi| + \frac{1}{12|\xi|}} \sqrt{2\pi|\xi|}.
\] (300)

Denoting \( \tau = \xi - \lfloor \xi \rfloor \), thanks to (300),
\[
\frac{\xi^{\lfloor \xi \rfloor + 1}}{\lfloor \xi \rfloor !} 2e^{-\xi} \geq \frac{2\xi^{1/2}}{\sqrt{2\pi |\xi|}} e^{-13/12} \geq \frac{1}{4\xi^{1/2}},
\] (301)

where (302) follows as \( 0 \leq \tau < 1 \), and \( 1 \leq |\xi| \leq \xi \). Combining (299) and (302) yields (296). \( \blacksquare \)

**Lemma 10.** Let \( M \) be a Poisson distributed random variable with mean \( \mu \geq 1 \), then
\[
P[M = \lfloor \mu \rfloor] > \frac{1}{8|\mu|^{1/2}}.
\] (304)

**Proof.** Let \( \tau = \lfloor \mu \rfloor - \mu \), using Stirling approximation as in (300),
\[
P[M = \lfloor \mu \rfloor] = \frac{\mu^{[\mu]} [\mu]! e^{-\mu}}{[\mu]!} \geq \frac{e^{\tau - \frac{13}{12} |\mu|}}{\sqrt{2\pi |\mu|}} \left(1 - \frac{\tau}{|\mu|}\right)^{[\mu]} \geq \frac{1}{8|\mu|^{1/2}},
\] (306)

where (307) follows from the facts that \( \log_e (1 - x) \geq -x - \frac{x^2}{2} \) for \( x \in [0, 1] \), \( \tau < 1 \), and \( \mu \geq 1 \). \( \blacksquare \)

**Lemma 11.** Let \( M \) be a Poisson distributed random variable with mean \( \mu \),
\[
P \left[ \left\| P_{Y^n|M} - P_{Y^n} \right\|_1 - \mathbb{E} \left[ \left\| P_{Y^n|M} - P_{Y^n} \right\|_1 \right] \right] \geq t \right] \leq 2 \exp \left( -\mu \left(1 - e^{-t^2/2}\right) \right) \leq 2 \exp \left( -\frac{\mu t^2}{2} \right).
\] (308)

**Proof.** For the sake of notational convenience, let
\[
T_n(M) = \left\| P_{Y^n|M} - P_{Y^n} \right\|_1, \quad V_n(M) = T_n(M) - \mathbb{E}[T_n(M)].
\] (310)
Conditioned on $M = m$, by Lemma 1,
\[
\mathbb{P}[|V_n(M)| \geq t | M = m] \leq 2 \exp \left( -\frac{mt^2}{2} \right). \tag{312}
\]
Hence, by the total probability law,
\[
\mathbb{P}[|V_n(M)| \geq t] \leq 2 \mathbb{E} \left[ \exp \left( -\frac{Mt^2}{2} \right) \right] \tag{313}
\]
\[
= 2 \exp \left( -\mu \left( 1 - e^{-t^2/2} \right) \right). \tag{314}
\]

To see (309), simply note that $x \geq 1 - e^{-x}$.

**Lemma 12.** Let $M$ be a Poisson distributed random variable with mean $\mu$,
\[
\mathbb{P}[[T_n([\mu]) - \mathbb{E}[T_n(M)]] \geq t] \leq 16 \lceil \mu \rceil \frac{1}{2} \exp \left( -\frac{\mu t^2}{2} \right), \tag{315}
\]
where $T_n(m) = \| P_{Y^n|C^n_m} - P_{Y^n} \|_1$.

**Proof.** Let $\tilde{M}$ be an independent copy of $M$, and observe that
\[
\mathbb{P}[[T_n([\mu]) - \mathbb{E}[T_n(M)]] \geq t] = \frac{\mathbb{P}[[T_n(\tilde{M}) - \mathbb{E}[T_n(M)]] \geq t, \tilde{M} = [\mu]]}{\mathbb{P}[\tilde{M} = [\mu]]} \tag{316}
\]
\[
= \frac{\mathbb{P}[[T_n(\tilde{M}) - \mathbb{E}[T_n(\tilde{M})]] \geq t, \tilde{M} = [\mu]]}{\mathbb{P}[\tilde{M} = [\mu]]} \tag{317}
\]
\[
\leq \frac{\mathbb{P}[[T_n(M) - \mathbb{E}[T_n(M)]] \geq t]}{\mathbb{P}[M = [\mu]]}, \tag{318}
\]
the result is immediate from Lemmas 10 and 11.

**II-B. Additional Lemmas for the Proof of Theorem 2**

This section contains some additional non-asymptotic results that are needed in Appendix VI, which contains the full proof of Theorem 2.

**Remark 42.** This section uses some additional notation which is defined in Appendix VI, see Definitions 21 and 22 therein.

**Remark 43.** Throughout Appendix II-B, $m \in \mathbb{N}$ is always fixed and $n \in \mathbb{N}$ is always a multiple of $m$. That is $n = km$ for some $k \in \mathbb{N}$, i.e., $n \in m\mathbb{N}$.

**Lemma 13.** Suppose $y^n \in T^n_{Q_Y}$, and $Q_{X|\bar{Y}} \in \mathcal{P}_n(\mathcal{X}|Q_{\bar{Y}}; P_{\bar{X}})$. Then,
\[
\sum_{x^n \in T^n_{P_{\bar{X}}}} 1 \left\{ x^n \in T^n_{Q_{X|\bar{Y}}}(y^n) \right\} = \sum_{x^n \in T^n_{P_{\bar{X}}}} 1 \left\{ (x^n, y^n) \in T^n_{Q_{X|\bar{Y}}} \right\} \tag{319}
\]
\[
= \frac{|T^n_{Q_{X|\bar{Y}}}|}{|T^n_{Q_{\bar{Y}}}|}, \tag{320}
\]
where $Q_{\bar{X}\bar{Y}} = Q_{X|\bar{Y}}Q_{\bar{Y}} = P_{\bar{X}}Q_{\bar{Y}|\bar{X}}$ for some conditional type $Q_{\bar{Y}|\bar{X}}$ given $x^n \in T^n_{P_{\bar{X}}}$. 
Proof. It is easy to get (319):

\[
\sum_{x^n \in T^n_{P_X}} 1 \{ x^n \in T^n_{Q^u_{X|Y}}(y^n) \} = \sum_{x^n \in T^n_{P_X}} 1 \{ x^n \in T^n_{Q^u_{X|Y}}(y^n) \} 1 \{ y^n \in T^n_{Q^y_Y} \} \\
= \sum_{x^n \in T^n_{P_X}} 1 \{ (x^n, y^n) \in T^n_{Q^u_{X|Y}} \}. 
\]

(321)

(322)

To establish (320), observe that

\[
|T^n_{Q^u_{X|Y}}| = \sum_{(a^n, b^n) \in \mathcal{X}^n \times \mathcal{Y}^n} 1 \{ (a^n, b^n) \in T^n_{Q^u_{X|Y}} \} = \sum_{b^n \in T^n_{Q^y_Y}} \sum_{a^n \in T^n_{P_X}} 1 \{ (a^n, b^n) \in T^n_{Q^u_{X|Y}} \} = \sum_{b^n \in T^n_{Q^y_Y}} \sum_{a^n \in T^n_{P_X}} 1 \{ (a^n, y^n) \in T^n_{Q^u_{X|Y}} \} = |T^n_{Q^y_Y}| \sum_{x^n \in T^n_{P_X}} 1 \{ (x^n, y^n) \in T^n_{Q^u_{X|Y}} \}. 
\]

(323)

(324)

(325)

(326)

(327)

where (324) follows because \( \mathcal{X} \)- and \( \mathcal{Y} \)-marginals of \( Q_{X|Y} \) are fixed to be \( P_X \) and \( Q_Y \); (326) follows because the summand in the right of (325) depends on \( b^n \) only through its type \( Q_Y \) and \( y^n \in T^n_{Q^y_Y} \).

Lemma 14. Suppose \( y^n \in T^n_{Q^y_Y} \), and \( Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q^y_Y; P_X) \). Then,

\[
\bar{p}_{Q_{X|Y}}(y^n) = \frac{|T^n_{Q^u_{X|Y}}|}{|T^n_{P_X}| |T^n_{Q^y_Y}|}, 
\]

(328)

where \( Q_{X|Y} = Q_{X|Y} Q^u_{Y|X} = P_X Q^y_Y \) for some conditional type \( Q^u_{Y|X} \) given \( x^n \in T^n_{P_X} \).

Proof.

\[
\bar{p}_{Q_{X|Y}}(y^n) = \mathbb{P} \left[ \hat{X}^n \in T^n_{Q^u_{X|Y}}(y^n) \right] \\
= \frac{1}{|T^n_{P_X}|} \sum_{x^n \in T^n_{P_X}} 1 \{ x^n \in T^n_{Q^u_{X|Y}}(y^n) \} \\
= \frac{1}{|T^n_{P_X}|} \sum_{x^n \in T^n_{P_X}} 1 \{ (x^n, y^n) \in T^n_{Q^u_{X|Y}} \} \\
= \frac{|T^n_{Q^u_{X|Y}}|}{|T^n_{P_X}| |T^n_{Q^y_Y}|},
\]

(329)

(330)

(331)

(332)

where (331) and (332) both follow from Lemma 13.

Lemma 15. Given \( y^n \in T^n_{Q^y_Y} \), and \( Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q^y_Y; P_X) \), let \( \bar{Z}_{Q_{X|Y}}(y^n) \) be the random variable as defined in (429), and let \( \bar{R}_{XY} \), be the channel output distribution defined in (417). Then,

\[
R_{XY}(y^n) \mathbb{E} \left[ \left| \bar{Z}_{Q_{X|Y}}(y^n) - \mathbb{E}[\bar{Z}_{Q_{X|Y}}(y^n)] \right| \right] \leq P_{Y^n|X^n}(y^n|x^n_{Q^u_{X|Y}}) \hat{\mathcal{Q}}(M, Q_{X|Y}), 
\]

(333)

where \( x^n_{Q^u_{X|Y}} \) represents an element from the type class \( T^n_{Q^u_{X|Y}} \), and \( \hat{\mathcal{Q}}(M, Q_{X|Y}) \) is as defined in (430).
Proof. Tailoring the proof of Lemma 3 by replacing

\[ Z_{Q,X|Y}(y^n) \leftarrow \tilde{Z}_{Q,X|Y}(y^n), \]  
\[ I_{Q,X|Y}(y^n) \leftarrow \tilde{I}_{Q,X|Y}(y^n), \]  
\[ p_{Q,X|Y}(y^n) \leftarrow \tilde{p}_{Q,X|Y}(y^n), \]  
\[ P_{Y^n}(y^n) \leftarrow R_{Y^n}(y^n), \]  

the result follows after invoking Lemma 14.  

