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Optimal Control of a One-Dimensional Storage Process[1]

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Abstract. We consider the discounted and ergodic optimal control problems related to a one-dimensional storage process. The existence and uniqueness of the corresponding Bellman equation and the regularity of the optimal value is established. Using the Bellman equation an optimal feedback control is constructed. Finally we show that under this optimal control the origin is reachable.

Introduction

We investigate the optimal control of a one-dimensional storage process. This problem arises in the economic planning of a nonrenewable natural resource (such as oil, mineral deposits or energy) in a socially managed economy. K. Arrow in [1] modelled the level of natural resource as a controlled jump-process. The randomness of the process was due to the uncertainty in the exploration of the natural resource. S. D. Deshmukh and S. R. Pliska studied a similar model in [5] under the assumption that the unexplored area has an infinite area. Let us briefly explain this model.

Let $y(t)$ be the current level of the natural process at time $t \geqslant 0$. At each time t the planner determines the consumption rate $c(t) \in [0, c_0]$, under which the storage level decreases with the rate $c(t)$. Since the resource level is always non-negative, $c(t) = 0$ is the only choice whenever $y(t) = 0$. In addition to the consumption rate, the planner also determines the exploration rate $e(t) \in [0, e_0]$ which is the intensity of search effort to discover additional sources of the

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resource. Under this policy the resource level has the jump rate $\lambda(e(t))$ and the jump-size distribution $G(e(t), \cdot)$. Note that $G(e, \cdot)$ has support on $[0, \infty)$.

In this paper we use the above model with feedback strategies. Let $\pi(x) = (e(x), c(x))$ be a Borel measurable map of $[0, \infty)$ into $[0, e_0] \times [0, c_0]$. The map π is an admissible strategy if (i) there is a unique storage process $y(t)$ with the consumption rate $c(y(t))$ and the exploration rate $e(y(t))$ and (ii) $c(0) = 0$. For each admissible strategy π consider a discounted cost $J^\alpha(x, \pi)$ with discount factor $\alpha > 0$.

$$J^\alpha(x, \pi) = E\left[\int_0^\infty e^{-\alpha t}(u(c(y(t))) - h(e(y(t))) - f(y(t)))\, dt \,|\, y(0) = x\right] \tag{0.1}$$

The optimal value $v^\alpha(x)$ is the supremum of $J^\alpha(x, \pi)$ over all admissible strategies. This problem is studied by S. R. Pliska [8] and S. D. Deshmukh and S. R. Pliska [5] in the case of no-holding cost f. Heuristically v^α satisfies the Bellman equation

$$\sup_{\substack{0 \le e \le e_0 \\ 0 < c \le c_0}} [(A^\pi v^\alpha)(x) + u(c) - h(e)] = f(x) + \alpha v^\alpha(x) \tag{0.2}$$

where A^π is the infinitesimal generator of the storage process. Under (1.10)–(1.15) it is shown that the optimal value is bounded, continuous with bounded continuous derivative on $[0, \infty)$, this class of functions is denoted by $C_b^1([0, \infty))$. Moreover v^α solves the integro-differential equation:

$$\frac{d}{dx} v^\alpha(x) = \sup_{\substack{0 \le e \le e_0 \\ 0 < c \le c_0}} \left\{ \frac{1}{c}\left[u(c) - f(x) - \alpha v^\alpha(x) - h(e) + \int_0^\infty (v^\alpha(x+y) - v^\alpha(x))\lambda(e) G(e, dy) \right] \right\}; \qquad x > 0 \tag{0.3}$$

with boundary condition

$$\alpha v^\alpha(0) = \sup_{0 \le e \le e_0} \left\{ -h(e) + \int_0^\infty (v^\alpha(y) - v^\alpha(0))\lambda(e) G(e, dy) \right\} \tag{0.4}$$

Note that (0.3)–(0.4) is in fact equivalent to (0.2). The unusual form of the boundary condition is caused by the state-space constraint.

By standard selection theorems one can choose $\pi^* = (e^*, c^*)$ so that for all $x \ge 0\, e^*(x), c^*(x)$ maximizes (0.3)–(0.4). The properties of v^α yield that π^* is admissible. Moreover if the consumption utility rate u is twice continuously differentiable around the origin then under the optimal strategy π^*, the origin is reachable.

In Section 4 we consider the corresponding ergodic control problem. For an admissible strategy let $\theta(x,\pi)$ be

$$\theta(x,\pi) = \limsup_{t\to\infty}\frac{1}{t}E\left[\int_0^t u(c(y(s)) - h(e(y(s))) - f(y(t))\,ds\,|\,y(0)=x\right] \tag{0.5}$$

and θ be the supremum of $\theta(x,\pi)$ over all admissible controls. A standard technique of solving this problem [2,7] is to consider the function $g^\alpha(x) = v^\alpha(x) - v^\alpha(0)$. In Section 4 we show that $g^\alpha(\cdot)$ and $\alpha v^\alpha(0)$ converge to $g(\cdot)$ and θ on a subsequence. Moreover (g,θ) solves the limiting equation of (0.3), (0.4), i.e.,

$$\frac{d}{dx}g(x) = \sup_{\substack{0\leqslant e\leqslant e_0\\ 0<c\leqslant c_0}}\left\{\frac{1}{c}\left[u(c)-f(x)-h(e)-\theta + \int_0^\infty (g(x+y)-g(x))\lambda(e)G(e,dy)\right]\right\};\quad x>0 \tag{0.6}$$

$$\theta = \sup_{0\leqslant e\leqslant e_0}\left\{-h(e)+\int_0^\infty g(y)\lambda(e)G(e,dy)\right\} \tag{0.7}$$

As suggested by the notation θ is in fact the optimal value if $f(\infty)$ is larger than $u(c_0)$. Additionally one can use (0.6)–(0.7) to obtain an optimal strategy. To motivate the assumption that $f(\infty)$ is sufficiently large suppose the holding cost rate f is identically zero. Then $g(x) = xu'(0)$ is a solution of (0.6)–(0.7) and $\pi(x) = (e^*, 0)$ maximizes the expressions in (0.6)–(0.7) which is clearly not an optimal strategy in general.