Lemma 16. Suppose that \( M \) is a Poisson distributed random variable with mean \( \mu \). Given \( y^n \in Y^n \),

\[ \mathbb{E}[M \tilde{L}_{\mathcal{D}_n}(y^n)] = \mu. \]  

In particular, if \( P_{Y^n}(y^n) > 0 \),

\[ \mathbb{E}[M \tilde{L}_{\mathcal{D}_n}(y^n)] = \mathbb{E} \left[ \sum_{j=1}^M P_{Y^n|X^n}(y^n|x^n_j) \tilde{X}_n \right] / R_{\tilde{Y}^n}(y^n) \]  
\[ = \sum_{Q_{X|Y} \in \mathcal{P}_n(X|QY;P_X)} \tilde{I}_{Q,X|Y}(y^n) \mathbb{E} \left[ \tilde{N}_{Q,X|Y}(y^n) \right] \]  
\[ = \mu. \]  

Proof. If \( R_{\tilde{Y}^n}(y^n) = 0 \), then \( \tilde{L}_{\mathcal{D}_n}(y^n) = 1 \), and

\[ \mathbb{E}[M \tilde{L}_{\mathcal{D}_n}(y^n)] = \mathbb{E}[M] \]  
\[ = \mu. \]  

Suppose \( R_{\tilde{Y}^n}(y^n) > 0 \), then by definition of \( \tilde{L}_{\mathcal{D}_n}(y^n) \),

\[ \mathbb{E}[M \tilde{L}_{\mathcal{D}_n}(y^n)] = \mathbb{E} \left[ \sum_{j=1}^M P_{X^n|Y^n}(y^n|x^n_j) \tilde{X}_n \right] / R_{\tilde{Y}^n}(y^n) \]  
\[ = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{j=1}^M P_{X^n|Y^n}(y^n|x^n_j) \tilde{X}_n \right] | M \right] \]  
\[ = \mathbb{E}[M] \]  
\[ = \mu, \]  

where (345) follows from the tower property of expectation.

Note that (340) is thanks to the linearity of expectation and the fact that both \( P_{Y^n|X^n}(y^n|x^n) \) and \( R_{\tilde{Y}^n}(y^n) \) depend on \((x^n, y^n)\) through its joint type, see (423) and the discussion therein.  

Lemma 17. Let \( M \) be a Poisson distributed random variable with mean \( \mu \),

\[ \mathbb{P} \left[ \left\| P_{Y^n|X^n} - R_{\tilde{Y}^n} \right\|_1 \geq \epsilon \right] \leq 2 \exp \left( -\mu \left( 1 - e^{-\epsilon^2/2} \right) \right) \leq 2 \exp \left( -\frac{\mu \epsilon^2}{2} \right). \]
Proof. For the sake of notational convenience, let
\[
\tilde{T}_n(M) = \left\| P_{Y^n|\tilde{P}_Y} - R_{\tilde{Y}^n} \right\|_1, \tag{350}
\]
\[
\tilde{V}_n(M) = \tilde{T}_n(M) - E[\tilde{T}_n(M)]. \tag{351}
\]
Following the idea in the proof of Lemma 11; by Lemma 2, and the total probability law,
\[
P(\tilde{V}_n(M) \geq t) \leq 2E \left[ \exp\left(-\frac{Mt^2}{2}\right) \right] = 2 \exp\left(-\mu \left(1 - e^{-t^2/2}\right)\right). \tag{352}
\]
To see (349), simply note that \(x \geq 1 - e^{-x}\). \(\blacksquare\)

**Lemma 18.** Let \(M\) be a Poisson distributed random variable with mean \(\mu\),
\[
P(\tilde{T}_n([\mu]) - E[\tilde{T}_n(M)] \geq t) \leq 16\left[\mu\right]^{2} \exp\left(-\frac{\mu t^2}{2}\right), \tag{354}
\]
where \(\tilde{T}_n(m) = \left\| P_{Y^n|\tilde{P}_Y^n} - R_{\tilde{Y}^n}\right\|_1\).

**Proof.** Tailoring the proof of Lemma 12 by replacing \(T_n(m) \leftarrow \tilde{T}_n(m)\),
the result is immediate from Lemmas 10 and 17. \(\blacksquare\)

### III. Asymptotic Exponents of the Key Quantities

This section provides the asymptotic\(^{25}\) exponents of the several key quantities that play a central role in the proofs of Theorems 1 and 2.

**Lemma 19.** Fix \((x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n\), and let \(Q_{\bar{X}\bar{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})\) denote its type.
\[
P_{Y^n|X^n}(y^n|x^n) = \exp(-nE[\mathbb{I}_{P_{Y^n|X^n}}(\bar{Y}|\bar{X})]), \tag{355}
\]
where \((\bar{X}, \bar{Y}) \sim Q_{\bar{X}\bar{Y}}\).

**Proof.** Since \(P_{Y^n|X^n}\) is an \(n\)-shot stationary discrete memoryless channel,
\[
P_{Y^n|X^n}(y^n|x^n) = \prod_{i=1}^{n} P_{Y|X}(y_i|x_i) \tag{356}
\]
\[
= \prod_{(a,b) \in \mathcal{X} \times \mathcal{Y}} P_{Y|X}^{nQ_{\bar{X}\bar{Y}}(a,b)}(b|a) \tag{357}
\]
\[
= \exp(-nE[\mathbb{I}_{P_{Y^n|X^n}}(\bar{Y}|\bar{X})]), \tag{358}
\]
where in (357) \(nQ_{\bar{X}\bar{Y}}(a,b)\) denotes the number of times that \((a,b) \in \mathcal{X} \times \mathcal{Y}\) appears in \(\{(x_i, y_i)\}_{i=1}^{n}\). \(\blacksquare\)

\(^{25}\)Non-asymptotic exponents are given wherever possible which are then used in proving the finite block-length results contained in Appendix VII.
Lemma 20. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite alphabets. The set of all joint $n$-types on $\mathcal{X} \times \mathcal{Y}$, namely $\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$, satisfies

$$1 \leq |\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})| \leq (n + 1)^{|\mathcal{X}||\mathcal{Y}|}. \tag{359}$$

Proof. See [18, Lemma 2.2].

Lemma 21. Let $T^n_{Q_Y}$ denote the set of all $y^n \in \mathcal{Y}^n$ whose type is $Q_Y \in \mathcal{P}_n(\mathcal{Y})$,

$$(n + 1)^{-|\mathcal{Y}|} \exp(nH(Q_Y)) \leq |T^n_{Q_Y}| \leq \exp(nH(Q_Y)). \tag{360}$$

Proof. See [18, Lemma 2.3].

Lemma 22. Fix $y^n \in \mathcal{Y}^n$, and let $T^n_{Q_{X|Y}}(y^n)$ denote the set of sequences $x^n \in \mathcal{X}^n$ having conditional type $Q_{X|Y}$ given $y^n$,

$$(n + 1)^{-|\mathcal{X}||\mathcal{Y}|} \exp(nH(\bar{X}|\bar{Y})) \leq |T^n_{Q_{X|Y}}(y^n)| \leq \exp(nH(\bar{X}|\bar{Y})). \tag{361}$$

Proof. See [18, Lemma 2.5].

Lemma 23. Fix $y^n \in \mathcal{Y}^n$, and let $Q_Y \in \mathcal{P}_n(\mathcal{Y})$ denote its type. For any $Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_Y)$

$$p_{Q_{X|Y}}(y^n) = \mathbb{P} \left[ X^n \in T^n_{Q_{X|Y}}(y^n) \right] = \exp \left( -n \mathbb{E}[T_{P_X}(\bar{X})] \right) |T^n_{Q_{X|Y}}(y^n)|, \tag{362}$$

where $p_{Q_{X|Y}}(y^n)$ is defined in (103), $\{X_i\}_{i=1}^n$ are i.i.d. according to $P_X$, and $\bar{X} \sim Q_{\bar{X}}$ with $Q_{\bar{X}}$ denoting the $\mathcal{X}$-marginal of the joint $n$-type $Q_{\bar{X}|\bar{Y}} \bar{Q}_Y$.

Proof. Note that

$$\mathbb{P} \left[ X^n \in T^n_{Q_{X|Y}}(y^n) \right] = \sum_{x^n \in T^n_{Q_{X|Y}}(y^n)} P_{X^n}(x^n) \tag{364}$$

$$= \sum_{x^n \in T^n_{Q_{X|Y}}(y^n)} \prod_{i=1}^n P_{X_i}(x_i) \tag{365}$$

$$= \sum_{x^n \in T^n_{Q_{X|Y}}(y^n)} \prod_{a \in \mathcal{X}} P^n_{X}(a) \tag{366}$$

$$= \exp \left( -n \mathbb{E}[T_{P_X}(\bar{X})] \right) |T^n_{Q_{X|Y}}(y^n)|, \tag{367}$$

where in (366) $nQ_X(a) \in \{0, 1, \ldots, n\}$ denotes the number of times that $a \in \mathcal{X}$ appears in $\{x_i\}_{i=1}^n$.

Lemma 24. Let $\mathbb{Q}(M, Q_{\bar{X}|\bar{Y}})$ be as defined in (105). Assuming$^{26}$ $M = \exp(nR)$, for any $Q_{\bar{X}|\bar{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$

$$D(Q_{\bar{X}|\bar{Y}}||P_XQ_Y) + \frac{1}{2} \left[ R - D(Q_{\bar{X}|\bar{Y}}||P_XQ_Y) \right]_+ \tag{105}$$

$^{26}$For the ease of presentation, the fact that $M$ is an integer is ignored. A more careful analysis with $M = \lceil \exp(nR) \rceil$ results in $\kappa_n = \frac{|\mathcal{X}||\mathcal{Y}|}{n} \log(n + 1) + \frac{1}{n} \log(2\sqrt{2})$ as $\exp(nR) \leq \lceil \exp(nR) \rceil \leq 2 \exp(nR)$.  

46
\[
\begin{align*}
\text{where } Q_Y & \text{ is the } Y\text{-marginal of } Q_{XY}, \text{ and} \\
[f]_+ &= \max\{0, f\}, \\
\kappa_n &= \frac{|X||Y|}{n} \log(n + 1) + \frac{1}{n} \log 2. 
\end{align*}
\]

Proof. Noting that
\[
\mathbb{E}[t_{P_X}(\bar{X})] - H(\bar{X}|\bar{Y}) = D(Q_{XY}\|P_XQ_Y),
\]
where \((\bar{X}, \bar{Y}) \sim Q_{XY}Q_Y\), (368) is a direct consequence of Lemma 23 and the upper bound in (361). To see (369), observing
\[
\frac{1}{2} \mathcal{D}(M, Q_{XY}) = p_{Q_{XY}}(y^n) \min \left\{ 1, \frac{1}{2} M^{-\frac{1}{2}} M^{-\frac{1}{2}} p_{Q_{XY}}(y^n) \right\},
\]
and applying Lemmas 22 and 23 suffices.

\[\blacksquare\]

Lemma 25. Given a joint \(n\)-type \(Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})\), suppose that \((\bar{X}, \bar{Y}) \sim Q_{XY}\). Let \(\mathcal{D}(M, Q_{XY})\) be as defined in (105), then
\[
\lim_{n \to \infty} -\frac{1}{n} \log \max_{Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} \left\{ \frac{1}{2} \mathcal{D}^n_{Q_{XY}}(\mathbb{E}[t_{P_Y|X}(\bar{Y}|\bar{X})]) \mathcal{D}(M, Q_{XY}) \right\} = \inf_{Q_{XY} \in \mathcal{P}_\infty(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY}\|P_X) + \frac{1}{2} [R - D(Q_{XY}\|P_XQ_Y)]_+ \right\},
\]
where
\[
\mathcal{P}_\infty(\mathcal{X} \times \mathcal{Y}) = \bigcup_{n=1}^\infty \mathcal{P}_n(\mathcal{X} \times \mathcal{Y}), \\
[f]_+ &= \max\{0, f\}.
\]