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1. Controlled Process

Let e_0 and c_0 be positive numbers and $E = [0, e_0]$, $C = [0, c_0]$. An admissible strategy $\pi = (e, c)$ is a Borel measurable map of $[0,\infty)$ into $E\times C$ satisfying (i) $c(0) = 0$ (ii) for any $x \geqslant 0, s \geqslant 0$ the equation

$$\frac{d}{dt}y_0(x,s;t,\pi) = -c(y_0(x,s;t,\pi)) \qquad t>s \tag{1.1}$$

with initial data $y_0(x, s; s, \pi) = x$, has a unique solution. Further let λ be a map of E into $[0, \lambda_0]$ satisfying (1.13) and G be a map of E into the set of probability measures on $[0,\infty)$ satisfying (1.14).

Set $T_0 = 0$ and $Y_0 = x$. We now can construct a probability space (Ω, P), a sequence of random times $\{T_n : n = 0,1,2,\dots\}$ and a sequence of positive random numbers $\{Y_n : n = 0,1,\dots\}$, corresponding to the jumps in the storage level, satisfying

$$P(T_{n+1} - T_n \geq t | T_1, \dots, T_n; Y_1, \dots, Y_n)$$
$$= \exp\left\{ - \int_{T_n}^{T_n + t} \lambda\big(e\big(y_0(Y_n, T_n; \tau, \pi)\big)\big)\, d\tau \right\} \quad (1.2)$$

$$P(Y_{n+1} - y_0(Y_n, T_n; T_{n+1}, \pi) \in A) | T_1, \dots, T_{n+1}, Y_1, \dots, Y_n)$$
$$= G\big(e\big(y_0(Y_n, T_n; T_{n+1}, \pi)\big), A\big) \quad (1.3)$$

Above identity holds for all Borel subset A of $[0, \infty)$. For more information see [5]. Now define the storage process $y(x; t, \pi)$ as

$$y(x; t, \pi) = y_0(Y_n, T_n; t, \pi) \quad \text{if } t \in [T_n, T_{n+1}) \quad (1.4)$$

The process $y(x, t, \pi)$ is a strong Markov process with infinitesimal generator A^π. Put $\beta(e, dy) = \lambda(e) G(e, dy)$ then A^π is given by

$$A^\pi \varphi(x) = -c(x) \frac{d}{dx} \varphi(x) + \int_0^\infty (\varphi(x+y) - \varphi(x)) \beta(e(x), dy) \quad (1.5)$$

with domain of A^π containing at least continuously differentiable functions on $[0, \infty)$ with bounded derivative. More precise description of A^π is given in [5].

In this paper we consider two different control problems. Let $\mathscr{A}$ be the set of all admissible strategies and α positive.

$$J^\alpha(x, \pi) = E \int_0^\infty e^{-\alpha t} [u(c(y(x, t, \pi)))$$
$$- h(e(y(x, t, \pi))) - f(y(x, t, \pi))]\, dt \quad (1.6)$$

$$v^\alpha(x, \pi) = \sup_{\pi \in \mathscr{A}} J^\alpha(x, \pi) \quad (1.7)$$

We refer to v^α as the optimal value of the discounted problem. The ergodic control problem is defined as

$$\theta(x, \pi) = \limsup_{T \to \infty} \frac{1}{T} E \int_0^T [u(c(y(x, t, \pi)))$$
$$- h(e(y(x, t, \pi))) - f(y(x, t, \pi))]\, dt \quad (1.8)$$

$$\theta = \sup_{x \in [0, \infty)} \sup_{\pi \in \mathscr{A}} \theta(x, \pi) \quad (1.9)$$

We assume the following throughout the paper.

u, f, h are continuous (1.10)

u is concave on $[0, c_0]$ with $u(0) = 0$ and differentiable at the origin (1.11)

f, h are non-decreasing with $f(0) = h(0) = 0$. Moreover f is bounded with $f(\infty) = \lim_{x \to \infty} f(x)$. (1.12)

λ is a nonnegative continuous function on E with $\lambda(0) = 0$ and it is bounded by $\lambda_0 < \infty$. (1.13)

G is a weakly continuous map of E, i.e., for any φ bounded continuous on $[0, \infty)$ (1.14)

$$\lim_{e \to \bar{e}} \left| \int_0^\infty \varphi(y) G(e, dy) - \int_0^\infty \varphi(y) G(\bar{e}, dy) \right| = 0$$

$$\int_0^\infty y G(e, dy) < \infty \quad \text{for every } e \text{ in } E. \tag{1.15}$$

Remark. An application of Dini's theorem together with (1.13)–(1.15) yields $\sup_{e \in [0, e_0]} \beta(e, [0, \infty)) < \infty$ and $\sup_{e \in [0, e_0]} \int_0^\infty y\beta(e, dy) < \infty$. Moreover

$$\lim_{M \to \infty} \sup_{e \in [0, e_0]} \beta(e, [M, \infty)) = 0$$

$$\lim_{M \to \infty} \sup_{e \in [0, e_0]} \int_M^\infty y\beta(e, dy) = 0 \tag{1.16}$$

2. Bellman Equation

We will show that v^α is in $C_b^1([0, \infty))$, the set of bounded continuous functions with bounded continuous first derivatives, and v^α solves the equation (0.3)–(0.4). To simplify the notation we suppress α. Following S. Pliska [8] we consider the following equation for any $\delta > 0$.

$$\frac{d}{dx} v_\delta(x) = \sup_{\substack{e \in [0, e_0] \\ c \in [\delta, c_0]}} \left\{ \frac{1}{c} \left[\left(u(c) - f(x) - h(e) \right. \right. \right.$$
$$\left. \left. \left. + \int_0^\infty (v_\delta(x+y) - v_\delta(x)) \beta(e, dy) \right. \right. \right.$$
$$\left. \left. - \alpha v_\delta(x) \right] \right\}; x > 0 \tag{2.1}$$

$$\alpha v_\delta(0) = \sup_{e \in [0, e_0]} \left[-h(e) + \int_0^\infty (v_\delta(y) - v_\delta(0)) \beta(e, dy) \right] \tag{2.2}$$

Theorem 2 in [8] yields that there is a unique $C_b^1((0,\infty))\cap C_b([0,\infty))$ function v_δ and

$$\lim_{x\to\infty} \alpha v_\delta(x) = u(c_0) - f(\infty) \tag{2.3}$$

Moreover the following estimate is derived in [8].

$$-f(\infty) \leqslant \alpha v_\delta(x) \leqslant u(c_0) \quad \text{for all } x \in [0,\infty). \tag{2.4}$$

In addition, by using Lipschitz continuity of v_δ and (1.16) one can show that the integral term in (2.1) is continuous in x uniformly with respect to e, (see (2.19) also). Hence left-hand side of (2.1) is continuous, so $v_\delta \in C_b^1([0,\infty))$.

We need an estimate of the first derivative of v_δ which is independent of δ to pass the limit as δ tends to zero. The next two lemmas will be used to derive the estimate. Let "$'$" denote the spatial derivative.

Lemma 2.1. *Let v_δ be the solution of* (2.1)–(2.2). *If $v_\delta'(z)=0$ for some $z \geqslant 0$ then $v_\delta'(x) \leqslant 0$ for all $x \geqslant z$.*

Proof. Suppose not, then there is z_0 such that $v_\delta'(z_0)=0$ and $v_\delta'(x)>0$ on $(z_0, z_0+\varepsilon)$ for some ε positive. Set $\gamma = v_\delta(z_0+\varepsilon) - v_\delta(z_0)$, note that γ is positive. Consider the set

$$\Gamma = \{x > z_0 : v_\delta'(x) = 0 \quad \text{and} \quad v_\delta(x) \geqslant v_\delta(z_0) + \gamma\} \tag{2.5}$$

Since v_δ' is continuous Γ is closed. Since $v_\delta'(z_0)=0$, $u(c_0)-f(\infty) \leqslant \alpha v_\delta(z_0) < \alpha v_\delta(z_0+\varepsilon)$ and $v_\delta'(z_0+\varepsilon) \geqslant 0$. Hence (2.3) yields that there has to be at least one zero of v_δ' larger than z_0 and the first one must be in Γ. So Γ is non-empty, also (2.3)–(2.4) imply that Γ must be bounded. Now let $x_0 = \sup\{x : x \in \Gamma\}$. Then x_0 is in Γ and finite. We claim that $v_\delta(x_0) \geqslant v_\delta(x)$ for all $x \geqslant x_0$. Suppose not, i.e. there is $y > x_0$ such that $v_\delta(y) > v_\delta(x_0)$. Then one of the following should hold:

(i) $v_\delta'(y) = 0$; then $y \in \Gamma$ and this contradicts the fact that $x_0 = \sup\{x : x \in \Gamma\}$.

(ii) $v_\delta'(y) < 0$; define $y_1 = \sup\{x \leqslant y : v_\delta'(x) = 0\}$. Since $v_\delta(y) > v_\delta(x_0) y_1$ must be larger than x_0 and $v_\delta(y_1) \geqslant v_\delta(y) > v_\delta(x_0)$. Hence $y_1 \in \Gamma$ which is a contradiction.

(iii) $v_\delta'(y) > 0$; define $y_2 = \inf\{x \geqslant y : v_\delta'(y) = 0\}$. A similar argument yields a contradiction.

Hence $v_\delta(x_0) \geqslant v_\delta(x)$ for all $x \geqslant x_0$. In equation (2.1) the integral term at x_0 is negative, thus maximum is achieved by choosing $e = 0$.

$$0 = v_\delta'(x_0) = \sup_{\delta \leqslant c \leqslant c_0} \left\{ \frac{1}{c}[u(c) - f(y_0) - \alpha v_\delta(x_0)] \right\}$$

One can easily conclude that $\alpha v_\delta(x_0) = u(c_0) - f(x_0)$. Choose $e = 0$ and $c = c_0$ at z_0 in equation (2.1) to obtain

$$0 = v_\delta'(z_0) \geqslant \frac{1}{c_0}[u(c_0) - f(z_0) - \alpha v_\delta(z_0)]$$

So $\alpha v_\delta(z_0) \geqslant u(c_0) - f(z_0) \geqslant u(c_0) - f(x_0) = \alpha v_\delta(x_0)$. This contradicts the fact that x_0 is in Γ, hence the result. □

In the case there is no holding cost v_δ is concave. This is no longer true when f is not zero, but we have the following analog of it. Let

$$z_0 = \inf\{x \geqslant 0 : v_\delta'(x) = 0\} \quad \text{or } +\infty \text{ if } v_\delta'(x) > 0 \text{ for all } x \geqslant 0 \tag{2.6}$$

Lemma 2.2. *Suppose there is* $\varepsilon \geqslant 0$ *such that* $\beta(e[0, \varepsilon]) = 0$ *for every* e *in* E. *Then* $v_\delta'(x)$ *is decreasing on* $[0, z_0]$.

Proof. Suppose z_0 is finite. To simplify the calculations let for φ bounded

$$I(x, e, \varphi) = -h(e) + \int_0^\infty (\varphi(x+y) - \varphi(x))\beta(e, dy) \tag{2.7}$$

$$I(x, \varphi) = \sup_{e \in E} I(x, e, \varphi) \tag{2.8}$$

Rewrite equation (2.1) as

$$\frac{d}{dx} v_\delta(x) = \sup_{c \in [\delta, c_0]} \left\{ \frac{1}{c} [u(c) - f(x) - \alpha v_\delta(x) + I(x, v_\delta)] \right\}; \qquad x > 0 \tag{2.9}$$

On $[0, z_0]$ αv_δ is increasing and f is non-decreasing. Therefore it is enough to show $I(x, v_\delta)$ is non-increasing on $[0, z_0]$. The previous lemma yields that for $x \in [z_0 - \varepsilon, \infty) I(x, e, v_\delta) = 0$ for all $e \in E$. Thus $I(x, v_\delta) = 0$ and $v_\delta'(x)$ is decreasing on $x \in [z_0 - \varepsilon, \infty)$. Define x_0 as

$$x_0 = \inf\{x \geqslant 0 : I(y, v_\delta) \quad \text{is non-increasing on } [x, \infty)\}. \tag{2.10}$$

Above calculation shows $x_0 \leqslant z_0 - \varepsilon$ and on $[x_0, z_0] v_\delta'(x)$ is decreasing. If $x_0 = 0$ we are done. Suppose $x_0 > 0$, since $v_\delta'(x)$ is decreasing on $[x_0, z_0]$ and it is continuous. There is $\gamma > 0$ such that for all $x \in [x_0 - \gamma, x_0]$

$$v_\delta'(x) > v_\delta'(y) \quad \text{for all } y \in [x + \varepsilon, \infty) \tag{2.11}$$

Take $x_1 < x_2$ in $[x_0 - \gamma, x_0]$.

$$I(x_2, e, v_\delta) - I(x_1, e, v_\delta) = \int_\varepsilon^\infty \int_{x_1}^{x_2} (v_\delta'(y+t) - v_\delta'(t))\, dt\, \beta(e, dy) \leqslant 0 \tag{2.12}$$

The last inequality follows (2.11). Thus we concluded that $I(x, e, v_\delta)$ is non-increasing on $[x_0 - \gamma, x_0]$ which contradicts the choice of x_0. Hence x_0 must be zero.