Proof. Using Lemmas 21 and 24, and the fact that
\[
D(Q_{XY}\|P_XQ_Y) - H(Q_Y) + \mathbb{E}[t_{P_Y|X}(\bar{Y}|\bar{X})] = D(Q_{XY}\|P_{XY}),
\]
it follows that, for any fixed \(n\),
\[
\begin{align*}
\min_{Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY}\|P_{XY}) + \frac{1}{2} [R - D(Q_{XY}\|P_XQ_Y)]_+ \right\} & \leq -\frac{1}{n} \log \max_{Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} \left\{ \frac{1}{2} \mathcal{D}^n_{Q_{XY}}(\mathbb{E}[t_{P_Y|X}(\bar{Y}|\bar{X})]) \mathcal{D}(M, Q_{XY}) \right\} \\
& \leq \min_{Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY}\|P_{XY}) + \frac{1}{2} [R - D(Q_{XY}\|P_XQ_Y)]_+ \right\} + \kappa_n + \frac{|Y|}{n} \log(n + 1),
\end{align*}
\]
where \(\kappa_n\) is as defined in (371). Taking \(n \to \infty\) yields the desired result as \(\kappa_n \to 0\).
Lemma 26. Given an \(m\)-type \(P_X \in \mathcal{P}_m(\mathcal{X})\) and a conditional type\(^{27}\) \(Q_{Y|X} \in \mathcal{P}_n(\mathcal{Y}|P_X)\) given \(x^n \in \mathcal{T}_{P_X}^n\), suppose \((\hat{X}, \hat{Y}) \sim P_X Q_{Y|\hat{X}}\). Let \(M = \exp(nR)\), and \(\hat{\mathcal{F}}(M, P_X Q_{Y|\hat{X}})\) be as defined in (430), then

\[
\lim_{n \to \infty} -\frac{1}{n} \log \max_{Q_{Y|X} \in \mathcal{P}_n(\mathcal{Y}|P_X)} \left\{ \frac{1}{2} |T_{Q_Y}^n| \exp(-n\mathbb{E}[t_{P_Y|X}(\hat{Y} | \hat{X})]) \hat{\mathcal{F}}(M, P_X Q_{Y|\hat{X}}) \right\} = \inf_{Q_{Y|X} \in \mathcal{P}_n(\mathcal{Y}|P_X)} \left\{ D(P_X Q_{Y|X}||X) + \frac{1}{2} [R - D(P_X Q_{Y|X}||X)]_+ \right\},
\]

where

\[
\mathcal{P}_\infty(\mathcal{Y}|P_X) = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n(\mathcal{Y}|P_X),
\]

\[
P_{XY} = P_X P_Y|X,
\]

\[
[f]_+ = \max\{0, f\}.
\]

Proof. First, note that

\[
H(P_X) + \mathbb{E}[t_{P_Y|X}(\hat{Y} | \hat{X})] - H(P_X Q_{Y|X}) = D(P_X Q_{Y|X}||X),
\]

\[
H(P_X) + H(Q_Y) - H(P_X Q_{Y|X}) = D(P_X Q_{Y|X}||X),
\]

Using Lemma 21 tailored for the type classes \(\mathcal{T}_{P_X}^n, \mathcal{T}_{Q_Y}^n,\) and \(\mathcal{T}_{P_X Q_{Y|X}}^n\), it follows for any fixed \(n\) that

\[
\min_{Q_{Y|X} \in \mathcal{P}_n(\mathcal{Y}|P_X)} \left\{ D(P_X Q_{Y|X}||X) + \frac{1}{2} [R - D(P_X Q_{Y|X}||X)]_+ \right\} - \bar{\iota}_n
\]

\[
\leq -\frac{1}{n} \log \max_{Q_{Y|X} \in \mathcal{P}_n(\mathcal{Y}|P_X)} \left\{ \frac{1}{2} |T_{Q_Y}^n| \exp(-n\mathbb{E}[t_{P_Y|X}(\hat{Y} | \hat{X})]) \hat{\mathcal{F}}(M, P_X Q_{Y|\hat{X}}) \right\}
\]

\[
\leq \min_{Q_{Y|X} \in \mathcal{P}_n(\mathcal{Y}|P_X)} \left\{ D(P_X Q_{Y|X}||X) + \frac{1}{2} [R - D(P_X Q_{Y|X}||X)]_+ \right\} + \bar{\kappa}_n,
\]

where

\[
\bar{\iota}_n = \frac{\mathcal{X} \log(n+1)}{2n} - \frac{1}{n} \log 2, \quad \bar{\kappa}_n = \frac{|\mathcal{X}| + 2|\mathcal{Y}|}{2n} \log(n+1) + \frac{1}{n} \log 2.
\]

Taking \(n \to \infty\) yields the desired result as both \(\bar{\iota}_n \to 0\), and \(\bar{\kappa}_n \to 0\).

\[\blacksquare\]

IV. Optimization over Types in the Limit

IV-A. Optimization over Joint Types in the Limit

Lemma 27. Let \(\mathcal{P}_\infty(\mathcal{X} \times \mathcal{Y}) = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})\). Then,

\[
\inf_{Q_{XY} \in \mathcal{P}_\infty(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY}||X) + \frac{1}{2} [R - D(Q_{XY}||X)]_+ \right\}
\]

\[
= \min_{Q_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY}||X) + \frac{1}{2} [R - D(Q_{XY}||X)]_+ \right\}.
\]

\(^{27}\)We assume \(n \in m\mathbb{N}\).

\(^{28}\)For the ease of presentation, the fact that \(M\) is an integer is ignored. A more careful analysis with \(M = \lceil\exp(nR)\rceil\) results in \(\bar{\kappa}_n = \frac{|\mathcal{X}| + 2|\mathcal{Y}|}{2n} \log(n+1) + \frac{1}{n} \log(2\sqrt{2})\) as \(\exp(nR) \leq \lceil\exp(nR)\rceil \leq 2\exp(nR)\).
Proof. First of all, since \( P_n(\mathcal{X} \times \mathcal{Y}) \subseteq \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \) for all \( n \in \mathbb{N} \), \( \geq \) is trivial in (390). To establish \( \leq \), let \( Q^*_{XY} \) be the minimizer in the right side of (390). We may assume that \( Q^*_{XY} \ll P_{XY} \), otherwise \( D(Q^*_{XY} \| P_{XY}) = +\infty \) which contradicts the minimality of \( Q^*_{XY} \). Since for every \( Q_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \) either \( Q_{XY} \in P_{\infty}(\mathcal{X} \times \mathcal{Y}) \) or \( Q_{XY} \) is a limit point of \( P_{\infty}(\mathcal{X} \times \mathcal{Y}) \), it follows that \( P_{\infty}(\mathcal{X} \times \mathcal{Y}) \) is dense in \( \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \). Hence, one can find a sequence of types \( \{ Q^*_{XY[k]} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \}_{k \in \mathbb{N}} \) such that

\[
\lim_{k \to \infty} \left\| Q_{XY} - Q^*_{XY[k]} \right\|_1 = 0. \tag{391}
\]

We may assume \( Q^*_{XY[k]} \ll P_{XY} \) as well. Note that, for all \( k \in \mathbb{N} \),

\[
\inf_{Q_{XY} \in \mathcal{P}_{\infty}(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q_{XY} \| P_X Q_Y) \right]_+ \right\} \\
\leq D(Q^*_{XY[k]} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q^*_{XY[k]} \| P_X Q^*_{Y[k]} \) \right]_+, \tag{392}
\]

where

\[
Q^*_{XY[k]}(y) = \sum_{x \in \mathcal{X}} Q^*_{XY[k]}(x, y). \tag{393}
\]

Since both \( D(Q_{XY} \| P_{XY}) \) and \( D(Q_{XY} \| P_X Q_Y) \) are convex functions of \( Q_{XY} \) on the finite dimensional space \( \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \), they are continuous in \( Q_{XY} \) over the set of discrete distributions that are absolutely continuous with respect to \( P_{XY} \), see, e.g., [37, Section 7.9]. Therefore,

\[
\inf_{Q_{XY} \in \mathcal{P}_{\infty}(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q_{XY} \| P_X Q_Y) \right]_+ \right\} \\
\leq \lim_{k \to \infty} D(Q^*_{XY[k]} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q^*_{XY[k]} \| P_X Q^*_{Y[k]} \) \right]_+ \tag{394}
\\
= D(Q^*_{XY} \| P_{XY}) + \frac{1}{2} \left[ R - D(Q^*_{XY} \| P_X Q^*_{Y} \) \right]_+, \tag{395}
\]

where (394) is due to (392); and in (395) \( Q^*_Y(y) = \sum_{x \in \mathcal{X}} Q^*_{XY}(x, y) \).

**IV-B. Optimization over Conditional Types in the Limit**

**Lemma 28.** Given an \( m \)-type \( P_X \in \mathcal{P}_m(\mathcal{X}) \), let \( \mathcal{P}_\infty(\mathcal{Y}|P_X) = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n(\mathcal{Y}|P_X) \). Then,

\[
\inf_{Q_{Y|X} \in \mathcal{P}_\infty(\mathcal{Y}|P_X)} \left\{ D(P_X Q_{Y|X} \| P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|X} \| P_X Q_Y) \right]_+ \right\} \\
= \min_{Q_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \left\{ D(P_X Q_{Y|X} \| P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|X} \| P_X Q_Y) \right]_+ \right\}, \tag{396}
\]

where \( P(\mathcal{Y}|\mathcal{X}) \) denotes the set of all random transformations from \( \mathcal{X} \) to \( \mathcal{Y} \), \( P_{XY} = P_X P_{Y|X} \), and \( Q_Y \) is such that \( P_X \rightarrow Q_{Y|X} \rightarrow Q_Y \).

**Proof.** Since \( \mathcal{P}_n(\mathcal{Y}|P_X) \subseteq \mathcal{P}(\mathcal{Y}|\mathcal{X}) \) for all \( n \in m\mathbb{N} \), \( \geq \) is trivial in (396). To establish \( \leq \), let \( Q^*_{Y|X} \) be the minimizer in the right side of (396). We may assume that \( P_X Q^*_{Y|X} \ll P_{XY} \), otherwise \( D(P_X Q^*_{Y|X} \| P_{XY}) = +\infty \), which contradicts the minimality of \( Q^*_{Y|X} \). Since for every probability transition matrix \( Q^*_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X}) \)
either $Q_{Y|\mathcal{X}} \in \mathcal{P}_\infty(\mathcal{Y}|P_X)$ or $Q_{Y|X}$ is a limit point of $\mathcal{P}_\infty(\mathcal{Y}|P_X)$, it follows that $\mathcal{P}_\infty(\mathcal{Y}|P_X)$ is dense in $\mathcal{P}(\mathcal{Y}|\mathcal{X})$. Hence, we can find a sequence of conditional types $\{Q_{Y|\mathcal{X}}^*[k]\} \subset \mathcal{P}_\infty(\mathcal{Y}|P_X)$ such that

$$\lim_{k \to \infty} \left\| Q_{Y|\mathcal{X}}^*[k] - Q_{Y|\mathcal{X}}^* \right\|_1 = 0. \quad (397)$$

We may assume $P_X Q_{Y|\mathcal{X}}^*[k] \ll P_{XY}$ as well. Note that, for all $k \in \mathbb{N}$,

$$\inf_{Q_{Y|\mathcal{X}} \in \mathcal{P}_\infty(\mathcal{Y}|P_X)} \left\{ D(P_X Q_{Y|\mathcal{X}} \parallel P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|\mathcal{X}} \parallel P_X Q_Y) \right]_+ \right\} \leq D \left( P_X Q_{Y|\mathcal{X}}^*[k] \parallel P_{XY} \right) + \frac{1}{2} \left[ R - D \left( P_X Q_{Y|\mathcal{X}}^*[k] \parallel P_X Q_Y^*[k] \right) \right]_+, \quad (398)$$

where

$$Q_{Y|\mathcal{X}}^*[k] = \sum_{x \in \mathcal{X}} P_X(x) Q_{Y|\mathcal{X}}^*[k](\cdot|x). \quad (399)$$

Since both $D(Q_{Y|\mathcal{X}} \parallel P_{Y|\mathcal{X}} P_X)$ and $D(Q_{Y|\mathcal{X}} \parallel Q_Y P_X)$ are convex in $Q_{Y|\mathcal{X}}$ on the finite dimensional space of discrete distributions, it follows that they are continuous in $Q_{Y|\mathcal{X}}$, see, e.g., [37, Section 7.9]. Therefore, we can find a sequence of unconditional types $\{P_X^*[k]\} \subset \mathcal{P}_\infty(\mathcal{X})$ such that

$$\inf_{Q_{Y|\mathcal{X}} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \left\{ D(P_X Q_{Y|\mathcal{X}} \parallel P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|\mathcal{X}} \parallel P_X Q_Y) \right]_+ \right\} \leq \lim_{k \to \infty} \left\{ D \left( P_X Q_{Y|\mathcal{X}}^*[k] \parallel P_{XY} \right) + \frac{1}{2} \left[ R - D \left( P_X Q_{Y|\mathcal{X}}^*[k] \parallel P_X Q_Y^*[k] \right) \right]_+ \right\} \leq D \left( P_X Q_{Y|\mathcal{X}}^* \parallel P_{XY} \right) + \frac{1}{2} \left[ R - D \left( P_X Q_{Y|\mathcal{X}}^* \parallel P_X Q_Y^* \right) \right]_+, \quad (400)$$

and

$$= \min_{Q_{Y|\mathcal{X}} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \left\{ D(P_X Q_{Y|\mathcal{X}} \parallel P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|\mathcal{X}} \parallel P_X Q_Y) \right]_+ \right\}, \quad (401)$$

where (400) follows from (398); and in (401) $Q_Y = \sum_{x \in \mathcal{X}} P_X(x) Q_{Y|\mathcal{X}}^*(\cdot|x)$.

### V. Probability of $Q_{X|Y}$-shell Under $P_X^n$

As discussed in Remark 22, by separating the minimization over joint distributions, one might alternatively write the soft-covering exponent as

$$\min_{Q_{Y}} \left\{ D(Q_Y \parallel P_Y) + \min_{Q_{X|Y}} \left\{ D(Q_{X|Y} \parallel P_{X|Y} Q_Y) + \frac{1}{2} \left[ R - D(Q_{X|Y} \parallel P_X Q_Y) \right]_+ \right\} \right\}. \quad (403)$$

Within the inner minimization in (403), in light of Lemmas 22 and 23 in Appendix III, $D(Q_{X|Y} \parallel P_X Q_Y)$ corresponds to the limiting exponent of $p_{Q_{X|Y}}(y^n) = \mathbb{P}[X^n \in \bar{T}_{Q_{X|Y}}^n(y^n)]$. Thanks to the bound in (262), it is easy to see that the conditional types $Q_{X|Y}$ with $p_{Q_{X|Y}}(y^n) = 1$ has no affect in finding the exact soft-covering exponent. Therefore, constraining the inner optimization over conditional types with strictly positive $D(Q_{X|Y} \parallel P_X Q_Y)$ should not have an effect on the inner minimization. This phenomenon is formally established in Lemma 29 below.

**Lemma 29.** Given a fixed $Q_Y$, let $\Lambda = D(P_X Q_Y).$ Then,

$$\min_{Q_{X|Y}} \left\{ D(Q_{X|Y} \parallel P_{X|Y} Q_Y) + \frac{1}{2} \left[ R - D(Q_{X|Y} \parallel P_X Q_Y) \right]_+ \right\}$$

50
\[
\min_{D(Q_{X|Y}) \leq A} \left\{ D(Q_{X|Y} \parallel P_{X|Y}) + \frac{1}{2} [R - D(Q_{X|Y} \parallel P_{X})]_+ \right\}.
\] (404)

**Proof.** We may assume that \( \Lambda > 0 \), otherwise the statement is a tautology. The \( \leq \) is clear in (404). To see the \( \geq \), assume \( Q^*_X \mid Y \) to be the minimizer on the left side of (404) such that \( D(Q^*_X \mid Y) < A \), otherwise there is nothing to show. Since
\[
D(Q^*_X \mid Y) > 0,
\]
\[
[R - D(Q^*_X \mid Y)]_+ \geq [R - A]_+,
\]
the choice \( Q_X \mid Y = P_X \mid Y \) on the right side of (404) would give us \( > \) in (404). \( \blacksquare \)

### VI. Proof of Theorem 2

This section proves Theorem 2. In doing so, some additional notions, namely, the set of joint types with fixed \( X \)- and \( Y \)-marginals, and the set of conditional types with fixed marginals, will be of use. The following definitions set the notation.

**Definition 21.** Set of Joint Types with Fixed Marginals. Consider the set of joint \( n \)-types \( P_n(\mathcal{X} \times \mathcal{Y}) \). The subset \( P_n(\mathcal{X} \times \mathcal{Y}; Q_X \times Q_Y) \subset P_n(\mathcal{X} \times \mathcal{Y}) \) denotes the set of all joint \( n \)-types whose \( X \)-marginal is fixed to be \( Q_X \) and \( Y \)-marginal is fixed to be \( Q_Y \). That is
\[
P_n(\mathcal{X} \times \mathcal{Y}; Q_X \times Q_Y) = \left\{ Q_{\mathcal{X} \times \mathcal{Y}} : \sum_{b \in \mathcal{Y}} Q_{\mathcal{X} \times \mathcal{Y}}(\cdot, b) = Q_X \text{ and } \sum_{a \in \mathcal{X}} Q_{\mathcal{X} \times \mathcal{Y}}(a, \cdot) = Q_Y \right\}.
\] (407)

Similarly, the subset \( P_n(\mathcal{X} \times \mathcal{Y}; Q_X \times \cdot) \) (respectively, \( P_n(\mathcal{X} \times \mathcal{Y}; \cdot \times Q_Y) \)) denotes the set of joint \( n \)-types on \( X \times \mathcal{Y} \) whose \( X \)-marginal is fixed to be \( Q_X \) (respectively, \( Y \)-marginal is fixed to be \( Q_Y \)). That is,
\[
P_n(\mathcal{X} \times \mathcal{Y}; Q_X \times \cdot) = \left\{ Q_{\mathcal{X} \times \mathcal{Y}} : \sum_{b \in \mathcal{Y}} Q_{\mathcal{X} \times \mathcal{Y}}(\cdot, b) = Q_X \right\},
\] (408)
\[
P_n(\mathcal{X} \times \mathcal{Y}; \cdot \times Q_Y) = \left\{ Q_{\mathcal{X} \times \mathcal{Y}} : \sum_{a \in \mathcal{X}} Q_{\mathcal{X} \times \mathcal{Y}}(a, \cdot) = Q_Y \right\}.
\] (409)

**Definition 22.** Set of Conditional Types with Fixed Marginals. Consider \( P_n(\mathcal{X} \mid Q_Y) \), the set of all conditional types given \( y^n \in \mathcal{T}_{Q_Y}^n \). The subset \( P_n(\mathcal{X} \mid Q_Y; P_X) \subset P_n(\mathcal{X} \mid Q_Y) \) denotes the set of conditional types given \( y^n \in \mathcal{T}_{Q_Y}^n \) with a fixed \( X \)-marginal \( P_X \). That is,
\[
P_n(\mathcal{X} \mid Q_Y; P_X) = \left\{ Q_{\mathcal{X} \mid Y} : Q_Y \rightarrow Q_{\mathcal{X} \mid Y} \rightarrow P_X \right\}
\] (410)
\[
= \left\{ Q_{\mathcal{X} \mid Y} : \sum_{b \in \mathcal{Y}} Q_{\mathcal{X} \mid Y}(\cdot | b) Q_Y(b) = P_X \right\}.
\] (411)

**Remark 44.** \( P_n(\mathcal{X} \mid Q_Y; P_X) \) depends on \( y^n \in \mathcal{T}_{Q_Y}^n \) only through its type \( Q_Y \). The subscript \( n \) in its notation is to denote that \( P_X \) and \( Q_Y \) are \( n \)-types. The elements of \( P_n(\mathcal{X} \mid Q_Y; P_X) \) are conditional types, which are not necessarily \( n \)-types, see Remark 4.
Remark 45. Using the coset notation and the definitions above, the following identities are immediate:

\[
P_n(\mathcal{X} \times \mathcal{Y}; P_{\bar{X}} \times \cdot) = \bigsqcup_{Q_Y \in \mathcal{P}_n(Y)} P_n(\mathcal{X} | Q_Y; P_{\bar{X}}) \times Q_Y
\]

where the notation \( \sqcup \) emphasizes that the unionization is disjoint.

Remark 46. Throughout Appendices VI-A and VI-B, the block-length \( n \) is multiple of a fixed integer \( m \). The underlying meaning of the limit as \( n \to \infty \) is, in fact, \( n = km \) and \( k \to \infty \).

VI-A. Proof of the Lower Bound in Theorem 2

This section is devoted to the proof of

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{E}\left[ \left\| P_{Y^n|\mathcal{Y}^n} - R_{Y^n} \right\|_1 \right] \geq \mathbb{R}(R, P_{\bar{X}}; P_{Y|\mathcal{X}}). 
\]

Indeed, using the finite block-length analysis, we shall prove the following stronger claim (see Theorem 4 in Appendix VII):

\[
-\frac{1}{n} \log \mathbb{E}\left[ \left\| P_{Y^n|\mathcal{Y}^n} - R_{Y^n} \right\|_1 \right] \leq \min_{Q_Y|\mathcal{X} \in \mathcal{P}(\mathcal{Y}|P_{\bar{X}})} \left\{ D(P_{\bar{X}} Q_{Y|\bar{X}} \| P_{\bar{X}}) + \frac{1}{2} \left[ R - D(P_{\bar{X}} Q_{Y|\bar{X}} \| P_{\bar{X}} Q_Y) \right] \right\} - \tilde{\eta}_n,
\]

where the vanishing constant \( \tilde{\eta}_n \) depends only on the block-length \( n \) and the alphabet sizes \(|\mathcal{X}|\) and \(|\mathcal{Y}|\).

For a predetermined \( m \)-type \( P_{\bar{X}} \in \mathcal{P}_m(\mathcal{X}) \), suppose that \( n \) is a multiple of \( m \), and let

\[
R_{\bar{X}^n}(x^n) = \frac{1}{|T_{P_{\bar{X}}}|} 1 \left\{ x^n \in T_{P_{\bar{X}}} \right\}
\]

be a constant-composition distribution based on \( P_{\bar{X}} \) and suppose \( R_{\bar{X}^n} \to P_{Y^n|\mathcal{X}^n} \to R_{Y^n} \). It is easy to see that

\[
R_{Y^n} = \frac{1}{|T_{P_{\bar{X}}}|} \sum_{x^n \in T_{P_{\bar{X}}}^n} P_{Y^n|X^n = x^n}.
\]

Given \( y^n \in \mathcal{Y}^n \), define

\[
\tilde{L}_{\mathcal{Y}^n}(y^n) = \begin{cases} 
\frac{P_{Y^n|\mathcal{Y}^n}(y^n)}{R_{Y^n}(y^n)} & \text{if } R_{Y^n}(y^n) > 0, \\
1 & \text{otherwise.}
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{M} \sum_{j=1}^{M} P_{Y^n|X^n}(y^n | \bar{X}^n_j) & \text{if } R_{Y^n}(y^n) > 0, \\
1 & \text{otherwise.}
\end{cases}
\]

\[29\]Abuse of terminology. \( \mathcal{P}_n(\mathcal{X}|Q_Y; P_{\bar{X}}) \) does not have a group structure.
Note that, similar to its counterpart \( L_{\mathcal{E}_M^n}(y^n) \) defined in (99), \( \hat{L}_{\mathcal{D}_M^n}(y^n) \) is also a random variable because it depends on the random constant-composition codebook \( \mathcal{D}_M^n \), and by construction,

\[
E[\hat{L}_{\mathcal{D}_M^n}(y^n)] = 1. \tag{420}
\]