Suppose $z_0 = +\infty$, i.e. $v'_\delta(x) > 0$ for every $x \in [0, \infty)$ and because of (2.3) $\lim_{x \to \infty} v'_\delta(x) = 0$. For each $y_0 \in [0, \infty)$ define $A(y_0)$ as

$$A(y_0) = \{x \geqslant y_0 : v'_\delta(x) > v'_\delta(y_0)\} \tag{2.13}$$

then $A(y_0)$ is an open bounded subset of (y_0, ∞). Suppose $A(y_0)$ is non-empty, there is N (finite or infinite) and a sequence of disjoint intervals $\{(x_i, y_i): i = 1, 2, \dots\}$ such that

$$A(y_0) = \bigcup_{i=1}^{N} (x_i, y_i) \quad \text{and} \quad y_i < x_{i+1}$$

If N is infinite choose m such that $y_n - x_m < \varepsilon$, for every $n \geqslant m$. Then on (x_m, y_m) $v'_\delta(x)$ satisfies (2.11). Thus v'_δ is decreasing on (x_m, y_m), in particular $v'_\delta(x_m) > v'_\delta(y_m)$ which contradicts the choice of x_m, y_m. If N is finite repeat the argument (2.10)–(2.12) on the interval (x_N, y_N), to conclude v'_δ is decreasing on (x_N, y_N). This is again a contradiction. Hence $A(y_0)$ is empty which implies $I(x, v_\delta)$ is nonincreasing. □

Suppose the conclusion of lemma 2.2 holds. For any φ bounded define $K(x, \varphi)$ as follows

$$K(x, \varphi) = -f(x) - \varphi\alpha(x) + I(x, \varphi) \tag{2.14}$$

Then $K(x, v_\delta)$ is decreasing on $[0, z_0]$ and equation (2.2) yields $K(0, v_\delta) = 0$. Thus for $x \in [0, z_0]$

$$0 \leqslant v'_\delta(x) = \sup_{c \in [\delta, c_0]} \left\{ \frac{1}{c} [u(c) + K(x, v_\delta)] \right\} \leqslant \sup_{c \in [\delta, c_0]} \frac{u(c)}{c} \leqslant u'(0)$$

On the other hand $I(x, v_\delta) = 0$ on $[z_0, \infty)$ and (2.4) yields $-\alpha v_\delta(x) \geqslant -u(c_0)$. Thus for $x \in [z_0, \infty)$

$$0 \geqslant v'_\delta(x) = \sup_{c \in [\delta, c_0]} \left\{ \frac{1}{c} [u(c) - f(x) - \alpha v_\delta(x)] \right\}$$

$$\geqslant \sup_{c \in [\delta, c_0]} -\frac{f(x)}{c} \geqslant -f(\infty) c_0^{-1}$$

Hence we obtained the estimate

$$-f(\infty) c_0^{-1} \leqslant v'_\delta(x) \leqslant u'(0) \quad \text{for all } x \geqslant 0 \tag{2.15}$$

Lemma 2.2 and the estimate (2.15), which is independent of ε, suggest the following approximation to (2.1) and (2.2). For n positive integer let β^n be

$$\beta^n(e, B) = \beta\left(e, B \cap \left(\frac{1}{n}, \infty\right)\right) + \beta\left(e, \left[0, \frac{1}{n}\right]\right) \chi_B\left(\frac{1}{n}\right) \tag{2.16}$$

Here χ_B is the indicator function of B. Let $v_{\delta, n}$ be the solution of (2.1)–(2.2) with

β^n instead of β. Since β^n satisfies the hypothesis of lemma 2.2 $v_{\delta,n}$ satisfies (2.15) and also (2.4). Therefore Ascoli–Arzela yields that there is a subsequence denoted by n again and Lipschitz continuous function $\bar{v}_\delta$ such that $v_{\delta,n}$ converges to $\bar{v}_\delta$ uniformly on compact sets. Let $I^n(x,e,v_{\delta n})$ be defined as in (2.7) with β^n instead of β. Then for every $M>0$ we have

$$\begin{aligned}
&|I(x,e,\bar{v}_\delta)-I^n(x,e,v_{\delta n})|\\
&\quad\leqslant \left|v_{\delta n}\left(x+\frac{1}{n}\right)-v_{\delta n}(x)\right|\beta\left(e,\left[0,\frac{1}{n}\right]\right)\\
&\qquad+\int_0^{1/n}|\bar{v}_\delta(x+y)-\bar{v}_\delta(x)|\beta(e,dy)\\
&\qquad+2\|v_{\delta,n}-\bar{v}_\delta\|_{L^\infty([x,M+x])}\beta(e,[0,M])\\
&\qquad+\left[\|v'_{\delta,n}\|_{L^\infty([0,\infty))}+\|\bar{v}'_\delta\|_{L^\infty([0,\infty))}\right]\beta(e,[M,\infty)).
\end{aligned}\tag{2.17}$$

Use the estimates (2.4)–(2.15) to conclude

$$\begin{aligned}
|I(x,e,\bar{v}_\delta)-I^n(x,e,v_{\delta,n})| &\leqslant C\frac{1}{n}+C\|v_{\delta,n}-v_\delta\|_{L^\infty([x,M+x])}\\
&\quad+C\sup_{e\in E}\beta(e,[M,\infty)).
\end{aligned}$$