Recycling our notation from before, given \( y^n \in \mathcal{Y}^n \) of type \( Q_Y \) let the conditional type \( Q_{X|Y} \) of \( x^n \in T_{P_X}^n \) be defined such that the joint type \( Q_{X|Y} \) of the sequence \((x^n, y^n)\) satisfies\(^{30}\)

\[
Q_{X|Y}(a, b) = Q_{X|Y}(a|b)Q_Y(b). \tag{421}
\]

Further, let

\[
P_n(\mathcal{X}|Q_Y; P_X) = \{ Q_{X|Y} : Q_{X|Y} \rightarrow Q_{X|Y} \rightarrow P_X \} \tag{422}
\]

be the set of conditional types \( Q_{X|Y} \) given \( y^n \in T_{Q_Y}^n \) with a fixed \( \mathcal{X} \)-marginal \( P_X \), see Definition 22. Observing that \( P_{Y^n|X^n}(y^n|x^n) \) and \( R_{Y^n}(y^n) \) depend on \((x^n, y^n)\) only through its joint type, once again using the type enumeration method [27, 28]:

\[
\hat{L}_{\mathcal{D}_M^n}(y^n) = \frac{1}{M} \sum_{Q_{X|Y} \in P_n(\mathcal{X}|Q_Y; P_X)} \tilde{N}_{Q_{X|Y}}(y^n)\tilde{I}_{Q_{X|Y}}(y^n), \tag{423}
\]

where

\[
\tilde{I}_{Q_{X|Y}}(y^n) = \frac{P_{X^n|Y^n}(y^n|x^n_{Q_{X|Y}})}{R_{Y^n}(y^n)} \tag{424}
\]

for some \( x^n_{Q_{X|Y}} \in T_{Q_{X|Y}}^n(y^n) \), and the random variable

\[
\tilde{N}_{Q_{X|Y}}(y^n) = \left| \left\{ \hat{X}^n \in \mathcal{D}_M^n : \hat{X}^n \in T_{Q_{X|Y}}^n(y^n) \right\} \right| \tag{425}
\]

\[
= \sum_{\hat{X}^n \in \mathcal{D}_M^n} 1 \left\{ \hat{X}^n \in T_{Q_{X|Y}}^n(y^n) \right\} \tag{426}
\]

denotes the number of random codewords in \( \mathcal{D}_M^n \) which have conditional type \( Q_{X|Y} \) given \( y^n \in T_{Q_Y}^n \). Just like before, since \( \mathcal{D}_M^n \) contains \( M \) independent codewords, \( \tilde{N}_{Q_{X|Y}}(y^n) \) is a binomial random variable with cluster size \( M \) and success probability

\[
\tilde{p}_{Q_{X|Y}}(y^n) = \mathbb{P} \left[ \hat{X}^n \in T_{Q_{X|Y}}^n(y^n) \right] \tag{427}
\]

\[
= \frac{|T_{Q_{X|Y}}^n|}{|T_{P_X||Q_Y}^n|}, \tag{428}
\]

where (428) follows from Lemma 14.

It is crucial to note that changing the setting from the random i.i.d. codebook \( \mathcal{E}_M^n \) to random constant-composition codebook \( \mathcal{D}_M^n \) does not alter the fact that both \( \tilde{I}_{Q_{X|Y}}(y^n) \) and \( \tilde{p}_{Q_{X|Y}}(y^n) \) depend on \( y^n \) only through its type.

\(^{30}\)Since \( P_X \) is fixed, \( Q_{X|Y} = P_XQ_{Y|X} \) for some conditional type \( Q_{Y|X} \). However, the proof utilizes \( Q_{X|Y} = Q_{X|Y}Q_Y \), where \( Q_{X|Y} \) is a conditional type with fixed \( X \)-marginal \( P_X \).
Given \( y^n \in \mathcal{T}^n_{Q_Y} \) and \( Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_Y;P_X) \), define
\[
\hat{Z}_{Q_{X|Y}}(y^n) = \frac{1}{M} \hat{N}_{Q_{X|Y}}(y^n) \hat{I}_{Q_{X|Y}}(y^n),
\]
(429)
\[
\tilde{\mathcal{Y}}(M, Q_{X\bar{Y}}) = \frac{|\mathcal{T}^n_{Q_X}|}{|\mathcal{T}^n_{P_X}| \cdot |\mathcal{T}^n_{Q_Y}|} \min \left\{ 2, M^{-\frac{1}{2}} \frac{|\mathcal{T}^n_{P_X}|^{\frac{1}{2}} |\mathcal{T}^n_{Q_Y}|^{\frac{1}{2}}}{|\mathcal{T}^n_{Q_X}|^{\frac{1}{2}}} \right\},
\]
(430)
and observe that
\[
\mathbb{E} \left[ \left\| P_{Y^n|\mathcal{X}^n} - R_{Y^n} \right\|_1 \right] = \sum_{y^n \in Y^n} R_{Y^n}(y^n) \mathbb{E} \left[ \left\| \hat{L}_{Q_X}(y^n) - 1 \right\| \right]
= \sum_{y^n \in Y^n} R_{Y^n}(y^n) \mathbb{E} \left[ \min_{Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_Y;P_X)} Z_{Q_{X|Y}}(y^n) - \mathbb{E}[\hat{Z}_{Q_{X|Y}}(y^n)] \right]
\leq \sum_{y^n \in Y^n} R_{Y^n}(y^n) \max_{Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_Y;P_X)} \left\{ \frac{1}{|\mathcal{T}^n_{Q_X}|} \exp(-n \mathbb{E}[\hat{L}_{P_Y}(\hat{Y}|\hat{X})]) \tilde{\mathcal{Y}}(M, Q_{X\bar{Y}}) \right\}
\leq \mathbb{P}_n(\mathcal{X} \times \mathcal{Y}) \max_{Q_{Y|X} \in \mathcal{P}_n(\mathcal{Y}|P_X)} \left\{ \frac{1}{|\mathcal{T}^n_{Q_X}|} \exp(-n \mathbb{E}[\hat{L}_{P_Y}(\hat{Y}|\hat{X})]) \tilde{\mathcal{Y}}(M, P_{\bar{X}} Q_{\bar{Y}|\bar{X}}) \right\},
\]
(431)
(432)
(433)
(434)
(435)
(436)
where (431) follows from the definition of \( \hat{L}_{Q_X}(y^n) \) in (418); (432) follows from (423) and the definition of \( \hat{Z}_{Q_{X|Y}}(y^n) \) in (429); (433) follows from the triangle inequality; (434) follows from Lemma 15 in Appendix II-B and the fact that
\[
\mathcal{Y}^n = \bigcup_{Q_Y \in \mathcal{P}_n(\mathcal{Y})} \mathcal{T}^n_{Q_Y};
\]
(437)
in (435) \( \mathcal{P}_n(\mathcal{X} \times \mathcal{Y}; P_{\bar{X}} \times \cdot) \subset \mathcal{P}_n(\mathcal{X} \times \mathcal{Y}) \) denotes the set of all joint types whose \( \mathcal{X} \)-marginal is fixed to be \( P_{\bar{X}} \), see Definition 21, and equality follows from\(^{31}\) Lemma 19 in Appendix III; finally, in (436) the maximization is over the set of all conditional types given \( x^n \in \mathcal{T}^n_{P_X} \), and \( Q_{\bar{Y}} \) denotes the type induced by \( P_{\bar{X}} \) on \( Q_{\bar{Y}|\bar{X}} \), that is \( P_{\bar{X}} \rightarrow Q_{\bar{Y}|\bar{X}} \rightarrow Q_{\bar{Y}} \).

Denoting
\[
\mathcal{P}_\infty(\mathcal{Y}|P_{\bar{X}}) = \bigcup_{n \in \infty} \mathcal{P}_n(\mathcal{Y}|P_{\bar{X}}),
\]
(438)
it follows from (436) that
\[
\liminf_{n \to \infty} - \frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{X}^n} - R_{Y^n} \right\|_1 \right] \geq \inf_{Q_{Y|X} \in \mathcal{P}_\infty(\mathcal{Y}|P_X)} \left\{ D(P_X Q_{\bar{Y}|\bar{X}} P_{\bar{X}}) + \frac{1}{2} \left[ R - D(P_{\bar{X}} Q_{\bar{Y}|\bar{X}} P_{\bar{X}} Q_{\bar{Y}}) \right] \right\}
\]
(439)
\footnote{Also see Remarks 44 and 45.}

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\[
\begin{align*}
&= \min_{Q_{Y|X} \in \mathcal{P}(X|Y)} \left\{ D(P_X Q_{Y|X} \| P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|X} \| P_Y Q_Y) \right]_+ \right\} \tag{440} \\
&= \aleph(R, P_X, P_{Y|X}), \tag{441}
\end{align*}
\]

where (439) is a consequence of Lemmas 20 and 26 in Appendix III; (440) is due to Lemma 28 in Appendix IV-B.

\[\blacksquare\]

VI-B. Proof of the Upper Bound in Theorem 2

This section is devoted to the proof of

\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ \| P_{Y^n|\mathcal{Y}^n} - R_{Y^n} \|_1 \right] \leq \aleph(R, P_X, P_{Y|X}). \tag{442}
\]

Indeed, using the finite block-length analysis, we shall prove the following stronger claim (see Theorem 4 in Appendix VII):

\[
-\frac{1}{n} \log \mathbb{E} \left[ \| P_{Y^n|\mathcal{Y}^n} - R_{Y^n} \|_1 \right] \\
\leq \min_{Q_{Y|X} \in \mathcal{P}(Y|P_X)} \left\{ D(P_X Q_{Y|X} \| P_{XY}) + \frac{1}{2} \left[ R - D(P_X Q_{Y|X} \| P_Y Q_Y) \right]_+ \right\} + \bar{\upsilon}_n, \tag{443}
\]

where the vanishing constant \( \bar{\upsilon}_n \) depends on the block-length \( n \), the alphabet sizes \(|X|\) and \(|Y|\), and the joint distribution \( P_{X} P_{Y|X} \).

Similar to its i.i.d. counterpart, the biggest obstacle in showing (442) is the mutual dependences of the random variables\(^{32}\) \( \hat{N}_{Q_{X|Y}}(y^n) \). Borrowing the technique from the proof of the upper bound in the random i.i.d. codebook case presented in Section V, we shall first treat the number of codewords \( M \in \mathcal{D}^n \) as if it were a Poisson distributed random variable with mean \( \mu_n = \exp(nR) \). Observe that, if \( M \) were Poisson, then \( \{ \hat{N}_{Q_{X|Y}}(y^n) \} \) would become the Poisson Splitting of the codewords in \( \mathcal{D}^n \), meaning that, given two distinct conditional types \( Q_{X|Y} \) and \( R_{X|Y} \), the random variables \( \hat{N}_{Q_{X|Y}}(y^n) \) and \( \hat{N}_{R_{X|Y}}(y^n) \) would correspond to two distinct Poisson splits and they would become independent from one another. Once again, this will be the main ingredient of the upper bound proof. The part of the proof where the bound is shown in the case when \( M \) is Poisson distributed will be called the “pseudo-upper bound proof”. To prove the upper bound for the actual statement in Theorem 2, the auxiliary assumption that the codebook \( \mathcal{D}^n \) contains a random number of codewords needs to be eliminated, which can be done using Lemma 2. As already mentioned in Remark 14, it is possible to prove a result similar to that in Lemma 2 with the assumption that \( M \) is Poisson distributed, see Lemma 17 in Appendix II-B. Utilizing this result, it can be shown that it is immaterial whether \( M \) is Poisson distributed or \( M = \lceil \exp(nR) \rceil \) that (442) holds.

As was the case in Section V, the upper bound proof is divided into three subsections: Appendix VI-B.1 introduces the auxiliary assumption that the codebook size \( M \) is Poisson distributed with mean \( \mu_n = \)}

\[\text{One quick way to see these mutual dependences is that the sum of } \hat{N}_{Q_{X|Y}}(y^n) \text{ over all conditional types } Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_{Y}:P_X) \text{ is equal to } M.\]
exp(nR), Appendix VI-B.2 provides the pseudo-upper bound proof under the assumption that $M$ is Poisson distributed, and finally, Appendix VI-B.3 shows that, removing the auxiliary assumption by conditioning on $[\mu_n]$, one still cannot do better than $\&R(P_X, P_{Y|X})$.

### VI-B.1 Poissonization

Suppose, for the moment, that $M$ is Poisson distributed with mean $\mu_n = \exp(nR)$. In that case, using the established notation so far, for each $y^n \in T^n_{Q_Y}$ and each $Q_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_{Y}; P_X)$, the random variable

$$\tilde{N}_{Q_{X|Y}}(y^n) = \sum_{\hat{X}^n \in T^n_{Q_{X|Y}}(y^n)} \{\hat{X}^n \in T^n_{Q_{X|Y}}(y^n)\}$$

(444)

is a Poisson splitting of $M$ with mean

$$\mu_n \tilde{N}_{Q_{X|Y}}(y^n) = \exp(nR) \frac{|T^n_{Q_{X|Y}}|}{|T^n_{P_X}||T^n_{Q_Y}|}.$$  

Moreover, as the random variables $\tilde{N}_{Q_{X|Y}}(y^n)$ and $\tilde{N}_{R_{X|Y}}(y^n)$ correspond to different bins defined by different conditional types $Q_{X|Y}$ and $R_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_{Y}; P_X)$, they are independent from each other.

Let $\delta \in (0, 1)$, and note that for any $y^n \in Y^n$ an application of Lemma 4 in Appendix II-A with

$$W \leftarrow M \left[ P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right],$$

(446)

$$X \leftarrow M,$$

(447)

$$c \leftarrow (1 + \delta)\mu_n,$$

(448)

yields

$$(1 + \delta)\mu_n \mathbb{E} \left[ P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right]$$

(449)

On one hand, regarding the first term in the right side of (449), the triangle inequality implies

$$\mathbb{E} \left[ M \left| P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right| \right]$$

$$\geq \mathbb{E} \left[ MP_{Y^n|\mathcal{P}_M^n}(y^n) - \mu_n R_{Y^n}(y^n) \right] - \mathbb{E} \left[ |M - \mu_n| R_{Y^n}(y^n) \right]$$

(450)

Moreover, as the random variables $\tilde{N}_{Q_{X|Y}}(y^n)$ and $\tilde{N}_{R_{X|Y}}(y^n)$ correspond to different bins defined by different conditional types $Q_{X|Y}$ and $R_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_{Y}; P_X)$, they are independent from each other.

Let $\delta \in (0, 1)$, and note that for any $y^n \in Y^n$ an application of Lemma 4 in Appendix II-A with

$$W \leftarrow M \left[ P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right],$$

(446)

$$X \leftarrow M,$$

(447)

$$c \leftarrow (1 + \delta)\mu_n,$$

(448)

yields

$$(1 + \delta)\mu_n \mathbb{E} \left[ P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right]$$

(449)

On one hand, regarding the first term in the right side of (449), the triangle inequality implies

$$\mathbb{E} \left[ M \left| P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right| \right]$$

$$\geq \mathbb{E} \left[ MP_{Y^n|\mathcal{P}_M^n}(y^n) - \mu_n R_{Y^n}(y^n) \right] - \mathbb{E} \left[ |M - \mu_n| R_{Y^n}(y^n) \right]$$

(450)

Moreover, as the random variables $\tilde{N}_{Q_{X|Y}}(y^n)$ and $\tilde{N}_{R_{X|Y}}(y^n)$ correspond to different bins defined by different conditional types $Q_{X|Y}$ and $R_{X|Y} \in \mathcal{P}_n(\mathcal{X}|Q_{Y}; P_X)$, they are independent from each other.

Let $\delta \in (0, 1)$, and note that for any $y^n \in Y^n$ an application of Lemma 4 in Appendix II-A with

$$W \leftarrow M \left[ P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right],$$

(446)

$$X \leftarrow M,$$

(447)

$$c \leftarrow (1 + \delta)\mu_n,$$

(448)

yields

$$(1 + \delta)\mu_n \mathbb{E} \left[ P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right]$$

(449)

On one hand, regarding the first term in the right side of (449), the triangle inequality implies

$$\mathbb{E} \left[ M \left| P_{Y^n|\mathcal{P}_M^n}(y^n) - R_{Y^n}(y^n) \right| \right]$$

$$\geq \mathbb{E} \left[ MP_{Y^n|\mathcal{P}_M^n}(y^n) - \mu_n R_{Y^n}(y^n) \right] - \mathbb{E} \left[ |M - \mu_n| R_{Y^n}(y^n) \right]$$

(450)

where (451) follows from Lemma 5 in Appendix II-A. On the other hand, regarding the second term in the right side of (449),

$$\mathbb{E} \left[ M \{M > (1 + \delta)\mu_n]\right] \leq \mu_n a_n \sqrt{\mu_n - 1},$$

(452)

which\(^{\text{33}}\) is a consequence of Lemma 6 in Appendix II-A. Note that, in the right side of (452), $a_n$ is a constant that satisfies $a_n < 1$ for all $n \in (0, 1)$, which is explicitly defined in (276).

\(^{\text{33}}\)The bound in (452) is valid only when $\delta > \frac{1}{\mu_n}$. Even though the choice of $\delta \in (0, 1)$ does not depend on $\mu_n = \exp(nR)$, the applicability of Lemma 6 is guaranteed for large enough $n$.  

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As will be seen, the remaining terms in the right side of (454) are residual terms whose exponents are

\[ E \]

Capitalizing the result of previous subsection,

\[ M \]

the next subsection.

\[ 34 \]

Assembling (449), (451) and (452),

\[
(1 + \delta)\mathbb{E} \left[ \left| P_{\tilde{Y}^n | Y^m} (y^n) - R_{\tilde{Y}^n} (y^n) \right| \right] \\
\geq \frac{1}{\mu_n} \mathbb{E} \left[ \left| M P_{\tilde{Y}^n | Y^m} (y^n) - \mu_n R_{\tilde{Y}^n} (y^n) \right| \right] \\
- \frac{R_{\tilde{Y}^n} (y^n)}{\sqrt{\mu_n - 1}} - a \mu_n - \frac{\delta}{\mu_n}.
\]

(453)

The first term in the right side of (453) is the term of main interest whose in-depth analysis is provided in the next subsection.

VI-B.2 Pseudo-Upper Bound Proof Assuming \( M \) is Poisson Distributed

Capitalizing the result of previous subsection,

\[
(1 + \delta)\mathbb{E} \left[ \left| P_{\tilde{Y}^n | Y^m} - R_{\tilde{Y}^n} \right| \right] \\
= \sum_{y^n \in Y^n} (1 + \delta)\mathbb{E} \left[ \left| P_{\tilde{Y}^n | Y^m} (y^n) - R_{\tilde{Y}^n} (y^n) \right| \right] \\
\geq \sum_{y^n \in Y^n} \frac{1}{\mu_n} \mathbb{E} \left[ \left| M P_{\tilde{Y}^n | Y^m} (y^n) - \mu_n R_{\tilde{Y}^n} (y^n) \right| \right] \\
- \frac{1}{\sqrt{\mu_n - 1}} - |Y|^n a \mu_n - \frac{\delta}{\mu_n}.
\]

(454)

This section focuses on the summation in the right side of (454) and shows that its exponent is \( \mathbb{E}(R, P_X, P_Y | X) \).

As will be seen, the remaining terms in the right side of (454) are residual terms whose exponents are grater than \( 34 \mathbb{E}(R, P_X, P_Y | X) \), and therefore, they do not contribute to the overall exponential decay rate of \( \mathbb{E}[\left| P_{\tilde{Y}^n | Y^m} - R_{\tilde{Y}^n} \right|] \).

To this end, invoking the lemmas provided in Appendix II,

\[
\sum_{y^n \in Y^n} \frac{1}{\mu_n} \mathbb{E} \left[ \left| M P_{\tilde{Y}^n | Y^m} (y^n) - \mu_n R_{\tilde{Y}^n} (y^n) \right| \right] \\
= \sum_{y^n \in Y^n} \frac{R_{\tilde{Y}^n} (y^n)}{\mu_n} \mathbb{E} \left[ \left| M \tilde{L}_{\tilde{Y}^n | Y^m} (y^n) - \mu_n \right| \right] \\
= \sum_{y^n \in Y^n} \frac{R_{\tilde{Y}^n} (y^n)}{\mu_n} \mathbb{E} \left[ \max_{Q_X | Y \in \mathcal{Q}_X | Y} \left\{ \sum_{P_X | Y \in \mathcal{P}_X | Y} \mathbb{E} \left[ \left| \tilde{N}_{Q_X | Y} (y^n) - \mathbb{E} \left[ \tilde{N}_{Q_X | Y} (y^n) \right] \right| \right] \right\} \left( \tilde{N}_{Q_X | Y} (y^n) - \mathbb{E} \left[ \tilde{N}_{Q_X | Y} (y^n) \right] \right) \right] \\
\geq \sum_{y^n \in Y^n} \frac{1}{\mu_n} \mathbb{E} \left[ \max_{Q_X | Y \in \mathcal{Q}_X | Y} \left\{ \sum_{P_X | Y \in \mathcal{P}_X | Y} \mathbb{E} \left[ \left| \tilde{N}_{Q_X | Y} (y^n) - \mathbb{E} \left[ \tilde{N}_{Q_X | Y} (y^n) \right] \right| \right] \right\} \right] \\
= \frac{1}{4} \sum_{Q_Y \in \mathcal{Q}_Y} \max_{Q_X | Y \in \mathcal{Q}_X | Y} \left\{ \mathbb{E} \left[ \left| \tilde{N}_{Q_X | Y} (y^n) - \mathbb{E} \left[ \tilde{N}_{Q_X | Y} (y^n) \right] \right| \right] \right\} \\
= \frac{1}{4} \sum_{Q_Y \in \mathcal{Q}_Y} \max_{Q_X | Y \in \mathcal{Q}_X | Y} \left\{ \mathbb{E} \left[ \left| \tilde{N}_{Q_X | Y} (y^n) - \mathbb{E} \left[ \tilde{N}_{Q_X | Y} (y^n) \right] \right| \right] \right\}.
\]

\( ^{34}\)In the sense that they vanish with a faster rate with \( n \).