The fact (1.16) yields that $I(x,v_{\delta n})$ converges to $I(x,\bar{v}_\delta)$. Now it is easy to show

$$\lim_{n\to\infty}\bar{v}'_{\delta n}(x)=\sup_{c\in[\delta,c_0]}\frac{1}{c}[u(c)-f(x)+I(x,\bar{v}_\delta)-\alpha\bar{v}_\delta(x)];\ x>0 \tag{2.18}$$

Moreover, we claim that the right-hand side of (2.18) is continuous on $[0,\infty)$. Recall that $\bar{v}_\delta$ is Lipschitz continuous

$$\begin{aligned}
&|I(x,e,\bar{v}_\delta)-I(z,e,\bar{v}_\delta)|\\
&\quad\leqslant\int_0^\infty(|\bar{v}_\delta(x+y)-\bar{v}_\delta(z+y)|+|\bar{v}_\delta(x)-\bar{v}_\delta(z)|)\beta(e,dy)\\
&\quad\leqslant C|x-z|\beta(e,[0,\infty))\leqslant C|x-z|\lambda_0
\end{aligned}\tag{2.19}$$

Hence $I(x,\bar{v}_\delta)$ is continuous on $[0,\infty)$. Using this one can prove the claim. Thus we proved that $\bar{v}_\delta\in C_b^1([0,\infty))$ and $\bar{v}_\delta$ solves the equation (2.1)–(2.2). But (2.1)–(2.2) has a unique solution, so $v_\delta=\bar{v}_\delta$. We proved the following

Lemma 2.3. *The solution v_δ of* (2.1)–(2.2) *satisfies the estimates* (2.4) *and* (2.15).

We have obtained an estimate of v'_δ which is independent of δ. Using this we can now pass to the limit to solve the original equation in which we are interested.

Theorem 2.4. *There is a unique solution v of* (0.3)–(0.4) *in $C_b^1([0,\infty))$. Moreover v satisfies* (2.4) *and* (2.15).

Proof. Since v_δ satisfies (2.4) and (2.15) there is a subsequence denoted by δ again and a Lipschitz continuous function v such that v_δ converges to v

uniformly on compact subsets of $[0, \infty)$. Arguing as in (2.17)–(2.19) we can show that $I(x, v_\delta)$ converges to $I(x, v)$ and $I(x, v)$ is continuous in x. Let $K(x, v)$ be defined as in (2.14). Then $K(x, v_\delta)$ converges to $K(x, v)$, in particular $K(x, v) \leqslant 0$ with $K(0, v) = 0$.

Suppose $K(x, v) < 0$. Then there is $\eta > 0$ such that for sufficiently small δ we have

$$v'_\delta(x) = \sup_{c \in [\delta, c_0]} \left\{ \frac{1}{c} [u(c) + K(x, v_\delta)] \right\} = \sup_{c \in [\eta, c_0]} \left\{ \frac{1}{c} [u(c) + K(x, v_\delta)] \right\}$$

The last expression converges to

$$\sup_{c \in [\eta, c_0]} \left\{ \frac{1}{c} [u(c) + K(x, v)] \right\} = \sup_{c \in [0, c_0]} \left\{ \frac{1}{c} [u(c) + K(x, v)] \right\}.$$

Suppose $K(x, v) = 0$. Then $K(x, v_\delta)$ converges to zero. For $\varepsilon > 0$ choose $\bar{c} > 0$ so that $\left| \frac{u(\bar{c})}{\bar{c}} - u'(0) \right| \leqslant \frac{\varepsilon}{2}$ and choose $\bar{\delta} > 0$ so that $K(x, v_\delta) \geqslant -\frac{\varepsilon \bar{c}}{2}$ for all $\delta \leqslant \bar{\delta}$. For δ smaller than $\bar{\delta}$ and $\bar{c}$ we have

$$v'_\delta(x) \geqslant \frac{1}{\bar{c}} [u(\bar{c}) + K(x, v_\delta)] \geqslant -\varepsilon + u'(0)$$

Additionally $v'_\delta(x) \leqslant \sup_{c \in [0, c_0]} \frac{u(c)}{c} = u'(0)$. Therefore we have $\lim_{\delta \to 0} v'_\delta(x) = u'(0)$. Combining this with the other case, $K(x, v_\delta) < 0$, yields

$$\lim_{\delta \to 0} v'_\delta(x) = \sup_{c \in [0, c_0]} \left\{ \frac{1}{c} [u(c) + K(x, v)] \right\}; \qquad x > 0 \tag{2.20}$$

Arguing similarly one can prove that the right hand side of (2.20) is continuous in x. This shows that $v \in C_b^1([0, \infty))$ and solves (0.3)–(0.4).

Uniqueness can be proved by using the maximum principle as in [8] or it follows a verification theorem as in lemma 6 of [5]. □

3. Optimal Strategies

Since $v \in C_b^1([0, \infty))$ the argument (2.19) holds for $I(x, e, v)$. So $I(x, e, v)$ is continuous in x uniformly with respect to e. Moreover (1.13) and (1.14) yields that $I(x, e, v)$ is also continuous in e. Thus one can select $e^*(x)$ Borel measurable such that $I(x, e^*(x), v) = I(x, v)$ for all $x \geqslant 0$. Recall that $I(x, v)$ is non-increasing. Thus $K(x, v)$ is decreasing on $[0, z_0]$. In particular, $K(x, v) < K(0, v) = 0$. Therefore one can select $c^*(x)$ non-decreasing and $[u(c^*(x)) + K(x, v)] c^*(x)^{-1} = v(x)$ for all $x > 0$. Since c^* is monotone $\pi^* = (e^*, c^*)$ is an admissible strategy. Moreover $\pi^*(x) = (0, c_0)$ on $x \in [z_0, \infty)$. Now we can show $J(x, \pi^*) = v(x)$ as in lemma 6 of [5].

Suppose u is strictly concave. Then we claim c^* is continuous on $[0,\infty)$ and it is the only optimal consumption rate. To prove this assume that there are $c_2 > c_1 > 0$ such that

$$v'(x) = \frac{u(c_1)+K(x,v)}{c_1} = \frac{u(c_2)+K(x,v)}{c_2} \tag{3.1}$$

By using the strict concavity of u we obtain

$$\left(\frac{c_1+c_2}{2}\right)^{-1}\left[u\left(\frac{c_1+c_2}{2}\right)+K(x,v)\right]$$
$$> \frac{c_1}{c_1+c_2}\left[\frac{u(c_1)+K(x,v)}{c_1}\right]+\frac{c_2}{c_1+c_2}\left[\frac{u(c_2)+K(x,v)}{c_2}\right] = v_\delta'(x)$$

This contradicts the choice of c_1 and c_2.