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\[\begin{align*}
&\geq \frac{1}{4} \max_{Q_X, } \max_{\mu, Q_{XY}} \mathbb{E}\{[T_{Q_X}^n| \exp(-n\mathbb{E}[\ell_{P_{XY}}(\bar{Y}|\bar{X})])\mathbb{F}(\mu_n, Q_{XY})]\} \\
&= \frac{1}{4} \max_{Q_{XY}} \mathbb{E}\{[T_{Q_X}^n| \exp(-n\mathbb{E}[\ell_{P_{XY}}(\bar{Y}|\bar{X})])\mathbb{F}(\mu_n, P_X Q_{XY})]\}
\end{align*}\]

where (455) follows from the definition of \(\tilde{L}_{\infty}^n(y^n)\) in (418); (456) follows from the type enumeration method, see (423), and Lemma 16 in Appendix II-B; the key step in (457) follows from Lemma 8 in Appendix II-A and the definition of \(\tilde{L}_{\infty}^n(y^n)\) in (424); in (458) the function \(\tilde{L}(\mu_n, Q_{XY})\) is as defined in (430) and the bound follows from Lemma 9 in Appendix II-A; in (459) \((\bar{X}, \bar{Y}) \sim P_X Q_{XY}\) and we invoke\(^{35}\) Lemma 19 in Appendix III; and finally, (460) follows because each summand on the right side of (459) is non-negative.\(^{36}\)

Note that
\[
\lim_{n \to \infty} -\frac{1}{n} \log \max_{Q_{XY} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \mathbb{E}\{[T_{Q_X}^n| \exp(-n\mathbb{E}[\ell_{P_{XY}}(\bar{Y}|\bar{X})])\mathbb{F}(\mu_n, P_X Q_{XY})]\} = \infty.
\]

On the other hand, going back to (454), the fact that \(\mu_n = \exp(nR)\) and \(a_\epsilon < 1\) for all \(\epsilon \in (0, 1)\) implies
\[
\lim_{n \to \infty} -\frac{1}{n} \log \frac{1}{\sqrt{\mu_n - 1}} = \frac{R}{2},
\]
\[
\lim_{n \to \infty} -\frac{1}{n} \log \left(\mathbb{E}[|Y^n|\alpha_{\delta}^{\mu_n} - 1/\mu_n]\right) = \infty.
\]

Since the right side of (463) is strictly less than \(R/2\), see Remark 33, it follows from (454), and (463)–(466) that, when \(M\) is a Poisson distributed random variable with mean \(\mu_n = \exp(nR)\),
\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mathbb{E}\left[\left\|P_{Y^n|\tilde{P}_{XY}^n} - R_{\tilde{Y}^n}\right\|_1\right] \leq \mathbb{R}(R, P_X, P_{Y|X}).
\]

### VI-B.3 Depoissonization

To prove the upper bound in Theorem 2, it remains to show that the result established in (467) still holds when \(M = \lfloor \exp(nR) \rfloor\). To this end, once again utilizing the fact that \(\mathbb{R}(R, P_X, P_{Y|X}) < R/2\), choose \(r \in (\mathbb{R}, R/2)\), let \(\epsilon_n = \exp(-nr)\), define the random variable
\[
\tilde{T}_n(m) = \left\|P_{Y^n|\tilde{P}_{XY}^n} - R_{\tilde{Y}^n}\right\|_1,
\]
and consider the following three events:
\[
\tilde{A}_n = \{\left|\mathbb{E}[\tilde{T}_n(\lfloor \mu_n \rfloor)] - \mathbb{E}[\tilde{T}_n(M)]\right| < 2\epsilon_n\},
\]

\(^{35}\) Also see Remark 44.

\(^{36}\) Also see Remark 45.
where \( \hat{T}_n([\mu_n]) \) denotes the case when the codebook is assumed to have a deterministic number of codewords and \( \hat{T}_n(M) \) denotes the case when the codebook is assumed to have a random (Poisson) number of codewords.

Observe that

\[
\mathbb{P}[\hat{A}_n] \geq \mathbb{P}[\hat{B}_n \cap \hat{C}_n] \geq 1 - \mathbb{P}[\hat{B}_n^c] - \mathbb{P}[\hat{C}_n^c] \geq 1 - \left(2 + 16[\mu_n]^{1/2}\right) \exp\left(-\frac{\mu_n \epsilon_n^2}{2}\right),
\]

where (472) is because \( \hat{A}_n \supset \hat{B}_n \cap \hat{C}_n \); (473) is union bound; and (474) follows from Lemmas 2 and 18 in Section III and Appendix II-B, respectively. Thanks to the choice of \( \epsilon_n \), for large enough \( n \), the right side of (474) is strictly greater than 0. Moreover, since \( A_n \) is a deterministic event, \( \mathbb{P}[\hat{A}_n] > 0 \) implies that \( \mathbb{P}[\hat{A}_n] = 1 \). That is, for large enough \( n \), and \( r \in (\mathbb{N}, R/2) \),

\[
\mathbb{E}[\hat{T}_n([\mu_n])] > \mathbb{E}[\hat{T}_n(M)] - 2 \exp(-nr).
\]

Hence, it follows that

\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mathbb{E}[\hat{T}_n([\mu_n])] \leq \limsup_{n \to \infty} -\frac{1}{n} \log(\mathbb{E}[\hat{T}_n(M)] - 2 \exp(-nr)) \leq R(P_X, P_{Y|X}),
\]

where (477) is due to (467).

\[\text{VII. Finite Block-length Results}\]

Using simple algebra, the following finite block-length bounds can be deduced from the analysis provided in Sections IV and V and Appendix VI.

**Theorem 3.** Fix \( n \in \mathbb{N} \). Suppose \( P_{X^n} \to P_{Y^n|X^n} \to P_{Y^n} \), where the \( n \)-shot stationary memoryless channel \( P_{Y^n|X^n} \) is non-degenerate, i.e., \( P_{Y^n|X^n} \neq P_{Y^n} \). For any \( R > I(P_X, P_{Y|X}) \), let \( M = \exp(nR) \), and denote by \( P_{Y^n|\mathcal{E}_M^n}(y^n) \) the induced output distribution when a uniformly chosen codeword from the random codebook \( \mathcal{E}_M^n \) is transmitted through the channel, see Definitions 12 and 13. Then,

\[
\min_{Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \| P_X Q_Y)]_+ \right\} - \kappa_n \leq -\frac{1}{n} \log \mathbb{E} \left[ \| P_{Y^n|\mathcal{E}_M^n} - P_{Y^n} \|_1 \right]
\]

\[
\leq -\frac{1}{n} \log \mathbb{E} \left[ \| P_{Y^n|\mathcal{E}_M^n} - P_{Y^n} \|_1 \right]
\]

\[
\leq \min_{Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \| P_X Q_Y)]_+ \right\} + \nu_n
\]
where for \( r \in (\alpha R, P_X, P_{Y|X}), R/2 \) and a fixed \( \delta \in (0, 1) \) that is greater than \( \exp(-nR) \),

\[
\kappa_n = \frac{|\mathcal{X}||\mathcal{Y}|}{n} \log(n + 1) + \frac{1}{n} \log 2, \\
\alpha_n = \min_{Q_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} \left\{ D(Q_{XY} \| P_{XY}) + \frac{1}{2} [R - D(Q_{XY} \| P_X Q_Y)]_+ \right\}, \\
\rho_n = \frac{(|\mathcal{X}| + 1)|\mathcal{Y}|}{n} \log(n + 1) + \frac{1}{n} \log 4, \\
\mu_n = \exp(nR), \\
\alpha_e = \frac{e^\epsilon}{(1 + \epsilon)^{1+\epsilon}}, \\
\phi_n = \exp(n(\alpha_n + \rho_n)) \left( \frac{1}{\sqrt{\mu_n - 1}} + |\mathcal{Y}| a_n^{\mu_n} + 2(1 + \delta) \exp(-nr) \right), \\
v_n = \rho_n + \log e - \frac{\phi_n}{1 - \phi_n} + \frac{1}{n} \log(1 + \delta).
\]

**Proof.** The lower bound, (478), easily follows from (111), (359), and (378). To see the upper bound, first assemble (134), (140), (154) and (379) to get

\[
(1 + \delta) \mathbb{E} \left[ \left\| P_{Y^n|X^n} - P_{Y^n} \right\|_1 \right] \geq \exp(-n(\alpha_n + \rho_n))(1 - \phi_n). 
\]  

The result in (479) follows after taking \(-\frac{1}{n} \log \) both sides and noticing that

\[
\log(1 - x) \geq \frac{-x}{1 - x} \log e. 
\]

\[\blacksquare\]

Similarly, the following finite block-length bounds follow from the constant-composition case analysis in Appendix VI.

**Theorem 4.** Let \( m \) be a fixed integer and \( P_{\bar{X}} \in \mathcal{P}_m(\mathcal{X}) \) be a fixed \( m \)-type. Fix \( n \in m\mathbb{N} \). Suppose that \( R_{\bar{X}^n} \) is a constant-composition distribution based on \( P_{\bar{X}} \) defined as in (28), and let \( R_{\bar{X}^n} \rightarrow P_{Y^n|X^n} \rightarrow R_{\bar{Y}^n} \), where the \( n \)-shot stationary discrete memoryless channel \( P_{Y^n|X^n} \) is non-degenerate, i.e., \( P_{Y^n|X^n} \neq R_{\bar{Y}^n} \). For any \( R > I(P_{\bar{X}}, P_{Y|\bar{X}}) \), let \( M = \exp(nR) \), and denote by \( P_{Y^n|\mathcal{F}_{\bar{X}}^n} \) the constant-composition induced output distribution when a uniformly chosen codeword from the random constant-composition codebook \( \mathcal{F}_{\bar{X}}^n \) is transmitted through the channel, see Definitions 19 and 20. Then,

\[
\min_{Q_{Y|\bar{X}} \in \mathcal{P}(\mathcal{Y}|P_{\bar{X}})} \left\{ D(P_{\bar{X}} Q_{Y|\bar{X}} \| P_{\bar{X}Y}) + \frac{1}{2} [R - D(P_{\bar{X}} Q_{Y|\bar{X}} \| P_{\bar{X}Q_Y})]_+ \right\} - \tilde{\eta}_n \\
\leq -\frac{1}{n} \log \mathbb{E} \left[ \left\| P_{Y^n|\mathcal{F}_{\bar{X}}^n} - R_{\bar{Y}^n} \right\|_1 \right] \\
\leq \min_{Q_{Y|\bar{X}} \in \mathcal{P}(\mathcal{Y}|P_{\bar{X}})} \left\{ D(P_{\bar{X}} Q_{Y|\bar{X}} \| P_{\bar{X}Y}) + \frac{1}{2} [R - D(P_{\bar{X}} Q_{Y|\bar{X}} \| P_{\bar{X}Q_Y})]_+ \right\} + \tilde{\eta}_n, 
\]

where \( P_{\bar{X}} \rightarrow Q_{\bar{Y}|\bar{X}} \rightarrow Q_{\bar{Y}} \), and for \( r \in (\mathcal{N}(R, P_{\bar{X}}, P_{Y|\bar{X}}), R/2) \) and a fixed \( \delta \in (0, 1) \) that is greater than \( \exp(-nR) \),

\[
[f]_+ = \max\{0, f\},
\]  

60
\[ \eta_n = \frac{|\mathcal{X}|(2 + 3|\mathcal{Y}|)}{2n} \log(n + 1), \]  
(492)

\[ \kappa_n = \min_{Q_{Y|X}} \left\{ D(P_{X}Q_{Y|X} \| P_{Y|X}) + \frac{1}{2} |R - D(P_{X}Q_{Y|X} \| P_{X}Q_{Y})|_1 \right\}, \]  
(493)

\[ \tilde{\rho}_n = \frac{|\mathcal{X}| + 2|\mathcal{X}| |\mathcal{Y}|}{2n} \log(n + 1) + \frac{1}{n} \log 4, \]  
(494)

\[ \mu_n = \exp(nR), \]  
(495)

\[ a_{\epsilon} = e^{(1 + \epsilon)(1 + \epsilon)}, \]  
(496)

\[ \tilde{\phi}_n = \exp(n(\mathcal{N}_n + \tilde{\rho}_n)) \left( \frac{1}{\sqrt{\mu_n - 1}} + |\mathcal{Y}| n^{\mu_n} - \frac{1}{n} + 2(1 + \delta) \exp(-nr) \right), \]  
(497)

\[ \tilde{\nu}_n = \tilde{\rho}_n + \frac{\log e}{n} \frac{\tilde{\phi}_n}{1 - \tilde{\phi}_n} + \frac{1}{n} \log(1 + \delta). \]  
(498)

**Proof.** Combining (359), (386), and (436), the bound in (489) is immediate. To get the upper bound, first assemble (387), (454), (461), and (475) to get

\[ (1 + \delta) E \left\| P_{Y^*|X} - R_{Y^*|X} \right\|_1 \geq \exp(-n(\mathcal{N}_n + \rho_n)) \left( 1 - \tilde{\phi}_n \right). \]  
(499)

The result in (490) follows after taking \(-\frac{1}{n} \log\) both sides and noticing that

\[ \log(1 - x) \geq \frac{-x}{1 - x} \log e. \]  
(500)

**Remark 47.** The \(P_{X|Y}P_{Y|X}\) (respectively, \(P_{X}P_{Y|X}\)) dependence of the upper bound constant \(\nu_n\) (respectively, \(\tilde{\nu}_n\)) in Theorem 3 (respectively, in Theorem 4) is due to the discontinuity of the exponent in the degenerate channel case, see Remark 21.

**VIII. Lemmas for the Dual Representation and Exponent Comparisons**

**Lemma 30.** Let \(U \sim P, V \sim Q\) and assume that \(f\) is a real valued function that has no internal dependence on the distribution \(Q\),

\[ \min_Q \{ D(Q\|P) - E[f(V)] \} = -\log E[\exp(f(U))], \]  
(501)

and the minimizing distribution \(Q^*\) satisfies

\[ \iota_{Q^*\|P}(x) = f(x) - \log E[\exp(f(U))]. \]  
(502)

**Proof.** Thanks to Jensen’s inequality

\[ D(Q\|P) - E[f(V)] = E[\iota_{Q\|P}(V) - f(V)] \geq -\log E[\exp(-\iota_{Q\|P}(V) + f(V))] = -\log E[\exp(f(U))], \]  
(503)

where the inequality in (504) holds with equality when \(\iota_{Q\|P}(x) = f(x) - \log E[\exp(f(U))]. \)
Corollary 1. Suppose \((X, Y) \sim P_{X|Y}P_Y\), \((\tilde{X}, \tilde{Y}) \sim Q_{X|Y}Q_Y\), and \((\tilde{X}, \tilde{Y}) \sim P_{X|Y}Q_Y\), then for any \(\lambda \in \mathbb{R}\)

\[
\min_{Q_{X|Y}} \left\{ D(Q_{X|Y}\|P_{X|Y}Q_Y) - \lambda \mathbb{E} \left[ t_{X,Y}(\tilde{X}; \tilde{Y}) \right] \right\} = -\mathbb{E} \left[ \log \mathbb{E} \left[ \exp \left( \lambda t_{X,Y}(\tilde{X}; \tilde{Y}) \right) \right] | Y = \bar{Y} \right],
\]

and for a fixed \(y \in \mathcal{Y}\), the minimizing conditional distribution \(Q_{X|Y}^*\) satisfies

\[
t_{Q_{X|Y}^*P_{X|Y}}(x|y) = \lambda t_{X,Y}(x; y) - \log \mathbb{E} \left[ \exp \left( \lambda t_{X,Y}(\tilde{X}; \tilde{Y}) \right) \right] | Y = y.
\]

Proof. For a fixed \(y \in \mathcal{Y}\), an application of Lemma 30 with

\[
P(\cdot) \leftarrow P_{X|Y}(\cdot|y),
Q(\cdot) \leftarrow Q_{X|Y}(\cdot|y),
f(\cdot) \leftarrow \lambda t_{X,Y}(\cdot; y)
\]
yields

\[
\min_{Q_{X|Y}} \left\{ D(Q_{X|Y}(\cdot|y)\|P_{X|Y}(\cdot|y)) - \lambda \mathbb{E} \left[ t_{X,Y}(\tilde{X}; \tilde{Y}) \right] | Y = y \right\} = -\log \mathbb{E} \left[ \exp \left( \lambda t_{X,Y}(\tilde{X}; \tilde{Y}) \right) \right] | Y = y.
\]

Taking expectation on both sides of (508) with respect to \(\bar{Y} \sim Q_Y\) gives (506).

Corollary 2. Suppose \((X, Y) \sim P_{X|Y}P_Y\), and \((\tilde{X}, \tilde{Y}) \sim Q_{X|Y}Q_Y\), then for any \(\lambda \in \mathbb{R}\)

\[
\min_{Q_Y} \left\{ D(Q_Y\|P_Y) - \frac{1}{2} \mathbb{E} \left[ \log \mathbb{E} \left[ \exp \left( \lambda t_{X,Y}(\tilde{X}; \tilde{Y}) \right) \right] Y \right] \right\} = -\log \mathbb{E} \left[ \mathbb{E}^\frac{1}{2} \left[ \exp \left( \lambda t_{X,Y}(X; Y) \right) \right] Y \right],
\]

and the minimizing distribution \(Q_Y^*\) satisfies

\[
t_{Q_Y^*P_Y}(y) = \frac{1}{2} \log \mathbb{E} \left[ \exp \left( \lambda t_{X,Y}(X; Y) \right) \right] | Y = y - \log \mathbb{E} \left[ \mathbb{E}^\frac{1}{2} \left[ \exp \left( \lambda t_{X,Y}(X; Y) \right) \right] Y \right].
\]

Proof. Let \(P_{X|Y}\) be the fixed random transformation from \(\mathcal{Y}\) to \(\mathcal{X}\). Applying Lemma 30 with

\[
P(\cdot) \leftarrow P_Y(\cdot),
Q(\cdot) \leftarrow Q_Y(\cdot),
f(\cdot) \leftarrow \frac{1}{2} \log \mathbb{E} \left[ \exp \left( \lambda t_{X,Y}(X; Y) \right) \right] | Y = \cdot.
\]
gives the desired result.

Corollary 3. Suppose \(\lambda \in [0, \infty), (X, Y) \sim P_{X|Y}P_Y\), and \((\tilde{X}, \tilde{Y}) \sim Q_{X|Y}Q_Y\), then

\[
\min_{Q_{XY}} \left\{ D(Q_{XY}\|P_{XY}) - \lambda \mathbb{E} \left[ t_{X,Y}(\tilde{X}; \tilde{Y}) \right] \right\} = -\lambda D_{1+\lambda}(P_{XY}\|P_XP_Y).
\]

where \(D_{\alpha}(P\|Q)\) denotes the Rényi divergence (see, e.g., [26]) of order \(\alpha\) between \(P\) and \(Q\), and the minimizing distribution \(Q_{XY}^*\) satisfies

\[
t_{Q_{XY}^*P_{XY}}(x, y) = \lambda t_{X,Y}(x; y) - \lambda D_{1+\lambda}(P_{XY}\|P_XP_Y).
\]
Proof. Immediate consequence of Lemma 30 with
\[ Q \leftarrow Q_{XY}, \]
\[ P \leftarrow P_{XY}, \]
\[ f(x, y) \leftarrow \lambda_{X,Y}(x; y). \]

\[ \square \]

Corollary 4. Suppose \( \lambda' \in [0, \infty), (X, Y) \sim P_{X|Y}P_Y, \) and \((\tilde{X}, \tilde{Y}) \sim Q_{X|Y}Q_Y, \) then
\[
\min_{Q_{XY}} \left\{ D(Q_{XY}\|P_{XY}) - \frac{1}{2} D(Q_{XY}\|P_{X|Y}Q_Y) - \frac{\lambda'}{2} \mathbb{E}\left[i_{X,Y}(\tilde{X}; \tilde{Y})\right] \right\} = -\frac{\lambda'}{2} \bar{D}_{1+\lambda'}(P_{XY}\|P_XP_Y). \tag{513} \]
where \( \bar{D}_{1+\lambda'}(P_{XY}\|P_XP_Y) \) is defined in (180), and the minimizing distribution \( Q^*_{XY} \) satisfies
\[
i_{Q_{XY}}|P_{XY}(x, y) = \frac{1}{2} \log \mathbb{E}\left[ \exp\left(\lambda\lambda_{X,Y}(X; Y)|Y = y\right) + \lambda\lambda_{X,Y}(x; y) - \left(\log \mathbb{E}\left[ \mathbb{E}^{\frac{1}{2}} \left[ \exp\left(\lambda\lambda_{X,Y}(X; Y)|Y\right)\right] \right] + \log \mathbb{E}\left[ \exp\left(\lambda\lambda_{X,Y}(\tilde{X}; \tilde{Y})\right)|\tilde{Y} = y\right]\right).ight. \tag{514} \]
with \((\tilde{X}, \tilde{Y}) \sim P_{X|Y}Q_Y. \)

Proof. Observe that
\[
\min_{Q_{XY}} \left\{ D(Q_{XY}\|P_{XY}) - \frac{1}{2} D(Q_{XY}\|P_{X|Y}Q_Y) - \frac{\lambda'}{2} \mathbb{E}\left[i_{X,Y}(\tilde{X}; \tilde{Y})\right] \right\} \\
= \min_{Q_Y} \left\{ D(Q_{Y}\|P_Y) + \frac{1}{2} D(Q_{X|Y}\|P_{X|Y}Q_Y) - \frac{\lambda'}{2} \mathbb{E}\left[i_{X,Y}(\tilde{X}; \tilde{Y})\right] \right\} \tag{515} \\
= \min_{Q_Y} \left\{ D(Q_{Y}\|P_Y) + \frac{1}{2} \min_{Q_{XY}} \left\{ D(Q_{X|Y}\|P_{X|Y}Q_Y) - \lambda\mathbb{E}\left[i_{X,Y}(\tilde{X}; \tilde{Y})\right] \right\} \right\} \tag{516} \\
= -\log \mathbb{E}\left[ \mathbb{E}^{\frac{1}{2}} \left[ \exp\left(\lambda\lambda_{X,Y}(X; Y)|Y\right)\right] \right] \tag{517} \\
= -\frac{\lambda'}{2} \bar{D}_{1+\lambda'}(P_{XY}\|P_XP_Y). \tag{519} \]
where (517) is the result of Corollary 1; (518) is the result of Corollary 2; and (519) is the definition of \( \bar{D}_{1+\lambda'}(P_{XY}\|P_XP_Y). \)

\[ \square \]

Corollary 5. Suppose \((X, Y) \sim P_XP_{Y|X}, \) and \((X, \tilde{Y}) \sim P_XQ_{Y|X}, \) then for any \( \lambda \in \mathbb{R} \)
\[
\min_{Q_Y|X} \left\{ D(Q_Y|X\|P_Y|X|P_X) - \lambda \mathbb{E}\left[i_{P_XY\|P_XS_Y}(X, \tilde{Y})\right] \right\} = -\mathbb{E}\left[ \log \mathbb{E}\left[ \exp\left(\lambda i_{P_{X|Y}\|P_XS_Y}(X, Y)\right)\right]|X\right], \tag{520} \]
and for a fixed \( x \in \mathcal{X}, \) the minimizing conditional distribution \( Q^*_{Y|X} \) satisfies
\[
i_{Q^*_{Y|X}}|P_{Y|X}(y|x) = \lambda i_{P_{X|Y}\|P_XS_Y}(x, y) - \log \mathbb{E}\left[ \exp\left(\lambda i_{P_{X|Y}\|P_XS_Y}(X, Y)\right)\right]|X = x. \tag{521} \]
Proof. For a fixed $x \in X$, an application of Lemma 30 with

\[ P(\cdot) \leftarrow P_{Y|X}(\cdot|x), \]
\[ Q(\cdot) \leftarrow Q_{Y|X}(\cdot|x), \]
\[ f(\cdot) \leftarrow \lambda_{p_{XY}\parallel p_{X|S_Y}}(x, \cdot) \]

yields

\[
\min_{Q_{Y|X}} \left\{ D(Q_{Y|X}(\cdot|x)||P_{Y|X}(\cdot|x)) - \lambda \mathbb{E} \left[ t_{p_{XY}\parallel p_{X|S_Y}}(X, \tilde{Y}) \bigg| X = x \right] \right\}
\]
\[
= - \log \mathbb{E} \left[ \exp \left( \lambda_{p_{XY}\parallel p_{X|S_Y}}(X, Y) \right) \bigg| X = x \right] \tag{522}
\]

Taking expectation on both sides of (522) with respect to $X \sim P_X$ gives (520).

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