Proposition 3.1. *Suppose u is twice differentiable in a neighborhood of origin and strictly concave. There there is $\gamma > 0$ such that $c^*(x) \geqslant \gamma x^{1/2}$ for small x. In particular origin is reachable under the strategy π^*.*

Proof. Let $K(x,v)$ be as in (2.14). Recall that $K(0,v)=0$ and $K(x,v)$ decreasing on $[0, z_0]$

$$\begin{aligned} K(x,v) &= -f(x)-\alpha v(x)+\sup_{e\in E} I(x,e,v)-K(0,v) \\ &\leqslant -f(x)-\alpha v(x)+I(x,e^*(x),v)+\alpha v(0)-I(0,e^*(x),v) \\ &= \int_0^\infty\int_0^x (v'(y+t)-v'(t))\,dt\,\beta(e^*(x),dy)-\alpha\int_0^x v'(t)\,dt-f(x) \end{aligned} \tag{3.2}$$

The second inequality obtained by choosing $e^*(x)$ in $K(0,v)$. Since $K(x,v)$ is continuous and $K(0,v)=0$, one can show that z_0 is away from the origin. Thus $v'(y+t)-v'(t) \leqslant 0$ for small t and y positive, so the first integral in (3.2) is non-positive.

$$K(x,v) \leqslant -f(x)-\alpha x \inf\{v'(y): y\in(0,x]\}. \tag{3.3}$$

Equation (0.3) yields that $\lim_{y\to 0} v'(y)=u'(0)$, so we can choose x_0 small enough so that $v'(y) \geqslant \frac{1}{2}u'(0)$ for every $y \leqslant x_0$. Substitute this into (3.3) to obtain a $\eta > 0$ such that

$$K(x,v) \leqslant -f(x)-\eta x \quad \text{for small } x \geqslant 0. \tag{3.4}$$

Note that at $c^*(x)$ the mapping $[u(c)+K(x,v)]/c$ has an interior maximum.

Thus we have

$$c^*(x)u'(c^*(x)) - u(c^*(x)) - K(x,v) = 0 \tag{3.5}$$

But $u(c) \leqslant u(0) + cu'(0)$ for every x, plug this into (3.5)

$$c^*(x)[u'(c^*(x)) - u'(0)] \leqslant K(x,v) \leqslant -\eta x \tag{3.6}$$

Also $u'(c) - u'(0) = \int_0^c u''(t)\,dt \geqslant c\min\{u''(t): t \in [0,c]\}$. Strict concavity of u implies that there is $\tilde{\alpha} > 0$ such that $u'(c) - u'(0) \geqslant -c\tilde{\alpha}$ for every c. Therefore

$$-[c^*(x)]^2\tilde{\alpha} \leqslant -\nu x. \qquad \square$$

4. Ergodic Control Problem

Let v^α be the solution of (0.3)–(0.4). Consider the function $g^\alpha(x) = v^\alpha(x) - v^\alpha(0)$. The estimate (2.15) yields that there is $K > 0$ such that

$$\sup_{\alpha > 0,\, x \geqslant 0} \left| \frac{d}{dx} g^\alpha(x) \right| \leqslant K \tag{4.1}$$

Also $g^\alpha(0) = 0$ for every $\alpha > 0$. Thus Ascoli–Arzela implies there is a subsequence denoted by α again and $g \in C([0,\infty))$ so that g^α converges to g uniformly on every compact subset of $[0,\infty)$. Passing to the limit in (0.4) yields that there is $\bar{\theta}$ such that $\bar{\theta} = \lim_{\alpha \to 0} \alpha v^\alpha(0)$ and

$$\bar{\theta} = \sup_{e \in E} \left\{ -h(e) + \int_0^\infty g(y)\beta(e,dy) \right\} \tag{4.2}$$

Moreover $\lim_{\alpha \to 0} \alpha v^\alpha(0) = \lim_{\alpha \to 0} \alpha v^\alpha(x) = \bar{\theta}$, because of (2.15). So one can pass to the limit in (0.3) also and by arguing as in Theorem 1 one can conclude that $g \in C^1([0,\infty))$ and $g' \in C_b([0,\infty))$ and

$$\frac{d}{dx} g(x) = \sup_{\substack{e \in [0,e_0] \\ c \in [0,c_0]}} \left\{ \frac{1}{c}\left[u(c) - f(x) - \bar{\theta} + \int_0^\infty (g(x+y) - g(x))\beta(e,dy) - h(e) \right] \right\}; \quad x > 0 \tag{4.3}$$

with $g(0) = 0$. Since $I(x,g)$ is the limit of $I(x,v^\alpha)$ and $I(x,v^\alpha)$ is non-increasing on $[0,\infty)$, $I(x,g)$ must be non-increasing. Consider $\bar{K}(x,g)$ defined as

$$\bar{K}(x,g) = -f(x) - \bar{\theta} + I(x,g) \tag{4.4}$$

Note that $\bar{K}(x, g)$ is non-increasing. The equation (4.2) reads as $\bar{\theta} = I(0, g)$, so $\bar{K}(0, g) = 0$. In particular $\bar{K}(x, g) \leqslant -f(x)$. As in Section 3 we can select $\pi^*(x) = (e^*(x), c^*(x))$ which is an admissible strategy and for every x such that $\bar{K}(x, g) < 0$ we have

$$\left(c^*(x)\right)^{-1}\left[u\left(c^*(x)\right) + \bar{K}(x, g)\right] = \sup\left\{\frac{1}{c}\left(u(c) + \bar{K}(x, g)\right) : c \in \left[0, c_0\right]\right\}$$

and

$$I\left(x, e^*(x), g\right) = \sup\left\{I(x, e, g) : e \in \left[0, e_0\right]\right\} = I(x, g) \quad \text{for all } x \geqslant 0$$

Note that $\bar{K}(x, g)$ may be equal to zero on an interval like $[0, a]$ and $c^*(x) = 0$ on this interval.

Proposition 4.1. *$\bar{K}(x, g)$ and $I(x, g)$ are non-increasing with $\bar{K}(0, g) = 0$ and $I(0, g) = \bar{\theta}$. Suppose $f(\infty) > u(c_0)$. Then $z_0 = \inf\{x \geqslant 0 : g'(x) = 0\}$ is finite.*

Proof. Equation (4.3) reads as $g'(x) = \sup_{c \in [0, c_0]} \{c^{-1}[u(c) + \bar{K}(x, g)]\}$ and $\bar{K}(x, g) \leqslant -f(x)$. □

Theorem 4.2. *Suppose z_0 is finite then $\bar{\theta} = \theta = \theta(x, \pi^*)$ for every $x \geqslant 0$, where θ and $\theta(x, \pi^*)$ are defined in* (1.8)–(1.9). *In particular this holds if $f(\infty) > u(c_0)$.*

Proof. Since g' is bounded (1.5) implies g is in the domain of A^{π}. By Dynkin's formula

$$Eg\left(y(x, t, \pi^*)\right) = g(x) + E\int_0^t \left(A^{\pi^*}g\right)\left(y(x, \tau, \pi^*)\right) d\tau. \tag{4.5}$$

Note that equation (4.3)–(4.4) implies $A^{\pi^*}g(x) = f(x) + h(e^*(x)) - u(c^*(x)) + \bar{\theta}$. Substitute this into (4.5)

$$\begin{aligned}\bar{\theta} = {} & \frac{1}{t}\left[Eg\left(y(x, t, \pi^*)\right) - g(x)\right] + \frac{1}{t}E\int_0^t \left[u\left(c^*\left(y(x, \tau, \pi^*)\right)\right)\right. \\ & \left. - h\left(e^*\left(y(x, \tau, \pi^*)\right)\right) - f\left(y(x, \tau, \pi^*)\right)\right] d\tau.\end{aligned} \tag{4.6}$$

Since z_0 is finite $g(y) \leqslant g(z_0)$ for every $y \geqslant 0$. Pass to the limit in (4.6) to obtain

$$\bar{\theta} \leqslant \theta(x, \pi^*) \quad \text{for all } x \geqslant 0. \tag{4.7}$$

For large M positive there is $G_M \in C_b^1([0, \infty))$ satisfying (i) $G_M(x) = g(x)$ if $x \leqslant z_0 + M$, (ii) $G_M'(x) \geqslant g'(x)$ for all $x \geqslant 0$, and (iii) $G_M(x + y) - G_M(x) \leqslant 0$ whenever $x + y \geqslant z_0 + M$. In fact G_M is a smooth version of the function $h_M(x) = g(x)$ on $x \in [0, z_0 + M]$ and $h_M(x) = g(z_0 + M)$ on $x \in [z_0 + M, \infty)$.

Now use conditions (ii) and (iii) to obtain $I(x, G_M) = I(x, g) = 0$ on $x \in [z_0, \infty)$. The equation (4.3) together with (ii) above yield

$$-cG'_M(x) + \int_0^\infty (G_M(x+y) - G_M(x))\beta(e, dy) \leqslant \bar{\theta} + f(x) + h(e) - u(c),$$

$$\text{for all } e, c, x \in [z_0, \infty). \quad (4.8)$$

Use the estimate (2.15) and condition (iii) to conclude

$$\int_0^\infty (G_M(x+y) - G_M(x))\beta(e, dy) \leqslant \int_0^\infty (g(x+y) - g(x))\beta(e, dy)$$
$$+ \int_{z_0+M-x}^\infty \|g'\|_{L^\infty} y\beta(e, dy) \quad (4.9)$$

The remark in Section 1 and (1.16) imply that the last term in (4.9) tends to zero as M goes to infinity. It will be denoted by $h(M)$. The equation (4.3), condition (ii) above and (4.9) yield

$$-cG'_M(x) + \int_0^\infty (G_M(x+y) - G_M(x))\beta(e, dy)$$
$$\leqslant \bar{\theta} + f(x) + h(e) - u(c) + h(M),$$

$$\text{for all } e, c, x \in [0, z_0] \quad (4.10)$$

Apply Dynkin's formula to G_M and use equation (4.3) together with (4.8), (4.10) to obtain for all π in $\mathscr{A}$:

$$\bar{\theta} \geqslant \frac{1}{t} E(G_M(y(x,t,\pi)) - G_M(x))$$
$$+ \frac{1}{t} E\int_0^t [u(c(y(x,\tau,\pi))) - h(e(y(x,\tau,\pi)))$$
$$- f(y(x,\tau,\pi))]\, d\tau - h(M) \quad (4.11)$$

First send t to infinity then M to infinity to obtain $\bar{\theta} \geqslant \theta(x, \pi)$ for all π. □

Theorem 4.3. *There is only one solution* (θ, g) *in* $R \times C^1([0,\infty))$ *of the equation* (4.2)–(4.3) *with* (*i*) $g(0) = 0$, (*ii*) *there is* $z_0 < \infty$ *such that* $g'(x) \leqslant 0$ *for all* $x \geqslant z_0$.

Proof. Suppose (θ_1, g_1) solves (4.2)–(4.3). In the proof of Theorem 4.2 we used only the fact $\sup g(x) < \infty$, so $\theta_1 = \theta$.

Let $g(x)$ be the limit of $\alpha v^\alpha(x) - \alpha v^\alpha(0)$ and $z_0(z_1)$ be the first zero of g' (g'_1 respectively). Finally let $y_0 = \max\{z_0, z_1\}$. Then on $x \in [y_0, \infty)$ $I(x, g) = I(x, g_1) = 0$ and the equation (4.3) yields that $g'(x) = g'_1(x)$ on the same interval. Thus we have

$$g'(x) = g'_1(x) \leqslant 0 \quad \text{for all } x \in [y_0, \infty);\ g'(y_0) = g'_1(y_0) = 0. \quad (4.12)$$

Let $\bar{K}(x, g)$ be defined as in (4.4). Then it is continuous with $\bar{K}(y_0, g_1) = \bar{K}(y_0, g) = -u(c_0)$. Consider x_n defined by

$$x_n = \inf\{y \geqslant 0 : \bar{K}(z, g_1) \leqslant -1/n \quad \text{and } \bar{K}(z, g) \leqslant -1/n \quad \text{for all } z \in [y, \infty)\} \tag{4.13}$$

There is $\eta > 0$ such that $g'(x) = \sup_{c \in [\eta, c_0]} \{\frac{1}{c}[u(c) + \bar{K}(x, g)]\}$ on $x \in [x_n, \infty)$ and the same statement holds for g_1 also. Let γ to be chosen later and $\bar{x} \in [y_0 - \gamma, y_0]$ such that

$$m = g'(\bar{x}) - g_1'(\bar{x}) = \max\{g'(x) - g_1'(x) : x \in [y_0 - \gamma, y_0]\} \tag{4.14}$$

Let e^*, c^* be as in Theorem 4.2. Then $c^*(x) \geqslant \eta$ on $[x_n, \infty)$ and the equation (4.3) yields

$$\begin{aligned} m = g'(\bar{x}) - g_1'(\bar{x}) &\leqslant \frac{1}{c^*(\bar{x})} \int_0^\infty \int_0^y \big(g'(\bar{x} + t) - g_1'(\bar{x} + t)\big) \beta(e^*(\bar{x}), dy) \\ &\leqslant \frac{1}{\eta} \int_0^{y_0 - \bar{x}} my\beta(e^*(\bar{x}), dY) \\ &\leqslant m\frac{\gamma}{\eta} \sup_{e \in E} \beta(e, [0, \infty)) \end{aligned} \tag{4.15}$$

Choose γ so that the last expression is less than m. Thus $m \leqslant 0$, but the argument is symmetric so we conclude that $g'(x) = g_1'(x)$ for all $x \in [y_0 - \gamma, \infty)$. Repeat the same argument to cover the interval $[x_n, \infty)$. We have

$$g'(x) = g_1'(x) \quad \text{for all } x \in [x_\infty, \infty) \tag{4.16}$$

where $x_\infty = \lim_{n \to \infty} x_n$. Moreover $\bar{K}(x_\infty, g) = \bar{K}(x_\infty, g_1) = 0$ but $\bar{K}(x, g)$ is non-increasing with $\bar{K}(0, g) = 0$. Therefore

$$\bar{K}(x, g) = f(x) = 0,\ g'(x) = u'(0),\ I(x, g) = \theta \quad \text{for all } x \in [0, x_\infty] \tag{4.17}$$

For $x \leqslant x_\infty$ we have

$$0 \leqslant I(x_\infty, g_1) - I(x, g_1) \leqslant \int_0^\infty \int_x^{x_\infty} \big(g_1'(t + y) - g_1'(t)\big)\, dt\, \beta\big(e_1^*(x_\infty), dy\big) \tag{4.18}$$

where $e_1^*(x_\infty)$ maximizes $I(x_\infty, e, g_1)$, because of (4.16) one may pick $e_1^*(x_\infty) = e^*(x_\infty)$. Now we claim that $\beta(e^*(x_\infty), [0, x_\infty)) = 0$.

$$\begin{aligned} 0 = I(0, g) - I(x_\infty, g) &\geqslant I\big(0, e^*(x_\infty), g\big) - I\big(x_\infty, e^*(x_\infty), g\big) \\ &+ \int_0^\infty \int_0^y \big(g'(t) - g'(x_\infty + t)\big)\, dt\, \beta\big(e^*(x_\infty), dy\big) \end{aligned} \tag{4.19}$$

The integrand is non-negative and for $t \in [0, x_\infty)$; $g'(t) - g'(x_\infty + t) > 0$. Therefore the claim should hold. Substitute this into (4.18)

$$0 \leqslant I(x_\infty, g_1) - I(x, g_1) \leqslant \int_{x_\infty}^{\infty}\int_{x}^{x_\infty}\big(g_1'(t+y) - g_1'(t)\big)\beta\big(e^*(x_\infty), dy\big) \tag{4.20}$$

Let $\bar{x} = \inf\{y \leqslant x_\infty : g_1'(\tau) = u'(0)$ for all $\tau \in [y, x_\infty]\}$. If $\bar{x} > 0$ one can find $\gamma > 0$ such that

$$g_1'(t) \geqslant g_1'(t+y) \quad \text{for all } y \geqslant x_\infty \quad \text{and} \quad t \in [\bar{x} - \gamma, \infty). \tag{4.21}$$

Use this in (4.20) to get $\theta = I(x_\infty, g_1) = I(y, g_1)$ on $y \in [\bar{x} - \gamma, x_\infty]$. Subsequently $g_1'(y) = u'(0)$ on the same interval, contradicting the choice of $\bar{x}$. Thus $\bar{x} = 0$. □

Any continuously differentiable solution g_1 of (4.3) satisfies $g_1'(x) \leqslant u'(0)$. Thus

$$I(x, g_1) \leqslant u'(0) \sup_{0 \leqslant e \leqslant e_0} \int_0^\infty y\beta(e, dy) = \alpha.$$

Then $\bar{K}(x, g_1) \leqslant \alpha - f(x)$ since θ is non-negative. So if $f(\infty) > u(c_0) + \alpha$ then $g_1'(x)$ must be negative for sufficiently large x. We have the following result:

Corollary 4.4. *If* $f(\infty) > u(c_0) + u'(0) \cdot \sup_{0 \leqslant e \leqslant e_0} \int_0^\infty y\beta(e, dy)$ *then there is one solution of* (4.2)–(4.3).

Remark. For $x, z \geqslant 0$ and admissible strategy π define a random time $\tau_{x,z}^\pi$ as

$$\tau_{x,z}^\pi = \inf\{t \geqslant 0 : y(x, t, \pi) = z\}. \tag{4.22}$$

Suppose there is $z \geqslant 0$ such that $\tau_{x,z}^{\pi^*} < \infty$ almost surely for every x. An easy application of Dynkin's formula yields

$$g(x) = g(z) + E\int_0^{\tau_{x,z}^{\pi^*}} \big[u\big(c^*(y(x,t,\pi^*))\big) - h\big(e^*(y(x,t,\pi^*))\big) - f\big(y(x,t,\pi^*)\big) - \theta\big]\, dt. \tag{4.23}$$

See [2] for more information.

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